On ε-maps onto manifolds

by

T. Ganea (Bucureşti)

1. Introduction. Let ε be a positive number; a continuous map f of a compact metric space X into another space Y is called an *ε-map* if the inverse-image of each point of $f(X)$ has diameter less than ε.

Let E^n denote the closed n-dimensional ball of radius 1 and let

$$\varepsilon_n = \frac{2n+2-\sqrt{2n^2+2n}}{n+2} > 0 .$$

Kuratowski [7] proved that no ε_n-map of E^n onto an n-dimensional sphere S^n exists. Later, Ulam [10] raised the question whether there exists for every $\varepsilon > 0$ an ε-map of E^2 onto a 2-dimensional torus.

In this paper (¹) we present an extension of the topological contents of Kuratowski's result by means of which we also obtain the (negative) answer to Ulam's question.

It will be shown that a compact metric n-dimensional absolute neighborhood retract which may be mapped with arbitrarily small counter-images onto closed n-dimensional manifolds has many of their general properties. In particular, such a space is essential and has the homotopy type of a closed n-dimensional manifold; moreover, its separation properties by closed subsets are the same as for n-dimensional closed manifolds and, if $n = 2$, such a space is necessarily homeomorphic to a closed surface.

2. Preliminaries. By $H^q(X, A; G)$ we generally denote the qth Čech cohomology group of the compact pair (X, A) over the coefficient group G. By Z and Z_2 we respectively denote the group of integers and the group of integers mod 2.

We shall first establish two simple results.

(¹) Part of this work was done during a visit in Warsaw; the author wishes to express his gratitude to Professors Borsuk and Kuratowski for their interest and many stimulating discussions.

2.1. Lemma. *If* (X, A) *is a compact pair and* $\dim X = n$, *then, for every coefficient group,*

$$H^n(X, A; G) \approx H^n(X, A; Z) \otimes G .$$

Proof. The universal coefficient theorem for Čech cohomology groups of compact pairs is expressed by the exact sequence

$$0 \to H^n(X, A; Z) \otimes G \to H^n(X, A; G) \to \mathrm{Tor}\left(H^{n+1}(X, A; Z), G\right) \to 0$$

(see for instance [9], p. 257). Since $\dim X = n$, by [5], Theorem VIII 4, we have $H^{n+1}(X, A; Z) = 0$ and the result follows.

2.2. Lemma. *Let* X *be a locally connected compact space and* G *an Abelian group. If there exists a closed subset* F *of* X *such that* $H^q(X, F; G) \neq 0$, *then there also exists a closed subset* A *of* X *such that* $X-A$ *is connected and* $H^q(X, A; G) \neq 0$.

Proof. Since X is locally connected, the components U_λ $(\lambda \in L)$ of $X-F$ are open and the maps

$$H^q(X, X-U_\lambda; G) \to H^q(X, F; G)$$

induced by inclusion yield an injective representation of $H^q(X, F; G)$ as a direct sum ([4], p. 294, B 3). Since the latter group is non-vanishing, there exists at least a subscript $\mu \in L$ such that $H^q(X, X-U_\mu; G) \neq 0$ and the set $A = X - U_\mu$ behaves as required.

By a closed manifold we mean a compact connected locally Euclidean Hausdorff space; no triangulability assumption is made.

2.3. Lemma. *Let* X *be a compact metric absolute neighbourhood retract. Suppose that for every* $\varepsilon > 0$ *there exists an* ε*-map of* X *onto a closed* n*-dimensional manifold (depending on* ε*). Then, if* A *and* C *are proper closed subsets of* X *such that* $A \subset \mathrm{Int}\, C$, *there exists a closed* n*-dimensional manifold* Y, *a proper closed subset* B *of* Y *and two maps of pairs*

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (X, C)$$

such that the composition gf *is homotopic to the inclusion map*

$$\theta\colon (X, A) \subset (X, C) .$$

Moreover, if $X-A$ *is connected,* $Y-B$ *may also be assumed to be connected.*

Proof. Let $U = X - A$ and $W = X - C$. Since $A \subset \mathrm{Int}\, C$, we have $\overline{W} \subset U$.

If U is not connected, set $V = W$.

If U is connected, a theorem by R. L. Wilder (see for instance [8], p. 166) yields a sequence of open connected subsets U_k of X such that

$$U = \bigcup U_k \quad \text{and} \quad \overline{U}_k \subset U_{k+1}$$

for all $k \geqslant 1$. Since $\overline{W}$ is compact, there exists a subscript $m \geqslant 1$ such that $\overline{W} \subset U_m$; then set $V = U_m$.

Since $\overline{V} \subset U$, there exist open subsets P and Q of X such that

$$\overline{V} \subset Q \subset \overline{Q} \subset P \subset \overline{P} \subset U .$$

According to the definition of V, we may obviously assume that

(1) $\qquad Q$ *is connected if so is* U.

Now let ϱ denote the distance-function in X. There exists $\eta > 0$ such that

(2) $$S(\overline{V}, \eta) \subset Q ,$$

(3) $$S(\overline{Q}, \eta) \subset P ,$$

(4) $$S(\overline{P}, \eta) \subset U ,$$

where $S(M, \eta) = \{x \mid x \in X,\ \varrho(x, M) < \eta\}$.

Since X is an ANR, there exists $\omega > 0$ such that any two maps $\varphi, \psi\colon X \to X$ satisfying $\varrho(\varphi x, \psi x) < \omega$ for all $x \in X$ are η-homotopic.

Finally, according to a theorem by Eilenberg [3], there exists $\varepsilon = \varepsilon(X, \eta, \omega) > 0$ such that to every map $f\colon X \to Y$ satisfying

(5) $$f(X) = Y \quad \text{and} \quad \operatorname{diam} f^{-1}(y) < \varepsilon$$

for all $y \in Y$, there corresponds a map $g\colon Y \to X$ satisfying

(6) $$\varrho\left(x, gf(x)\right) < \min(\eta, \omega)$$

for all $x \in X$.

By assumption, there exists a map f of X onto a closed n-dimensional manifold Y satisfying (5). Let then $g\colon Y \to X$ satisfy (6).

By (6) and (3) we have

$$f(\overline{Q}) \subset g^{-1}(P) .$$

If U is not connected, set $R = g^{-1}(P)$.

If U is connected, then, by (1), $f(\overline{Q})$ is a connected subset of Y and R will denote the component of $g^{-1}(P)$ which contains $f(\overline{Q})$.

Since P is open, R is open and $B = Y - R$ is compact. Moreover, $Q \neq \emptyset$ implies $R \neq \emptyset$ and B is a proper subset of Y.

If $x \in X$ and $f(x) \in R$, then $gf(x) \in P$, whence, by (6) and (4), $x \in U$; since $A = X - U$, we obtain

$$f(A) \subset B .$$

Furthermore, for arbitrary $y \in B$ there exists $x \in X$ with $y = f(x)$; by (6) and (2), $gf(x) \in V$ implies $x \in Q$, whence, according to the definition of R, $f(x) \in R$. As a result, $g(y) = gf(x) \in X - V$ and therefore

$$g(B) \subset C .$$

This provides the required sequence of maps of pairs.

By (6), $\varrho\big(x, gf(x)\big) < \omega$ for all $x \in X$; therefore, there exists a homotopy $h_t\colon X \to X$ such that

$$h_0(x) = x\,, \tag{7}$$

$$h_1(x) = gf(x)\,, \tag{8}$$

$$\varrho\big(x, h_t(x)\big) < \eta, \tag{9}$$

for all $x \in X$, $0 \leqslant t \leqslant 1$. By (9) and (2) we have

$$h_t(x) \in X - V \subset C \quad \text{if} \quad x \in A = X - U$$

and the Lemma is proved.

3. General results. They are as follows:

3.1. Theorem. *Let X be a compact metric n-dimensional absolute neighborhood retract. Suppose that for every $\varepsilon > 0$ there exists an ε-map of X onto a closed n-dimensional manifold (depending on ε). Then*

(3.1.1) $H^n(X; Z_2) \neq 0$;

(3.1.2) $H^n(A; Z) = 0$ *for every proper closed subset A of X*;

(3.1.3) *X is essential*;

(3.1.4) *No proper closed subset A of X satisfying $H^{n-1}(A; Z_2) = 0$ separates X*;

(3.1.5) *There exists $\sigma = \sigma(X) > 0$ such that every closed subset A of X satisfying* $\operatorname{diam} A < \sigma$ *and $H^{n-1}(A; Z_2) \neq 0$ separates X*;

(3.1.6) *X is a Cantor-manifold.*

3.2. Corollary. *To every compact n-dimensional manifold X with non-empty boundary there corresponds a positive $\varepsilon = \varepsilon(X)$ such that no ε-map of X onto a closed n-dimensional manifold exists.*

We immediately pass to the proofs.

Proof of (3.1.1). We shall first distinguish two mutually exclusive cases for X.

Case 1. There exists a closed subset A of X such that $H^n(X, A; Z_2) \neq 0$. By (2.2) we may assume $X - A$ to be connected and we select a non-vanishing element $a \in H^n(X, A; Z_2)$.

Case 2. $H^n(X, F; Z_2) = 0$ for all closed subsets F of X. Since $\dim X = n$, by [5], Theorem VIII 4, there exists a closed subset A of X such that $H^n(X, A; Z) \neq 0$; by (2.2) we may assume $X - A$ to be connected and we select a non-vanishing element $a \in H^n(X, A; Z)$.

We shall return to these two cases in the last part of the proof; until then we proceed in the same way in both of them. We do not write explicitly the coefficient group, which is Z_2 in the first case and Z in the second.

Since $H^n(X, A) \neq 0$, A is necessarily a proper subset of X. By the continuity of the Čech theory ([4], p. 260-261, Theorems 2.6 and 3.1), there exists a proper closed subset C of X such that $A \subset \operatorname{Int} C$ and

$$a = \theta^*(c) \quad \text{for some} \quad c \in H^n(X, C)\,, \tag{10}$$

where $\theta\colon (X, A) \subset (X, C)$ is the inclusion map.

By (2.3) there exists a closed n-dimensional manifold Y, a proper closed subset B of Y and two maps of pairs

$$(X, A) \xrightarrow{f} (Y, B) \xrightarrow{g} (X, C)$$

such that the composition gf is homotopic to θ. Since $X - A$ is connected, we may assume that also $Y - B$ is connected.

Let $g_1\colon Y \to X$ be the map defined by g and consider the diagram

$$\begin{array}{ccccc} H^n(X, A) & \xleftarrow{f^*} & H^n(Y, B) & \xleftarrow{g^*} & H^n(X, C) \\ & & \downarrow j^* & & \downarrow j'^* \\ & & H^n(Y) & \xleftarrow[g_1^*]{} & H^n(X) \\ & & \downarrow i^* & & \\ & & H^n(B) & & \end{array}$$

Since gf is homotopic to θ, (10) implies

$$a = f^* g^*(c)\,. \tag{11}$$

We now return to the two cases which were distinguished at the beginning of the proof.

In the first the diagram is to be considered with Z_2 as coefficient group. Since Y and $Y - B$ are connected, by [4], p. 314 and 319, Theorem 6.8 and Remark, we have the isomorphisms

$$H^n(Y) \approx Z_2 \approx H^n(Y, B)\,. \tag{12}$$

Since B is a proper closed subset of an n-dimensional manifold, we have $H^n(B) = 0$ and, by exactness, j^* is onto; (12) then implies that

(13) *j^* is an isomorphism.*

Since $a \neq 0$, (11) implies $g^*(c) \neq 0$ and (13) further implies $j^* g^*(c) \neq 0$. By commutativity

$$g_1^* j'^*(c) = j^* g^*(c)\,,$$

whence $j'^*(c) \neq 0$ and finally $H^n(X; Z_2) \neq 0$.

In the second case we only consider the upper row of the diagram. The coefficient group is now Z and cohomology groups with unspecified coefficients are taken over Z. By (2.1) we have

$$H^n(X, C; Z_2) \approx H^n(X, C) \otimes Z_2 .$$

By assumption, the group on the left vanishes; therefore, every element of $H^n(X, C)$ is divisible by 2 ([4], p. 143). It follows that the elements $c \in H^n(X, C)$ and, therefore, also $g^*(c) \in H^n(Y, B)$ are *divisible by any power of* 2. Since $H^n(Y, B)$ is isomorphic either to Z or to Z_2 ([4], p. 315 and 319), this necessarily implies $g^*(c) = 0$ and (11) yields $a = 0$. This is a contradiction proving that, under the assumptions of (3.1), the second case is impossible.

Proof of (3.1.2). Let A be a closed proper subset of X and select an arbitrary element $a \in H^n(A; Z)$. By the continuity of the Čech theory there exists a closed proper subset C of X such that $A \subset \operatorname{Int} C$ and

$$a = \theta_2^*(c) \quad \text{for some} \quad c \in H^n(C; Z) ,$$

where θ_2: $A \subset C$ is the inclusion map.

Let (Y, B), f, g, θ be as in (2.3) and let f_2: $A \to B$ and g_2: $B \to C$ be the maps defined by f and g. Since gf is homotopic to θ, we have $a = f_2^* g_2^*(c)$; since B is a proper closed subset of an n-dimensional manifold, we have $H^n(B; Z) = 0$. Therefore, $g_2^*(c) = 0$ and $a = 0$, i. e. $H^n(A; Z) = 0$.

Proof of (3.1.3). This is an immediate consequence of (3.1.1), (2.1), and (3.1.2).

Proof of (3.1.4). Suppose that A separates X, i. e. that $X - A = U_1 \cup U_2$ where U_1 and U_2 are open, non-empty and without common points. Set $X_1 = U_1 \cup A$, $X_2 = U_2 \cup A$. With Z_2 as coefficient group, the Mayer-Vietoris sequence

$$H^{n-1}(A) \to H^n(X) \to H^n(X_1) + H^n(X_2)$$

is exact. By assumption $H^{n-1}(A) = 0$; by (3.1.2) and (2.1), we have $H^n(X_1) = H^n(X_2) = 0$. This implies $H^n(X) = 0$, which contradicts (3.1.1).

Proof of (3.1.5). Since X is a compact ANR, there exists $\sigma = \sigma(X) > 0$ such that every subset C of X satisfying $\operatorname{diam} C < 2\sigma$ is contractible in X.

We work with Z_2 as coefficient group.

Let A be a closed subset of X satisfying $\operatorname{diam} A < \sigma$ and $H^{n-1}(A) \neq 0$. Select a non-vanishing element $a \in H^{n-1}(A)$; by the continuity of the Čech theory, there exists a proper closed subset C of X such that $\operatorname{diam} C < 2\sigma$, $A \subset \operatorname{Int} C$ and

$$a = \theta_2^*(c) \quad \text{for some} \quad c \in H^{n-1}(C) ,$$

where θ_2: $A \subset C$ is the inclusion map.

Assume that $X - A$ is connected and let (Y, B), f, g, θ be as in (2.3); by (2.3) we may then suppose that also $Y - B$ is connected, and this will be shown to lead to a contradiction.

Let

$$f_1\colon X \to Y , \quad g_1\colon Y \to X ,$$
$$f_2\colon A \to B , \quad g_2\colon B \to C ,$$

be the maps defined by f and g respectively, and consider the diagram

$$\begin{array}{ccccc} H^{n-1}(X) & \xleftarrow{f_1^*} & H^{n-1}(Y) & \xleftarrow{g_1^*} & H^{n-1}(X) \\ & & \downarrow i^* & & \downarrow i'^* \\ H^{n-1}(A) & \xleftarrow[f_2^*]{} & H^{n-1}(B) & \xleftarrow[g_2^*]{} & H^{n-1}(C) \end{array}$$

Since gf is homotopic to θ, we have $a = f_2^* g_2^*(c)$; since $a \neq 0$, it follows that

$$H^{n-1}(B) \neq 0 . \tag{14}$$

On the other hand, $g_1 f_1$ is homotopic to the identity map θ_1 of X, Y is an n-dimensional closed manifold and, by (2.1) and (3.1.1), $H^n(X; Z) \neq 0$: by [1], these three facts imply that f_1 is a homotopy equivalence ([2]). As a consequence, f_1^* is an isomorphism and, since $f_1^* g_1^*$ is the identity map of $H^{n-1}(X)$, g_1^* is also an isomorphism. Therefore, every $b \in H^{n-1}(Y)$ is of the form $b = g_1^*(d)$ with $d \in H^{n-1}(X)$; by commutativity we have

$$i^*(b) = i^* g_1^*(d) = g_2^* i'^*(d) .$$

Since $\operatorname{diam} C < 2\sigma$, C is contractible in X and $i'^* = 0$; therefore

$$i^*(b) = 0 \quad \text{for all} \quad b \in H^{n-1}(Y) . \tag{15}$$

We now introduce the duality diagram

$$\begin{array}{ccccccccc} H_1(Y) & \xrightarrow{j_*} & H_1(Y, Y-B) & \xrightarrow{\partial} & H_0(Y-B) & \to & H_0(Y) & \to & H_0(Y, Y-B) \\ \updownarrow & & \updownarrow & & & & & & \\ H^{n-1}(Y) & \xrightarrow[i^*]{} & H^{n-1}(B) & & & & & & \end{array}$$

([2]) The theorem proved in [1] requires $H^n(X; Z) \neq 0$ in the singular sense; we may identify the singular groups with the Čech groups because X is a compact metric ANR (see for instance [4], p. xiii).

(see [2], 20-04; the upper row is the singular homology sequence of the pair $(Y, Y-B)$, the lower row consists of Čech cohomology groups, the vertical arrows are isomorphisms and the square commutes).

As a result of (15) we have $j_* = 0$ and (14) implies $H_1(Y, Y-B) \neq 0$. It follows that

(16) $$\operatorname{image} \partial \neq 0 .$$

Since $Y-B$ is assumed to be connected, we have $H_0(Y-B) \approx Z_2$ and (16) now implies that ∂ is onto. Since Y is connected, we have $H_0(Y, Y-B) = 0$ and exactness finally implies $H_0(Y) = 0$, which is absurd.

Thus, $Y-B$ is not connected and the same holds for $X-A$.

Proof of (3.1.6). This is an immediate consequence of (3.1.4) and of the fact that $\dim A \leqslant n-2$ implies $H^{n-1}(A; Z_2) = 0$.

Proof of (3.2). This an immediate consequence of (3.1.1) and of the fact that $H^n(X; Z_2) = 0$ if X is an n-dimensional manifold with boundary.

4. Homotopy types.

We shall prove

4.1. Theorem. *To every compact metric n-dimensional absolute neighborhood retract X there corresponds a positive $\varepsilon = \varepsilon(X)$ such that every ε-map (if any!) of X onto a closed n-dimensional manifold is a homotopy equivalence.*

This obviously implies

4.2. Corollary. *Let X be a compact metric n-dimensional absolute neighborhood retract. If for every $\varepsilon > 0$ there exists an ε-map of X onto a closed n-dimensional manifold (depending on ε), then X has the homotopy type of a closed n-dimensional manifold.*

Proof of (4.1). We shall again distinguish two mutually exclusive cases for X.

Suppose first that $H^n(X; Z) = 0$. Then, by (2.1) we also have $H^n(X; Z_2) = 0$ and, by (3.1.1), there exists $\varepsilon = \varepsilon(X) > 0$ such that no ε-map of X onto a closed n-dimensional manifold exists.

Suppose now that $H^n(X; Z) \neq 0$. By the previously quoted theorem of Eilenberg [3], there exists $\varepsilon = \varepsilon(X) > 0$ such that to every map f: $X \to Y$ satisfying

(17) $$f(X) = Y \quad \text{and} \quad \operatorname{diam} f^{-1}(Y) < \varepsilon$$

for all $y \in Y$ there corresponds a map $g\colon Y \to X$ such that

gf is homotopic to the identity map of X.

If Y is a closed n-dimensional manifold and f satisfies (17), the existence of the left homotopy inverse g of f implies by [1] that f is a homotopy equivalence ([2]).

5. The 2-dimensional case.

It will be completely solved by the following

5.1. Theorem. *Let X be a compact metric absolute neighborhood retract. If $\dim X = 2$ and if for every $\varepsilon > 0$ there exists an ε-map of X onto a closed 2-dimensional manifold (depending on ε), then X is necessarily a closed surface.*

Proof. Since the manifolds considered are assumed to be connected, it is easy to see that X must also be connected. Thus, X is a Peano continuum. Since $\dim X = 2$, X is not a local dendrite (in the sense of [8], p. 227) and, by [8], p. 228, Lemme 3, we infer that

(18) *X contains arbitrarily small 1-spheres.*

With $\sigma = \sigma(X) > 0$ defined as in (3.1.5), it further follows that

(19) *Every 1-sphere of diameter $< \sigma$ in X separates X.*

Finally, by (3.1.4),

(20) *No arc in X separates X.*

By a theorem of van Kampen [6], (18), (19) and (20) actually imply that X is a closed surface.

6. An example.

Throughout the paper, X has been assumed to be an absolute neighborhood retract. A simple example may be invoked to prove that this condition on X cannot be dispensed with.

Let X be a 2-sphere with countably many handles, which have decreasing diameters and converge to a point. Clearly, X is a locally connected continuum which may be mapped with arbitrarily small counter-images onto spheres with finitely many handles. However, X is not an ANR and is neither topologically nor homotopically equivalent to a closed surface.

References

[1] I. Berstein and T. Ganea, *Remark on spaces dominated by manifolds*, Fund. Math. this volume, p. 45-56.

[2] H. Cartan, *Cohomologie des groupes-Suites spectrales-Faisceaux*, Séminaire E. N. S., 1950-1951.

[3] S. Eilenberg, *Sur les transformations à petites tranches*, Fund. Math. 30 (1938), p. 92-95.

[4] — and N. Steenrod, *Foundations of algebraic topology*, Princeton 1952.

[5] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1941.

[6] E. R. van Kampen, *On some characterizations of 2-dimensional manifolds*, Duke Math. J. 1 (1935), p. 74-93.

[7] C. Kuratowski, *Sur les transformations des sphères en des surfaces sphériques*, Fund. Math. 20 (1933), 206-213.

[8] C. Kuratowski, *Topologie*, vol. II, Warszawa 1952.

[9] F. Peterson, *Some results on cohomotopy groups*, Amer. J. Math. 78 (1956), p. 243-258.

[10] S. Ulam, *Problem 21*, Scottish Book.

INSTITUTE OF MATHEMATICS, R. P. R. ACADEMY

Reçu par la Rédaction le 11. 8. 1958

Remark on spaces dominated by manifolds

by

I. Berstein and T. Ganea (Bucureşti)

1. Introduction and results. Let X, Y be arbitrary topological spaces and $f: X \to Y$ a continuous map. A map $g: Y \to X$ is called a *left (right) homotopy inverse* of f if $gf \simeq 1_X$ ($fg \simeq 1_Y$), where $\simeq 1_E$ means homotopic to the identity map of E. The map f is called a *homotopy equivalence* if there exists a map $g: Y \to X$ which is both a left and a right homotopy inverse of f; if f only has a left homotopy inverse, then Y is said *to dominate* X ([9], p. 214).

By a *manifold* we mean a connected locally Euclidean Hausdorff space; no triangulability assumptions are made. As usual, $H^n(X; Z)$ stands for the nth singular cohomology group of X with integer coefficients. Our result is expressed by

THEOREM 1. *Let $f: X \to Y$ be a continuous map of an arbitrary topological space X into a compact n-dimensional manifold Y. If $H^n(X; Z) \neq 0$ and if f has a left homotopy inverse, then f is a homotopy equivalence.*

Remark 1. If f is a homotopy equivalence, every left homotopy inverse of f also is a right homotopy inverse of f.

Remark 2. Denote by $\{X\}$ the homotopy type of the space X and write $\{X\} \prec \{Y\}$ if Y dominates X. This is a quasi-order ([4], p. 212) in the class of "all" homotopy types. Let $\mathcal{C}^n$ denote the subclass of all homotopy types of integral cohomological dimension $\geqslant n$. Our result then implies the

COROLLARY. *The homotopy types of compact n-dimensional manifolds are minimal elements in $\mathcal{C}^n$.*

2. Preliminaries. Since the manifold Y in Theorem 1 is arcwise connected and dominates X, the latter also is arcwise connected.

Let now $P(X)$ denote the singular polytope of X; this is a connected simplicial CW-complex and there is a map $\varphi: P(X) \to X$ which induces isomorphisms of homotopy groups in all dimensions ([5], Theorem VI). Since the compact manifold Y is dominated by a CW-complex, the same also holds for X and, by [9], Theorem 1, φ is a homotopy equivalence. As a consequence