Remark on spaces dominated by manifolds

by

I. Berstein and **T. Ganea** (Bucureşti)

1. Introduction and results. Let X, Y be arbitrary topological spaces and $f: X \to Y$ a continuous map. A map $g: Y \to X$ is called a *left (right) homotopy inverse* of f if $gf \simeq 1_X$ ($fg \simeq 1_Y$), where $\simeq 1_E$ means homotopic to the identity map of E. The map f is called a *homotopy equivalence* if there exists a map $g: Y \to X$ which is both a left and a right homotopy inverse of f; if f only has a left homotopy inverse, then Y is said *to dominate* X ([9], p. 214).

By a *manifold* we mean a connected locally Euclidean Hausdorff space; no triangulability assumptions are made. As usual, $H^n(X; Z)$ stands for the nth singular cohomology group of X with integer coefficients. Our result is expressed by

THEOREM 1. *Let $f: X \to Y$ be a continuous map of an arbitrary topological space X into a compact n-dimensional manifold Y. If $H^n(X; Z) \neq 0$ and if f has a left homotopy inverse, then f is a homotopy equivalence.*

Remark 1. If f is a homotopy equivalence, every left homotopy inverse of f also is a right homotopy inverse of f.

Remark 2. Denote by $\{X\}$ the homotopy type of the space X and write $\{X\} \preccurlyeq \{Y\}$ if Y dominates X. This is a quasi-order ([4], p. 212) in the class of "all" homotopy types. Let $\mathcal{C}^n$ denote the subclass of all homotopy types of integral cohomological dimension $\geqslant n$. Our result then implies the

COROLLARY. *The homotopy types of compact n-dimensional manifolds are minimal elements in $\mathcal{C}^n$.*

2. Preliminaries. Since the manifold Y in Theorem 1 is arcwise connected and dominates X, the latter also is arcwise connected.

Let now $P(X)$ denote the singular polytope of X; this is a connected simplicial CW-complex and there is a map $\varphi: P(X) \to X$ which induces isomorphisms of homotopy groups in all dimensions ([5], Theorem VI). Since the compact manifold Y is dominated by a CW-complex, the same also holds for X and, by [9], Theorem 1, φ is a homotopy equivalence. As a consequence

Remark 3. It is enough to prove Theorem 1 in the case where X is a connected CW-complex.

This enables us to consider connected covering spaces of X, generally denoted by (B, p) where $p: B \to X$ is the projection; in particular, X has a simply connected covering space. The cardinal number of the set $p^{-1}(x)$ is the same for all points $x \in X$ and is called the *number of sheets* of (B, p). The group of all homeomorphisms ξ of B onto itself satisfying $p\xi = p$ is called the *Deckbewegungsgruppe* of (B, p) and is denoted by $\Pi(B, p)$. If $\xi, \eta \in \Pi(B, p)$ and $\xi(b) = \eta(b)$ for some $b \in B$, then $\xi = \eta$. The covering space (B, p) is called *regular* if, for any two points $b_1, b_2 \in B$ satisfying $p(b_1) = p(b_2)$, there exists $\xi \in \Pi(B, p)$ such that $\xi(b_1) = b_2$.

Similar considerations also apply to the covering spaces of the (not necessarily triangulable) manifold Y.

Let (P, p) and (Q, q) be regular covering spaces of X and Y respectively; suppose that the diagram of spaces and maps

$$\begin{array}{ccc} P & \xrightarrow{e} & Q \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is commutative. Then, to every $\xi \in \Pi(P, p)$ there corresponds a unique element $\eta \in \Pi(Q, q)$ such that $e\xi = \eta e$; this defines a homomorphism

$$_1e: \Pi(P, p) \to \Pi(Q, q),$$

referred to as the homomorphism induced by e.

We shall frequently use the well-known

Monodromy principle. *Let X, Y be arcwise connected, locally arcwise connected, locally simply connected spaces. Let (B, p) be a connected covering space of Y. If $f_1: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is the homomorphism induced by a map $f: (X, x_0) \to (Y, y_0)$ and if $f_1\pi_1(X, x_0) \subset p_1\pi_1(B, b_0)$ with $p(b_0) = y_0$, then there exists a map $\varphi: (X, x_0) \to (B, b_0)$ such that $p\varphi = f$.*

A singular simplex in a space E is a map $T: \Delta \to E$ where Δ is the standard Euclidean simplex; the support $|T|$ of T is the subset $T(\Delta) \subset E$. For any Abelian group G, $H_r(E; G)$ and $H^r(E; G)$ stand for the ordinary singular homology and cohomology groups of E with coefficients in G ([4], chapter VII); Z and Z_2 respectively denote the group of integers and of integers mod 2.

Some of the arguments that follow could have been equally well presented within the framework of homology with local coefficients in the sense of Steenrod.

3. Homology in covering spaces.

We shall prove here

Proposition 1. *Let (P, p) and (Q, q) be regular covering spaces of the connected CW-complex X and of the compact n-dimensional manifold Y. Suppose that the diagram of spaces and maps*

$$\text{(1)} \qquad \begin{array}{ccc} P & \xrightarrow{e} & Q \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is commutative. Let G stand for Z or Z_2 and assume that

(2) *the homomorphism $_1e: \Pi(P, p) \to \Pi(Q, q)$ induced by e is an isomorphism,*

(3) $H_n(Y; G) \neq 0$,

(4) $f_*: H_n(X; G) \to H_n(Y; G)$ *is an epimorphism.*

Then, $e_*: H_r(P; G) \to H_r(Q; G)$ is an epimorphism for all $r \geqslant 0$.

Proof. Let Φ_Q denote the family of all compact subsets of the (possibly non-compact) manifold Q. We shall use

(I) Homology groups $\mathcal{H}_i(Q; G)$ based on (possibly infinite) singular chains c such that each set belonging to Φ_Q meets the supports of at most finitely many singular simplexes occuring in c with non-vanishing coefficient;

(II) Cohomology groups $\mathcal{H}^i(Q; G)$ based on ordinary singular cochains γ with the additional property that, for some $C = C(\gamma) \in \Phi_Q$, any singular simplex T satisfying $|T| \cap C = \emptyset$ yields $\gamma(T) = 0$.

Let further Φ_P denote the family of all closed subsets F of P such that $e(F)$ is contained in a set belonging to Φ_Q. Accordingly, introduce as above the auxiliary groups $\mathcal{H}_i(P; G)$ and $\mathcal{H}^i(P; G)$.

Notice that the groups $\mathcal{H}_i(Q; G)$ and $\mathcal{H}^i(Q; G)$ coincide with the groups of second kind described in [1], 5-06.

By the choice of the families Φ_Q and Φ_P, the map $e: P \to Q$ sends chains of type (I) into chains of type (I) and cochains of type (II) into cochains of type (II), therefore inducing homomorphisms

$$\varepsilon_*: \mathcal{H}_i(P; G) \to \mathcal{H}_i(Q; G),$$
$$\varepsilon^*: \mathcal{H}^i(Q; G) \to \mathcal{H}^i(P; G).$$

Since G is a ring, the cap-product of an s-cochain of type (II) and of an $(r+s)$-chain of type (I) may be defined in the usual manner (see for instance [3], 28.2, p. 436) and yields an ordinary finite r-chain. There results a pairing

$$\frown: \mathcal{H}^s \otimes \mathcal{H}_{r+s} \to H_r$$

with $\frown(a\otimes a)$ denoted by $a\frown a$, satisfying

$$e_*\big((\varepsilon^*\gamma)\frown c\big) = \gamma\frown(\varepsilon_* c) \tag{5}$$

for all $\gamma\in\mathcal{H}^s(Q;G)$, $c\in\mathcal{H}_{r+s}(P;G)$, see for instance [3], 27.2, p. 435 or [8], p. 119.

Since Q is locally compact, the groups $\mathcal{H}_i(Q;G)$ coincide with the groups $H_i^{\mathcal{F}}(Q;G)$, as defined for instance in [8], p. 118, where $\mathcal{F}$ is the family of all closed subsets of Q. Moreover, on a manifold, the singular groups $\mathcal{H}^i(Q;G)$ coincide with the groups $H^i_{\mathcal{C}}(Q;G)$ and may replace them in the duality theorem (see [1], 8-06 and [2], 20-01; $\mathcal{C}$ denotes the family of all compact subsets of Q).

Therefore, to every $z_r\in H_r(Q;G)$ there corresponds a dual class $\zeta^s = D(z_r)\in\mathcal{H}^s(Q;G)$ with $r+s=n$, and

$$\zeta^s\frown v = z_r \tag{6}$$

where $v\in\mathcal{H}_n(Q;G)$ is the fundamental class of the manifold Q ([8], p. 120). As follows from Lemma 1 below, there exists $u\in\mathcal{H}_n(P;G)$ such that $\varepsilon_*(u)=v$ and, by (5) and (6), we obtain

$$e_*\big((\varepsilon^*\zeta^s)\frown u\big) = \zeta^s\frown(\varepsilon_* u) = \zeta^s\frown v = z_r$$

with $(\varepsilon^*\zeta^s)\frown u\in H_r(P;G)$. Thus, Proposition 1 is proved provided we establish

LEMMA 1. *Under the assumptions of Proposition* 1, *the homomorphism* $\varepsilon_*\colon \mathcal{H}_n(P;G)\to\mathcal{H}_n(Q;G)$ *is onto.*

Proof. Since Δ is simply connected, the monodromy principle yields for every singular simplex $T\colon\Delta\to X$ a singular simplex $T'\colon\Delta\to P$ satisfying $pT'=T$; moreover, the regularity of (P,p) implies that any other $T''\colon\Delta\to P$ satisfying $pT''=T$ is of the form $T''=\xi T'$ with $\xi\in\Pi(P,p)$, where $\xi T'$ stands for the composition $\Delta\xrightarrow{T'}P\xrightarrow{\xi}P$. We may therefore introduce the (possibly infinite) chain

$$\sigma(T) = \Sigma\{\xi T' \mid \xi\in\Pi(P,p)\},$$

which does not depend on any particular choice of T'.

A quite similar procedure associates with every singular simplex $T\colon\Delta\to Y$ the (possibly infinite) chain

$$\tau(T) = \Sigma\{\eta T' \mid \eta\in\Pi(Q,q)\}$$

with $qT'=T$; as above, $T'\colon\Delta\to Q$ exists and $\tau(T)$ depends only on T. It is easy to check that

(7) $\tau(T)$ *is a chain of type* (I).

Furthermore, for any $T\colon\Delta\to X$ and $T'\colon\Delta\to P$ satisfying $pT'=T$, the commutativity of (1) implies $qeT'=fT$; therefore, (2) implies

$$\begin{aligned} e\sigma(T) &= \Sigma\{e\xi T' \mid \xi\in\Pi(P,p)\}\\ &= \Sigma\{\eta e T' \mid \eta\in\Pi(Q,q)\} = \tau f(T)\end{aligned} \tag{8}$$

and, recalling the definitions of Φ_P and Φ_Q, (7) now implies that also

(9) $\sigma(T)$ *is a chain of type* (I).

Since the singular simplexes in X and Y form a free G-base of the ordinary chain-groups, we may extend σ and τ to all finite chains. As readily seen, σ and τ commute with the corresponding boundary operators, therefore inducing homomorphisms σ_* and τ_* for which, by (8), the diagram

$$\begin{array}{ccc} \mathcal{H}_i(P;G) & \xrightarrow{e_*} & \mathcal{H}_i(Q;G)\\ {\scriptstyle\sigma_*}\uparrow & & \uparrow{\scriptstyle\tau_*}\\ H_i(X;G) & \xrightarrow[f_*]{} & H_i(Y;G)\end{array} \tag{10}$$

is commutative. By (4), f_* is an epimorphism for $i=n$ and Lemma 1 is an obvious consequence of

LEMMA 2. *Under the assumptions of Proposition* 1, τ_* *is an isomorphism for* $i=n$.

Proof. Select subsets M and N of Y such that the pair (M,N) be homeomorphic to the pair

$$(\{x \mid \|x\|\leqslant 2\},\ \{x\mid \|x\|\leqslant 1\})$$

in Euclidean n-space. Let B denote an arbitrary component of $q^{-1}(N)$ and let A denote the component of $q^{-1}(M)$ which contains B. Since M is homeomorphic to a closed n-cell, q maps A homeomorphically onto M and

$$\eta(A)\frown A = \emptyset \tag{11}$$

for all $\eta\in\Pi(Q,q)$ with $\eta\neq 1_Q$.

Since $q^{-1}(Y-N)\subset Q-B$, τ sends ordinary chains of $Y-N$ into chains of type (I) in $Q-B$ and there results a homomorphism τ'_* as indicated in the following diagram with coefficients in G:

$$\begin{array}{ccccccc}
H^0(Q) & \xrightarrow{i_1^*} & H^0(B) & & & & \\
\updownarrow & D_1 & \updownarrow & & & & \\
\mathcal{H}_n(Q) & \xrightarrow{\mu_*} & \mathcal{H}_n(Q, Q-B) & \xleftarrow{\omega} & H_n(Q, Q-B) & \xleftarrow{k_*} & H_n(A, A-B) \\
\tau_* \uparrow & L & \uparrow \tau'_* & & E & & d_* \uparrow \\
H_n(Y) & \xrightarrow[m_*]{} & H_n(Y, Y-N) & & \xleftarrow[l_*]{\qquad\qquad} & & H_n(M, M-N) \\
\updownarrow & D_2 & \updownarrow & & & & \\
H^0(Y) & \xrightarrow[i_2^*]{} & H^0(N) & & & &
\end{array}$$

Here ω is an isomorphism which results upon noticing that every chain c of type (I) in Q is of the form $c = c_1 + c_2$, where c_1 is an ordinary finite chain in Q and c_2 is of type (I) in $Q-B$. All other horizontal maps are induced by inclusions. Notice further that k_* and l_* are excisions, hence isomorphisms; moreover, i_1^* and i_2^* also are isomorphisms since Q, B, Y, N are all connected.

Since q maps the pair $(A, A-B)$ homeomorphically onto the pair $(M, M-N)$, the map

$$d = q^{-1}\colon (M, M-N) \to (A, A-B)$$

is defined and d_* is an isomorphism. Notice now that for any singular simplex T in M, the chain $\tau(T)$ is of the form

$$\tau(T) = d(T) + c$$

where c is a chain of type (I) in $Q-B$; for, (11) implies that any $\eta \in \Pi(Q, q)$ with $\eta \neq 1_Q$ sends $d(T)$ into a singular simplex of $Q-A \subset Q-B$. As a result, commutativity holds in square E.

The definitions of τ_* and τ'_* obviously imply commutativity in square L.

If $G = Z$, (3) implies that Y, hence also its covering manifold Q, are orientable; if $G = Z_2$, we are not concerned with orientability. Therefore, the duality theorem ([2], 20-04) applies yielding the commutative squares D_1 and D_2 in which the vertical arrows are isomorphisms.

Square E yields that

(12) *τ'_* is an isomorphism;*

the squares D_1 and D_2 respectively imply that

(13) *μ_* and m_* are isomorphisms.*

Commutativity in square L finally implies by (12) and (13) that τ_* is an isomorphism and Lemma 2 is proved.

4. The fundamental group.

The first result of this section and its proof are related to previous arguments of Hopf ([6], p. 585).

PROPOSITION 2. *Let $f\colon X \to Y$ be a continuous map of a connected CW-complex X into a compact n-dimensional orientable manifold Y. If $H_n(X; Z) \neq 0$ and if f has a left homotopy inverse, then the homomorphism $f_1\colon \pi_1(X) \to \pi_1(Y)$ induced by f is an isomorphism.*

Proof. Select base-points $x_0 \in X$ and $y_0 = f(x_0) \in Y$. Starting with an arbitrary left homotopy inverse g' of f, the homotopy extension theorem readily yields a map $g \simeq g'$ such that $g(y_0) = x_0$. Since $gf \simeq 1_X$, the composition

$$\pi_1(X, x_0) \xrightarrow{f_1} \pi_1(Y, y_0) \xrightarrow{g_1} \pi_1(X, x_0)$$

is an inner automorphism of $\pi_1(X, x_0)$ and, therefore, f_1 is a monomorphism. It only remains to show that f_1 is onto.

Let then (B, p) denote a covering space of Y such that $p_1\pi_1(B, b_0) = f_1\pi_1(X, x_0)$ for a suitable base-point $b_0 \in B$ with $p(b_0) = y_0$. The monodromy principle next yields a map φ such that

$$\begin{array}{ccc}
 & & (B, b_0) \\
 & \overset{\varphi}{\nearrow} & \downarrow p \\
(X, x_0) & \xrightarrow[f]{} & (Y, y_0)
\end{array}$$

is a commutative diagram; moreover,

$$\varphi_1\pi_1(X, x_0) = \pi_1(B, b_0)\,. \tag{14}$$

Passing to homology in dimension n with Z as coefficients, we obtain $f_* = p_*\varphi_*$. Since $gf \simeq 1_X$, f_* maps the non-vanishing group $H_n(X; Z)$ isomorphically onto a direct summand of $H_n(Y; Z)$; since Y is orientable, the latter is isomorphic to Z and, therefore, f_* is an isomorphism. As a consequence, p_* is an epimorphism; therefore, $H_n(B; Z) \neq 0$ (i. e. B is a compact n-dimensional manifold) and $|\text{degree}\, p| = 1$. Since

$$|\text{degree}\, p| = \text{number of sheets of } (B, p)\,,$$

we obtain $B = Y$, $\varphi = f$, and Proposition 2 follows now from (14).

PROPOSITION 3. *Let $f\colon X \to Y$ be a continuous map of a connected CW-complex X into a compact n-dimensional non-orientable manifold Y; let (Q, q) denote the 2-sheeted orientable covering space of Y. If $H_n(X; Z_2) \neq 0$ and if f has a left homotopy inverse, then there exists a map g, a 2-sheeted covering space (P, p) of X and two maps e, k such that*

(15) *commutativity holds in the diagram*

$$\begin{array}{ccccc} P & \xrightarrow{e} & Q & \xrightarrow{k} & P \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & X \end{array}$$

(16) $gf \simeq 1_X$,

(17) ${}_1e\colon \Pi(P, p) \to \Pi(Q, q)$ *is an isomorphism,*

(18) $ke \simeq 1_P$.

Proof. Select base-points $x_0 \in X$ and $y_0 = f(x_0) \in Y$. Starting with an arbitrary left homotopy inverse g' of f, the homotopy extension theorem readily yields a map $g \simeq g'$ such that $g(y_0) = x_0$.

Let Ω be the normal subgroup of orientation-preserving elements of $\pi_1(Y, y_0)$. Introduce further the normal subgroups

(19) $$\Gamma = f_1^{-1}(\Omega) \subset \pi_1(X, x_0) \quad \text{and} \quad \Lambda = g_1^{-1}(\Gamma) \subset \pi_1(Y, y_0)$$

where

$$f_1\colon \pi_1(X, x_0) \to \pi_1(Y, y_0) \quad \text{and} \quad g_1\colon \pi_1(Y, y_0) \to \pi_1(X, x_0)$$

are the homomorphisms induced by f and g. We have

(20) $$\Omega \cdot f_1\pi_1(X, x_0) = \pi_1(Y, y_0)\,.$$

For, since $\pi_1(Y, y_0)/\Omega \approx Z_2$, the contrary case implies $f_1\pi_1(X, x_0) \subset \Omega$ and the monodromy principle yields then a map φ such that the triangle

$$\begin{array}{ccc} & & Q \\ & {\scriptstyle \varphi}\nearrow & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is commutative; passing to homology in dimension n with Z_2 as coefficients we obtain $f_* = q_*\varphi_*$ and this is absurd since q_* is known to be trivial whereas $gf \simeq 1_X$ and $H_n(X; Z_2) \neq 0$ imply $f_* \neq 0$.

As a consequence of (20), we single out

(21) $$\pi_1(X, x_0)/\Gamma \approx f_1\pi_1(X, x_0)/\Omega \cap f_1\pi_1(X, x_0) \approx \Omega \cdot f_1\pi_1(X, x_0)/\Omega \approx Z_2\,.$$

Since $gf \simeq 1_X$, g_1f_1 is an inner automorphism of $\pi_1(X, x_0)$; therefore, g_1 is an epimorphism and (21) implies

(22) $$\pi_1(Y, y_0)/\Lambda \approx \pi_1(X, x_0)/\Gamma \approx Z_2\,;$$

since Γ is a normal subgroup, $g_1f_1(\Gamma) = \Gamma$ whence, by (19),

(23) $$f_1(\Gamma) \subset \Lambda\,.$$

By (21) and (22), the covering spaces (P, p) and (S, s) of X and Y which satisfy $p_1\pi_1(P) = \Gamma$ and $s_1\pi_1(S) = \Lambda$ are 2-sheeted; moreover, by (23) and (19), the monodromy principle yields maps e, k such that the diagram

$$\begin{array}{ccccc} P & \xrightarrow{e} & S & \xrightarrow{k} & P \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & X \end{array}$$

is commutative and

$${}_1e\colon \Pi(P, p) \to \Pi(S, s)$$

is an isomorphism. Finally, replacing if necessary the initial map k by ξk with suitable $\xi \in \Pi(P, p)$, the covering homotopy theorem enables us to assert that $ke \simeq 1_P$.

It only remains to prove that

(24) $$(S, s) = (Q, q)\,.$$

Since the orientable 2-sheeted covering space of a non-orientable manifold Y is uniquely determined, it is enough to prove that S is orientable. We shall assume that S is non-orientable and obtain a contradiction. To this end, consider the covering space (T, t) of Y which satisfies $t_1\pi_1(T) = \Lambda \cap \Omega$. Since $s_1\pi_1(S) = \Lambda \supset \Lambda \cap \Omega$, the monodromy principle yields a map τ such that the diagram

$$\begin{array}{ccccc} & {\scriptstyle \psi} & T & {\scriptstyle t} & \\ & \nearrow & \downarrow{\scriptstyle \tau} & \searrow & \\ P & \xrightarrow{e} & S & \xrightarrow{s} & Y \\ & {\scriptstyle p}\searrow & & \nearrow{\scriptstyle f} & \\ & & X & & \end{array}$$

is commutative. As easily seen, (T, τ) is a covering space of S. By (23) and (19) we have

$$s_1e_1\pi_1(P) = f_1p_1\pi_1(P) = f_1(\Gamma) \subset \Lambda \cap \Omega\,;$$

since s_1 is a monomorphism and

$$s_1\tau_1\pi_1(T) = t_1\pi_1(T) = \Lambda \cap \Omega\,,$$

we obtain

$$e_1\pi_1(P) \subset \tau_1\pi_1(T)$$

and the monodromy principle yields a map ψ satisfying $\tau\psi = e$. Passing to homology in dimension n with Z_2 as coefficients, we obtain

(25) $$e_* = \tau_*\psi_*\,.$$

Since $t_1\pi_1(T)\subset\Omega$, T is orientable and the assumed non-orientability of S now implies

(26) $$\tau_* = 0 .$$

On the other hand, $gf\simeq 1_X$ and $H_n(X; Z_2)\neq 0$ imply that f_*: $H_n(X; Z_2)\to H_n(Y; Z_2)$ is an epimorphism; by Proposition 1 with $G=Z_2$ it follows that

(27) e_*: $H_n(P; Z_2)\to H_n(S; Z_2)$ *is an epimorphism.*

Since (S, s) is 2-sheeted, S is compact and $H_n(S; Z_2)\neq 0$. Therefore, (26) and (27) contradict (25); thus, (24) is entirely proved.

5. Proof of Theorem 1. According to Remark 3 we may assume that X is a connected CW-complex.

Since f: $X\to Y$ has a left homotopy inverse g,

(28) f_*, *respectively* g^*, *maps the groups* $H_r(X; G)$, *respectively* $H^r(X; G)$, *isomorphically onto direct summands of the corresponding groups of* Y *for any coefficient group* G *and all* $r\geqslant 0$.

Suppose first that

(29) Y *is orientable.*

Then, $H_{n-1}(Y; Z)$ is torsion-free, hence free, and (28) implies that $H_{n-1}(X; Z)$ also is free. As a result, the second summand in the right side of the universal coefficient formula ([4], p. 161)

$$H^n(X; Z)\approx \mathrm{Hom}\big(H_n(X; Z), Z\big)+\mathrm{Ext}\big(H_{n-1}(X; Z), Z\big)$$

vanishes. Since by assumption the group on the left is non-vanishing, we obtain $H_n(X; Z)\neq 0$; moreover, (29) yields $H_n(Y; Z)\approx Z$ and, by (28), we now obtain that

(30) f_*: $H_n(X; Z)\to H_n(Y; Z)$ *is an isomorphism.*

As a consequence, Proposition 2 implies that also

(31) f_1: $\pi_1(X)\to\pi_1(Y)$ *is an isomorphism.*

Let now (P, p) and (Q, q) denote the simply connected covering spaces of X and Y; the monodromy principle yields a map e: $P\to Q$ satisfying $qe = fp$. Identifying fundamental groups and *Deckbewegungsgruppen* of the simply connected covering spaces, (31) implies that

$${}_1e\colon \Pi(P, p)\to\Pi(Q, q) \text{ is an isomorphism.}$$

By Proposition 1, this and (30) imply that

(32) e_*: $H_r(P; Z)\to H_r(Q; Z)$ *are epimorphisms for all* $r\geqslant 0$.

Since Q is simply connected, the monodromy principle yields a map k: $Q\to P$ satisfying $pk = gq$; replacing if necessary this k by ξk with a suitable $\xi\in\Pi(P, p)$, the covering homotopy theorem implies next that $ke\simeq 1_P$. Therefore

(33) e_*: $H_r(P; Z)\to H_r(Q; Z)$ *are monomorphisms for all* $r\geqslant 0$.

Finally, according to a theorem by J. H. C. Whitehead ([9], Theorem 3), (31), (32) and (33) imply that f: $X\to Y$ actually is a homotopy equivalence.

Suppose now that

(34) Y *is non-orientable.*

This implies $H^n(Y; Z)\approx Z_2$. Since $H^n(X; Z)\neq 0$, (28) implies $H^n(X; Z)\approx Z_2$ and the universal coefficient formula ([7], p. 257)

$$H^n(X; Z_2)\approx H^n(X; Z)\otimes Z_2+\mathrm{Tor}\big(H^{n+1}(X; Z), Z_2\big)$$

implies $H^n(X; Z_2)\neq 0$; since Z_2 is a field, also

(35) $$H_n(X; Z_2)\neq 0 .$$

We may now apply Proposition 3 to obtain (P, p), (Q, q), g, e, k. Since (34) yields $H_n(Y; Z_2)\approx Z_2$, (28) and (35) imply that

$$f_*\colon H_n(X; Z_2)\to H_n(Y; Z_2) \text{ is an isomorphism,}$$

and Proposition 1, with $G = Z_2$, now yields that

$$e_*\colon H_n(P; Z_2)\to H_n(Q; Z_2) \text{ is an epimorphism.}$$

As a consequence

(36) $$H_n(P; Z_2)\neq 0 .$$

Since Q is orientable, $H_{n-1}(Q; Z)$ is torsion-free and, since $ke\simeq 1_P$, the same holds for $H_{n-1}(P; Z)$. Therefore, the second summand on the right in the universal coefficient formula ([4], p. 161)

$$H_n(P; Z_2)\approx H_n(P; Z)\otimes Z_2+\mathrm{Tor}\big(H_{n-1}(P; Z), Z_2\big)$$

vanishes and (36) implies

(37) $$H_n(P; Z)\neq 0 .$$

This is the only homological assumption needed in the orientable case in order to obtain that

(38) e: $P\to Q$ *is a homotopy equivalence.*

As a consequence of (38), e induces isomorphisms of homotopy groups in all dimensions; since (P, p) and (Q, q) are covering spaces of X and Y, it follows that

(39) *$f\colon X \to Y$ induces isomorphisms of homotopy groups in all dimensions $\geqslant 2$.*

Since e_1 and ${}_1e$ are isomorphisms, commutativity and exactness in the diagram

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(P) & \xrightarrow{p_1} & \pi_1(X) & \to & \Pi(P, p) & \to & 1 \\ & & \downarrow e_1 & & \downarrow f_1 & & \downarrow {}_1e & & \\ 1 & \to & \pi_1(Q) & \xrightarrow[q_1]{} & \pi_1(Y) & \to & \Pi(Q, q) & \to & 1 \end{array}$$

imply that also

(40) *$f_1\colon \pi_1(X) \to \pi_1(Y)$ is an isomorphism.*

Finally, according to a theorem by J. H. C. Whitehead ([9], Theorem 1), (40) and (39) imply that $f\colon X \to Y$ actually is a homotopy equivalence.

References

[1] H. Cartan, *Topologie algébrique*, Séminaire E. N. S., 1948-1949.

[2] — *Cohomologie des groupes-Suite spectrale-Faisceaux*, Séminaire E. N. S., 1950-1951.

[3] S. Eilenberg, *Singular homology theory*, Annals of Math. 45 (1944), p. 407-447.

[4] — and N. Steenrod, *Foundations of algebraic topology*, Princeton 1952.

[5] J. B. Giever, *On the equivalence of two singular homology theories*, Annals of Math. 51 (1950), p. 178-191.

[6] H. Hopf, *Zur Topologie der Abbildungen von Mannigfaltigkeiten*, Zweiter Teil, Math. Annalen 102 (1930), p. 562-623.

[7] F. Peterson, *Some results on cohomotopy groups*, Amer. J. Math. 78 (1956), p. 243-258.

[8] R. Thom, *Espaces fibrés en sphères et carrés de Steenrod*, Annales Sci. Ecole Normale Supérieure 69 (1952), p. 109-182.

[9] J. H. C. Whitehead, *Combinatorial homotopy I*, Bull. Amer. Math. Soc. 55 (1949), p. 213-245.

INSTITUTE OF MATHEMATICS, R. P. R. ACADEMY

Reçu par la Rédaction le 11. 8. 1958

The first order properties of products of algebraic systems

by

S. Feferman and R. L. Vaught (USA)

Introduction

In modern algebra and set theory, a variety of ways of forming sums or products of finitely or infinitely many algebraic systems have been considered. Examples are cardinal and ordinal sums, ordinary and weak direct products, and ordinal products. In this paper we shall introduce a notion of *generalized product*, which comprehends (in a sense) all of these examples, and a number of other products. By means of its use, we shall investigate the relation between the elementary (i. e., first order) properties possessed by the product algebraic system and those possessed by its factors.

The method applied here has its origin in the work of Mostowski in [13] on finite or infinite, ordinary or weak, direct powers of algebraic systems. The first work on "products" other than direct was the study made by Beth [1] of the ordinal sum of finitely many ordered systems. Our own work has occurred in a series of steps over several years. Some of these were reported in abstracts [4], [5], and [28]. Recently, a summary of our results in nearly their present state was given in [6] and [7] (1).

In many cases the definition of a product operation, which is to be applied in the general situation to an indexed family of systems $\langle \mathfrak{A}^{(i)} \mid i \in I \rangle$, takes into account some sort of "structure" on the index set I in the form of another system $\langle I, R_0, R_1, \ldots \rangle$. (For example, when dealing with ordinal sums or products, it is necessary to consider systems $\langle I, R \rangle$ where R is a binary relation ordering the set I.) However, it has turned

(1) Many of the results described in the present paper were obtained while both authors were students of Professor Alfred Tarski. We are indebted to him for many stimulating remarks and suggestions. The work summarized in [28] formed a chapter of the second author's doctoral disseration at the University of California, Berkeley, 1954.

Part of the work of the first author on this paper was supported by a grant from the National Science Foundation.