# General continuum hypothesis and ramifications *

by

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**1. Introduction and summary.** Let $\mathcal{W}$ be a well-ordered set; for any set $S$, let

(1) $$S(\mathcal{W}) \quad \text{or} \quad S^{\mathcal{W}}$$

denote the system of all functions on $\mathcal{W}$ to $S$; in particular, if $\alpha, \beta$ are ordinal numbers, let $\alpha(\beta)$ be the set of all the $\beta$-sequences of ordinals $<\alpha$, i. e.

(2) $$\alpha(\beta) = I\alpha(I\beta).$$

For example, $2(\omega_1)$ is the set of all the $\omega_1$-sequences of digits 0, 1.

Let us put

(3) $$TS(\mathcal{W}) = \bigcup_X S(X),$$

$X$ running over all initial segments of $\mathcal{W}$. Consequently, $T2(\omega_1)$ is the set of all the dyadic sequences whose length is $\leqslant \omega_1$. The set (3) is regarded as ordered by the relation

$$\mathrel{=\!\!|} \quad \text{meaning:} \quad \textit{to be an initial portion of.}$$

In particular $\dashv$ means $\mathrel{=\!\!|}$ and $\neq$.

One easily proves that the set (3) is a *tree*, i. e. that for every point $x$ of (3) the set of all the elements each of which is $\dashv x$ is well-ordered.

The investigation of sets $T2(\omega_\sigma)$ and, in general, of sets of the form (3) is very important and involves enormous difficulties. In particular, we showed that the *problem whether every non countable subset of $T2(\omega_1)$ contains an uncountable chain or an uncountable antichain is equivalent to the Suslin problem* (cf. Kurepa, [1], p. 106, 124, 132, $P_4 \leftrightarrow P_5$).

In particular, the following two propositions are mutually equivalent:

(A) *Every subset $S$ of $T2(\omega_1)$ of cardinality $\aleph_1$ such that every antichain of $S$ is $\leqslant \aleph_0$ contains a chain of cardinality $\aleph_1$;*

(*) The second part of the results was presented 23. 12. 1953 in Beograd at the Mathematics Institute of the Serbian Academy of Sciences. For the first part see Kurepa [2].

(S) *Every linearly ordered dense set such that every system of its disjointed intervals is* $\leqslant\aleph_0$ *is similar to a set of real numbers ordered according to their magnitude.*

Now, it is extremely interesting that the continuum hypothesis can be equivalently expressed in terms of sets $T2(\omega_\sigma)$ and in connexion with the existence of some chains in subsets of $T2(\omega_\sigma)$. In particular we shall prove the following theorem (cf. Theorem 3.2).

THEOREM. *The continuum hypothesis* $2^{\aleph_0}=\aleph_1$ *is equivalent to this statement*

($D_0$) *If an initial portion* $P$ *of length* $\omega_2$ *of* $T2(\omega_2)$ *contains no chain with* $\aleph_1$ 1-*s, i. e. if for every chain* $C\subseteq P$ *the set* $\sup C$ *contains* $<\aleph_1$ *times the digit* 1, *then* $P$ *contains a chain of cardinality* $\aleph_2$ *(obviously composed mainly of* 0-*s).*

This theorem is a corollary to a general theorem dealing with analogous sets $T2(\omega_\sigma)$ (cf. the main theorem 3.1).

The proof of the theorem is based on a theorem (Theorem 2.1 below) on regressing functions proved in another paper (Kurepa [2]).

**2. Auxiliary theorems.** In another paper we have proved the following theorem ([2], Theorem 3.2).

THEOREM 2.1. *Let* $\omega_\sigma$ *be a regular initial uncountable ordinal number. Let* $S_{\omega'_\sigma}$ $(\omega'_\sigma<\omega_\sigma)$ *be a sequence of non-void pairwise disjoint sets such that*

$$kS_{\omega'_\sigma}<k\omega_\sigma\,.$$

*Let* $M$ *be a set of cardinality* $\aleph_\sigma$ *of ordinals* $<\omega_\sigma$ *such that in the space* $I\omega_\sigma$ *of ordinals* $<\omega_\sigma$ *the complement of* $M$ *contains no closed set of cardinality* $k\omega_\sigma$. *Let* $f$ *be a mapping of*

$$M_0=\bigcup S_\mu \quad (\mu\in M)$$

*into*

$$S=\bigcup S_{\omega'_\sigma}$$

*such that* $x\in S_\mu$, $\mu>0$, *imply* $fx\in S_{\beta(\mu x)}$ *with* $\beta(\mu,x)<\mu$ $(\mu\in M)$. *Then there exists a* $y\in fM_0$ *satisfying*

$$k\{f^{-1}y\}=k\omega_\sigma,$$

*i. e.* $f$ *is constant in a set of cardinality* $\aleph_\sigma$.

On the basis of this theorem we have proved the following theorem.

THEOREM 2.2. *Let* $T$ *be a system of sequences of ordinals* $<\omega_\sigma^-$ *such that every initial segment of every element of* $T$ *belongs to* $T$; *let* $\omega_\sigma$ *be such that* $\omega_\sigma^-$ *is regular and that*

$$1\leqslant kR_\alpha T<\aleph_\sigma \quad (\alpha<\omega_\sigma)\,.$$

*If no ordered chain of* $T$ *contains* $\aleph_\sigma^-$ *digits* $\neq 0$, *then* $T$ *contains a* $\omega_\sigma$-*chain (terminating necessarily with* 0'*s).* (Cf. Kurepa [2], Theorem 4.1.)

Remark. It is interesting to observe that the elements of sequences forming $T$ might be composed of digits $0,1$ only. Some elements of $T$ might be sequences of digits $\neq 0$ only; but what matters is the existence of a chain of cardinality $\aleph_\sigma$ of elements of $T$ all terminating with 0's.

THEOREM 2.3. *Let* $T$ *be a tree every node of which is* $<\aleph_\sigma^-$ *and* $1\leqslant kR_{\omega'_\sigma}T<\aleph_\sigma^-$. *Let us suppose that* $\aleph_\sigma^-$ *is regular and that there exists a mapping* $f$ *of* $T$ *into* $I\omega_\sigma^-$ *such that:*

1° $f$ *is one-to-one in every knot of* $T$;

2° *the set* $\{f^{-1}0\}$ *of points of* $T$ *each of which is transformed into* 0 *intersects every chain of* $T$ *whose cardinality is* $\aleph_\sigma^-$.

*Then the tree* $T$ *contains a chain of cardinality* $\aleph_\sigma$.

As a matter of fact the existence of the preceding function $f$ enables us to give a representation of $T$ in the form of a system of sequences occuring in theorem 2.2. Let $x\in T$ and $T(\cdot,x]=\{y \mid y\in T,\ y\leqslant x\}$. Then $fx$ as well as $fx'$ is an ordinal for every $x'\leqslant x$; then $fT(\cdot,x]$ is a sequence of ordinals $<\omega_\sigma^-$ and one easily proves that the system

$$S=\{fT(\cdot,x] \mid x\in T\}$$

is the required set of sequences: the mapping

$$x\to fT(\cdot,x]$$

is a similarity between $T$ and $S$.

The system $S$ is a tree of the kind we examined in Theorem 2.2, except that the length of every element of $S$ is an isolated ordinal; joining to $S$ also the initial portions of the second kind of every element of $S$, one gets a system $T_0$ like that in theorem 2.2. And one sees that $T_0$ contains a chain of cardinality $\aleph_\sigma$; therefore the tree $S$ as well as the given tree $T$ contains a chain of cardinality $\aleph_\sigma$, which is what was required.

And now we are going to prove the main result of this paper.

**3. Main theorem.**

THEOREM 3.1. *For any ordinal* $\alpha$ *the following statements* ($C_\alpha$) *and* ($D_\alpha$) *are mutually equivalent:*

($C_\alpha$) $2^{\aleph_\alpha}=\aleph_{\alpha+1}$.

($D_\alpha$) *Let* $D$ *be an initial portion of the tree* $T\omega_\alpha(\omega_{\alpha+2})$ *composed of all functions on* $I\omega_{\alpha+2}$ *into* $I\omega_\alpha$. *If the length of* $D$ *is* $\omega_{\alpha+2}$ *and if* $D$ *contains no chain with more than* $\aleph_\alpha$ *digits* $\neq 0$, *then* $D$ *contains a* $\omega_{\alpha+2}$-*sequence (obviously terminating with* 0'*s each).*

Proof of Theorem 3.1. ($C_\alpha$) *implies* ($D_\alpha$).

At first, let us prove the following lemma.

LEMMA 3.1. *Relation* $(C_\alpha)$ *implies*

$$kR_\nu D \leqslant \aleph_{\alpha+1} \qquad (\nu < \omega_{\alpha+2}). \tag{4}$$

As a matter of fact we infer by induction that first of all

$$kR_\nu D \leqslant \sum_{\nu' \leqslant \nu} \aleph_\alpha^{k\nu'};$$

therefore, in particular

$$kR_{\omega'_{\alpha+1}} D \leqslant \sum \aleph_\alpha^{k\omega'_{\alpha+1}} \leqslant \aleph_\alpha^{\aleph_\alpha} = \big(\text{by hypothesis } (C_\alpha)\big) = \aleph_{\alpha+1}.$$

Hence, relation (4) holds for $\nu < \omega_{\alpha+1}$. Suppose now that

$$\omega_{\alpha+1} \leqslant \zeta < \omega_{\alpha+2}$$

and that (4) holds for every $\nu < \zeta$; let us prove (4) also for $\nu = \zeta$. If $\zeta$ is isolated, all is obvious. If $\mathrm{cf}\zeta = \omega_{\alpha+1}$, then in virtue of the supposition in $(D_\alpha)$, every element $x$ of $R_\zeta D$, being a $\zeta$-sequence of ordinals of $I\omega_\alpha$, terminates with a $\omega_{\alpha+1}$-sequence of 0's; therefore

$$kR_\zeta D \leqslant \sum kR_{\zeta'} D \leqslant (\text{by induction hypothesis}) \leqslant \aleph_{\alpha+1} k\zeta = \aleph_{\alpha+1}.$$

It remains to prove the case

$$1 < \tau < \omega_{\alpha+1} \qquad \text{where} \qquad \tau = cf\zeta.$$

Then let $\beta_0 < \beta_1 < \ldots < \beta_{\tau'} < \ldots$ be an increasing $\tau$-sequence of ordinals converging to $\zeta$. Then every $x$ of $R_\zeta D$ is the supremum of a well-determinated $\tau$-sequence $x^{\beta_{\tau'}} \epsilon R_{\beta_{\tau'}} D$.

Now the number of all such $\tau$-chains is $\leqslant \prod_{\tau'} kR_{\beta_{\tau'}} D \leqslant$ (by hypothesis) $\leqslant \prod_{\tau' < \tau} \aleph_{\alpha+1} \leqslant \aleph_{\alpha+1}^{\aleph_\alpha} = \big(\text{by hypothesis } (C_\alpha)\big) = (2^{\aleph_\alpha})^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Consequently, relation (4) holds.

Now, the length of $D$ is, by hypothesis, $\omega_{\alpha+2}$; consequently, in virtue of (4) we get

$$1 \leqslant kR_\nu D < \aleph_{\alpha+2} \qquad (\nu < \omega_{\alpha+2}).$$

Moreover the cardinal $\aleph_{\alpha+1} = \aleph_{\alpha+2}^-$ is regular. And since by supposition on $(D_\alpha)$, $D$ contains no chain with more than the $\aleph_\alpha$ digits $\neq 0$, the hypotheses of Theorem 2.2 are satisfied; accordingly, the tree $D$ contains a chain of cardinality $kD = \aleph_{\alpha+2}$. The implication $(C_\alpha) \rightarrow (D_\alpha)$ is proved.

3.2. $(D_\alpha)$ *implies* $(C_\alpha.)$ Let us suppose on the contrary that $2^{\aleph_\alpha} > \aleph_{\alpha+1}$ although $(D_\alpha)$ holds. Then in particular the set $2(\omega_\alpha)$ of cardinality $2^{\aleph_\alpha}$ of all dyadic $\omega_\alpha$-sequences would contain a subset $X$ of cardinality $\aleph_{\alpha+2}$.



Let $s(x)$ $(x \epsilon X)$ be a one-to-one function of $X$ onto $I\omega_{\alpha+2}$; every such $x$ being a dyadic $\omega_\alpha$-sequence, let us consider the sequence

$$hx = x + \{0\}_{sx}$$

obtained by a juxtaposition of $x$ and the constant $s(x)$-sequence composed of 0's; of course the length $\gamma hx$ of $hx$ equals $\gamma x + s(x) = \omega_\alpha + s(x)$. To distinct elements $x$ of $X$ correspond in this way distinct sequences $h(x)$'s. Then let $D$ be the system of all initial segments of those $h(x)$'s, $x$ running over $X$. The set $D$ would be an initial portion of $2(<\omega_{\alpha+2})$ (1) and no supremum of a chain of $D$ would have more than $\aleph_\alpha$ digits 1 in its representation. According to the statement $(D_\alpha)$ the set $D$ would contain a $\omega_{\alpha+2}$-chain, which contradicts the fact that obviously every chain in $D$ is $<\aleph_{\alpha+2}$. The theorem 3.1 is completely proved.

As a particular case of theorem 3.1 we have the following one.

THEOREM 3.2. *The continuum hypothesis*

$$C_0 \ldots 2^{\aleph_0} = \aleph_1$$

*is equivalent to the following proposition:*

*Let $T$ be any tree of height $\omega_2$; if there is a mapping $f$ of $T$ into $I2 = \{0, 1\}$ such that $f$ is one-to-one in every node of $T$ and if $\{f_1^{-1}\}$ contains no chain of cardinality $> \aleph_0$, then $T$ (and in particular $\{f^{-1}0\}$) contains a chain of cardinality $\aleph_2$.*

**Notation.** For any number $\alpha$, $I\alpha$ denotes the set of numbers $<\alpha$. For any number $\alpha$, $\alpha'$ runs over $I\alpha$. $kX$ denotes the cardinality of $X$. $cf\alpha$ is the minimal ordinal $\beta$ such that $\alpha$ is the supremum of a $\beta$-sequence of numbers $<\alpha$; if $\alpha^- < \alpha$ exists, then $cf\alpha = 1$. $\alpha^-$ is the supremum of numbers $<\alpha$.

For an ordered set $S$ and an ordinal $\alpha$, $R_\alpha S$ denotes the set of all the points $x$ of $S$ such that the set $S(\cdot, x)$ is similar with $I\alpha$.

A node of $S$ is every maximal subset $X$ of $S$ such that the sets $S(\cdot, x)$ $(x \epsilon X)$ are equal mutually.

## References


[1] G. Kurepa, *Ensembles ordonnés et ramifiés*, Thèse Paris, 1935, et Publ. Math. Beograd 4 (1935), p. 1-138.

[2] — *On regressing functions*, Zeitschrift für math. Logik und Grundlagen der Math. 4 (1958), p. 148-157.


(1) Obviously, $\alpha(<\beta)$ denotes the union of all the sets $\alpha(\beta')$, $\beta'$ runinng over $I\beta$.

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