On a metrization of polytopes

by

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1. Convex spaces. Let X be a metric space and let $\varrho(x, y)$ denote the distance between two points $x, y \in X$. The point $z \in X$ is said to lie *between x and y* provided that

$$\varrho(x, y) = \varrho(x, z) + \varrho(z, y) . \tag{1}$$

The point $z \in X$ is said to be a *centre* of the pair x, y provided that

$$\varrho(x, z) = \varrho(y, z) = \tfrac{1}{2} \cdot \varrho(x, y) . \tag{2}$$

Evidently every centre of the pair x, y lies between x and y.

A space X is said to be *convex* (Megner [3], p. 81) provided that for each two distinct points x, y of it there exist a point $z \in X$ different from x and y which lies between x and y. It was proved by Menger ([3], p. 89, see also Aronszajn [1]) that in complete convex spaces X each two points $x, y \in X$ are joined by a *metric segment*, i. e. by a subset of X isometric with the real interval of length $\varrho(x, y)$. We shall denote metric segments by the letter L with a convenient index. The existence of metric segments with given endpoints is also ensured if X is complete and for every pair of points $x, y \in X$ there exists in X at least one centre.

2. Strongly convex spaces. By a *strongly convex space* we understand a space X in which for every two distinct points $x, y \in X$ the set of all points $z \in X$ lying between x and y is a metric segment. We shall denote this segment by $X(x, y)$. For complete spaces (in particular for compacta), strong convexity is equivalent to the condition that every pair of points $x, y \in X$ has exactly one centre.

We easily see that for a strongly convex space X there exists, for each pair of points $x, y \in X$ and every $0 \leqslant t \leqslant 1$, exactly one point $z \in X$ such that

$$\varrho(x, z) = t \cdot \varrho(x, y) \quad \text{and} \quad \varrho(z, y) = (1-t) \cdot \varrho(x, y) .$$

Setting $z = \varphi_{x,y}(t)$ we obtain a function of three arguments x, y, t, its values being points of X. One easily sees that if X is a compactum, then $\varphi_{x,y}(t)$ depends continuously on the triple x, y, t. Hence, in this case, if we fix $y = y_0$ and set

$$\psi_t(x) = \varphi_{x,y_0}(t) \quad \text{for every } x \in X \text{ and } 0 \leqslant t \leqslant 1,$$

then we obtain a family $\{\psi_t\}$ of continuous mappings ψ of X into itself, depending continuously on t and satisfying, for every $x \in X$, the conditions:

$$\psi_0(x) = x, \quad \psi_1(x) = y_0.$$

It means that the family $\{\psi_t\}$ constitutes a homotopy contracting the compactum X in itself to the point y_0. Consequently *every strongly convex compactum is contractible to a point.*

Let us mention that it is not every compactum contractible to a point that may be metrized in a strongly convex manner. Moreover, it has been shown by K. Sieklucki [4] that already among compact polytopes of dimension 2, there exists one that is contractible to a point but cannot be metrized in a strongly convex manner.

3. Local strong convexity. The notion of strong convexity may be localized in many ways. Let us formulate some conditions, each of which constitutes some manner of local strong convexity:

CONDITION 1. *For every point $x_0 \in X$ there exists a neighbourhood U such that for each pair of points $x, y \in U$ there exists exactly one centre $z \in X$.*

CONDITION 2. *For every point $x \in X$ there exists in X a strongly convex neighbourhood.*

CONDITION 3. *For every point $x \in X$ and every neighbourhood U of x there exists in X a strongly convex neighbourhood $V \subset U$.*

Evidently each of those conditions implies the preceding one. Moreover it is easy to see that for compacta, condition 3 is equivalent to the following

CONDITION 3'. *For every $x \in X$ and every $\varepsilon > 0$ there exists a strongly convex and compact neighbourhood of x with diameter $< \varepsilon$.*

For our aims, it is convenient to adopt the following definition of locally strong convexity:

DEFINITION. A space X is said to be *locally strongly convex* if it satisfies condition 3.

4. Some elementary properties of the n-sphere. By the n-sphere ($n \geqslant 1$) we understand the set S^n of all points of the $(n+1)$-dimensional Euclidean space E^{n+1} at distance 1 from the origin. In this note we always assume that S^n is metrized by the spherical metric ϱ_{S^n} assigning to every two points $x = (x_1, x_2, \ldots, x_{n+1})$, $y = (y_1, y_2, \ldots, y_{n+1}) \in S^n$ the distance

$$\varrho_{S^n}(x, y) = \arccos[x_1y_1 + x_2y_2 + \ldots + x_{n+1}y_{n+1}].$$

Let us observe that

1° *S^n is compact and convex but not strongly convex.*

2° *The metric segments in S^n coincide with the arcs of great circles not longer than the hemicircle.*

3° *S^n is locally strongly convex.*

More exactly, for each pair of points $x, y \in S^n$ with $\varrho_{S^n}(x, y) < \pi$, the set of points lying between x and y coincides with the smaller of two arcs determined by x and y on the great circle on S^n passing through x and y. Hence for $\varrho_{S^n}(x, y) < \pi$ there exists in S^n exactly one metric segment with endpoints x and y. We shall denote this segment by $S^n(xy)$ and call it *spherical segment with endpoints x and y*. In order to see that condition 3, characterizing locally strong convexity, is satisfied, let us remark that for every point $x \in S^n$ and every $0 < \eta < \frac{1}{2}\pi$ the locus of points $y \in S^n$ satisfying the inequality $\varrho_{S^n}(x, y) \leqslant \eta$ is a compact, strongly convex neighbourhood of x in S^n with diameter 2η.

Let $a_0, a_1, \ldots, a_{n+1}$ be the vertices of a regular $(n+1)$-dimensional simplex inscribed into S^n. The projections of the k-sides of this simplex from the centre of S^n on S^n constitute k-dimensional regular spherical simplex on S^n. Each of those spherical simplexes is strongly convex. In particular we obtain in this manner a decomposition of S^n into $n+2$ isometric regular n-dimensional spherical simplexes with disjoint interiors.

5. Some elementary properties of the 2-sphere. In the 2-sphere S^2 every three points x, y, z of S^2 lying in a hemisphere (in particular every three points with mutual distances $< \frac{2}{3}\pi$) determine spherical triangle with vertices x, y, z (which may degenerate to a segment) defined as the projection from centre of S^2 onto S^2 of the usual triangle in E^3 with vertices x, y, z. In particular the spherical triangles on S^2, being projections of the sides of a 3-dimensional regular simplex inscribed in S^2, constitute a decomposition of S^2 into four regular isometric spheric triangles, called *quarters* of S^2. It is clear that the size of each of the angles of a quarter of S^2 is equal to $\frac{2}{3}\pi$. Moreover, let us observe that the length of each side of a quarter is $> \frac{1}{2}\pi$.

In this note we shall need the following elementary properties of S^2:

I. *Let a_0, a_1, a_2 be points of S^2 such that*

$$(3) \qquad 0 < \varrho_{S^2}(a_0, a_i) \leqslant \tfrac{1}{2}\pi \quad \textit{for} \quad i = 1, 2.$$

Given a positive number ε, let us denote by a_i', for $i = 1, 2$, the point lying on the spheric segment $S^2(a_0a_i)$ at the distance $\operatorname{Min}(\varepsilon, \varrho_{S^2}(a_0, a_i))$ from a_0. Then

$$\varrho_{S^2}(a_1', a_2') \leqslant \varrho_{S^2}(a_1, a_2)\,. \tag{4}$$

Moreover, if

(5) a_0 *does not lie between* a_1 *and* a_2

then

(6) a_0 *does not lie between* a_1' *and* a_2'.

II. *Let $a_0, a_1, \ldots, a_m, a_{m+1}$ be points of S^2 lying at a distance $< \frac{1}{2}\pi$ from a point $b \in S^2$ and such that for every $i = 0, 1, \ldots, m$ the point b does not lie between a_i and a_{i+1}. Let α_i denote the size of the angle at the vertex b in the spherical triangle a_iba_{i+1}. Hence $0 \leqslant \alpha_i < \pi$. If $\sum_{i=0}^{m} \alpha_i \geqslant \pi$ then*

$$\sum_{i=0}^{m} \varrho_{S^2}(a_i, a_{i+1}) > \varrho_{S^2}(a_0, b) + \varrho_{S^2}(b, a_{m+1})\,. \tag{7}$$

Proof. It is evident that for every $i = 0, 1, \ldots, m$ there exists an isometric transformation φ_i of the spherical triangle a_iba_{i+1} onto a spherical triangle $a_i'ba_{i+1}'$ satysfying the following conditions:

$$\varphi_i(a_i) = a_i'; \quad \varphi_i(b) = b; \quad \varphi_i(a_{i+1}) = a_{i+1}'\,. \tag{8}$$

(9) If $i < j \leqslant m$ and if $\alpha_\nu = 0$, for every $i < \nu < j$, then the interiors of the spherical triangles $a_i'ba_{i+1}'$ and $a_j'ba_{j+1}'$ are disjoint.

Evidently α_i is equal to the size of the angle at the vertex b in the spherical triangle $a_i'ba_{i+1}'$. It follows, by the inequality $\sum_{i=0}^{m} \alpha_i \geqslant \pi$ and by (9), that for some index $i_0 \leqslant m$, there exists a point a' lying between two successive points a_{i_0}', a_{i_0+1}' such that

(10) b lies between a_0' and a', but does not lie between a_0' and a_{i_0}'.

We infer by (10) that

$$\sum_{i=0}^{i_0-1} \varrho_{S^2}(a_i', a_{i+1}') + \varrho_{S^2}(a_{i_0}', a') > \varrho_{S^2}(a_0', b) + \varrho_{S^2}(b, a')\,. \tag{11}$$

Moreover, we have

$$\sum_{i=i_0+1}^{m} \varrho_{S^2}(a_i', a_{i+1}') + \varrho_{S^2}(a', a_{i_0+1}') + \varrho_{S^2}(b, a') \geqslant \varrho_{S^2}(b, a_{m+1}')\,. \tag{12}$$

Applying (10) and (8), we obtain by (11) and (12), the required inequality (7).

III. *Let $a_0, a_1, \ldots, a_m, a_{m+1}$ be points of S^2 lying at a distance $\leqslant \frac{1}{2}\pi$ from a point $b \in S^2$ and such that for every $i = 0, 1, \ldots, m$ the point b does not lie between a_i and a_{i+1}. Let α_i denote the size of the angle at the vertex b in the spherical triangle a_iba_{i+1} and let us assume that $\sum_{i=0}^{m} \alpha_i < \pi$. Then there exists a point b' lying between b and a_0 such that both spherical segments $S^2(a_0b')$ and $S^2(a_{m+1}b')$ lie in the sum of spherical triangles a_iba_{i+1} and*

$$\varrho_{S^2}(a_0, b') + \varrho_{S^2}(b', a_{m+1}) < \varrho_{S^2}(a_0, b) + \varrho_{S^2}(b, a_{m+1})\,.$$

IV. *Let x, y, z be theree points belonging respectively to three sides $S^2(ab)$, $S^2(ac)$, $S^2(bc)$ of a quarter Q^2 of S^2. Then*

$$\varrho_{S^2}(x, y) + \varrho_{S^2}(y, z) \geqslant \varkappa > \tfrac{1}{2}\pi\,,$$

where $\varkappa$ denotes the length of each of the sides of Q^2.

For one easily sees that $\varrho_{S^2}(x, y) \geqslant \varrho_{S^2}(a, y)$ and $\varrho_{S^2}(y, z) \geqslant \varrho_{S^2}(c, y)$. By addition it follows that $\varrho_{S^2}(x, y) + \varrho_{S^2}(y, z) \geqslant \varrho_{S^2}(a, c) = \varkappa$.

6. Spherical metrization of a polytope. Let T be a triangulation of a polytope P of dimension $n \geqslant 1$. Without loss of generality we may assume that P lies in a Euclidean space E^p. Then every point x lying on a k-dimensional ($k \leqslant n$) simplex $\Delta \in T$ with vertices $b_0, b_1, \ldots, b_k$ may be represented uniquely by the formula

$$x = \sum_{i=0}^{k} t_ib_i\,, \quad \text{where} \quad t_i \geqslant 0 \quad \text{for} \quad i = 0, 1, \ldots, k \text{ and } \sum_{i=0}^{k} t_i = 1\,. \tag{13}$$

Consider now a regular n-dimensional spherical simplex Q^n, which is the central projection of an n-dimensional side of an $(n+1)$-dimensional regular simplex inscribed in S^n. Let $a_0, a_1, \ldots, a_n$ be the vertices of Q^n. Setting

$$\varphi_\Delta^n(x) = \frac{\sum_{i=0}^{k} t_i \cdot a_i}{\left|\sum_{i=0}^{k} t_i \cdot a_i\right|}\,, \tag{14}$$

we obtain a homeomorphism φ_Δ^n mapping Δ onto a k-dimensional side of the n-dimensional spherical simplex Q^n. Evidently φ_Δ^n maps every straight segment lying on Δ onto a spherical segment lying on S^n.

Now let us put

$$\varrho_\Delta(x, y) = \varrho_{S^n}[\varphi_\Delta^n(x), \varphi_\Delta^n(y)]$$

for every two points $x, y \in \Delta$. Evidently ϱ_Δ constitutes a distance-function for Δ and it does not depend on the choice of the vertices $a_0, a_1, \ldots, a_n$ of the regular spherical simplex Q^n. Moreover, let us remark that if Δ_1, Δ_2 are two simplexes of T and the points $x, y \in P$ belong to the common part of them, then

$$\varrho_{\Delta_1}(x, y) = \varrho_{\Delta_2}(x, y)\ .$$

Using the metrics ϱ_Δ, we now introduce a metric ϱ_T, for the whole polytope P. Suppose first that P is connected. Then for every two points $x, y \in P$ there exists a finite sequence of points

$$x = x_0, x_1, x_2, \ldots, x_m, x_{m+1} = y$$

such that every two successive points x_i, x_{i+1} belong to one simplex $\Delta_i \in T$. A sequence $x_0, x_1, \ldots, x_{m+1}$ of this sort will be called a *passage in the triangulation* T *from* x *to* y. The number

$$(15) \qquad |x_0 x_1 \ldots x_{m+1}| = \sum_{i=0}^{m} \varrho_{\Delta_i}(x_i, x_{i+1})$$

will be called the *length* of the passage $x_0, x_1, \ldots, x_{m+1}$.

Let us denote by $\varrho_T(x, y)$ the lower bound of the lengths of all passages in T from x to y. Evidently ϱ_T is a distance-function for the connected polytope P.

Let us show that P *metrized by* ϱ_T *is a convex space*. It suffices to show, that for each pair of points $x, y \in P$ there exists a centre. In order to do it, let us observe that if $x_0, x_1, \ldots, x_{m+1}$ is a passage in T from x to y, then there exists on one of the metric segments $\Delta_i(x_i, x_{i+1}) \subset \Delta_i$ a point z (called the *centre of the passage* $x_0, x_1, \ldots, x_{m+1}$) such that

$$|x_0 x_1 \ldots x_i z| = |z x_{i+1} \ldots x_{m+1}| = \tfrac{1}{2}\cdot|x_0 x_1 \ldots x_{m+1}|\ .$$

By the definition of the distance-function ϱ_T, for every $\nu = 1, 2, \ldots$ there exists a passage $x_{\nu,0} x_{\nu,1} \ldots x_{\nu,m_\nu+1}$ in T from x to y satisfying the inequality

$$\varrho_T(x, y) \leqslant |x_{\nu,0} x_{\nu,1} \ldots x_{\nu,m_\nu+1}| \leqslant \varrho_T(x, y) + 1/\nu\ .$$

Let z_ν denote the centre of this passage. One easily sees that

$$(16) \qquad \begin{aligned} \tfrac{1}{2}\cdot\varrho_T(x, y) &\leqslant \varrho_T(x, z_\nu) \leqslant \tfrac{1}{2}\cdot[\varrho_T(x, y) + 1/\nu]\ ,\\ \tfrac{1}{2}\cdot\varrho_T(x, y) &\leqslant \varrho_T(z_\nu, y) \leqslant \tfrac{1}{2}\cdot[\varrho_T(x, y) + 1/\nu]\ . \end{aligned}$$

Since P is compact, the sequence $\{z_\nu\}$ contains a subsequence convergent to a point $z \in P$. We infer by (16) that

$$\varrho_T(x, z) = \varrho_T(z, y) = \tfrac{1}{2}\cdot\varrho_T(x, y)\ ,$$

whence z is a centre of the pair x, y.

Now, suppose that the polytope P is not connected and let $P_1, P_2, \ldots, P_k$ be the components of P. The simplexes of the triangulation T of P lying on P_i constitute a triangulation T_i of P_i. We introduce in P_i the metric ϱ_{T_i} (using always the same regular spherical simplex Q^n, where $n = \dim P$). By this metric every component P_i has a finite diameter d_i. Let d denote the greatest of the numbers $d_1, d_2, \ldots, d_k$. Setting

$$\varrho_T(x, y) = \varrho_{T_i}(x, y) \quad \text{if} \quad x, y \in P_i\ , \quad \text{for} \quad i = 1, 2, \ldots, k$$

and

$$\varrho_T(x, y) = d + 1 \quad \text{if} \quad x, y \text{ belong to distinct components of } P\ ,$$

we obtain a distance-function ϱ_T for the whole polytope P. We call it the *spherical metric corresponding to the triangulation* T.

7. Main theorem. The purpose of this note is to prove the following

THEOREM. *Every polytope of dimension* $\leqslant 2$, *metrized by the spherical metric corresponding to one of its triangulations, is locally strongly convex.*

The problem whether the analogous statement holds also for polytopes of dimension > 2 remains open (comp. [2], p. 108 problem 6).

Let us observe first that the proof of the theorem may be reduced to the case in which the polytope has dimension 2 at each of its points. Let T be a triangulation of a polytope P of dimension $\leqslant 2$ and let P_2 denote the sum of all 2-dimensional simplexes (triangles) of T, and P_1 the closure of the set $P - P_2$. Then

$$P = P_1 \cup P_2\ ,$$

where P_1 is a polytope of dimension $\leqslant 1$, and P_2 a polytope which has dimension 2 at each of its points. Evidently the triangulation T contains a triangulation T_1 of the polytope P_1 and a triangulation T_2 of the polytope P_2.

Let us assume that P_2 metrized by the distance-function ϱ_{T_2} is locally strongly convex. Moreover we easily see that for every point $x_0 \in P_2$ there exists in P_2 a neighbourhood U_{x_0} such that for each pair of points $x, y \in U_{x_0}$ we have $\varrho_{T_2}(x, y) = \varrho_T(x, y)$. Now it is evident that for each point $x \in P - (P_1 \cap P_2)$ there exists a compact, strongly convex neighbourhood with an arbitrarily small diameter. If however $x \in P_1 \cap P_2$ then, for every $\varepsilon > 0$, there exists a compact strongly convex neighbourhood V_i^ε of x in P_i with diameter $< \frac{1}{2}\varepsilon$. It is clear that V_1^ε consists of a finite number of metric segments, with diameters $< \frac{1}{2}\varepsilon$, having x as the common endpoint. We can replace each of these segments by a subsegment containing x (as one of its endpoints) and having a diameter so small that the common part of it with P_2 contains only the point x. One easily sees that the sum of all segments thus obstained and of the set V_2^ε is a compact strongly convex neighbourhood of x in P and the diameter of this neighbourhood is $< \varepsilon$.

Thus we infer that in the proof of the main theorem we can restrict ourselves to the case when the polytope P has dimension 2 at every point.

Hence in the sequel, we shall always assume that the polytope P is 2-dimensional at every point.

8. Spherical metric ϱ_T on individual triangles of T. We shall prove the following

LEMMA 1. *If the points x, y belong to one triangle Δ of a triangulation T of P and $\varrho_T(x, y) \leqslant \frac{1}{2}\pi$ then $\varrho_T(x, y) = \varrho_\Delta(x, y)$.*

Proof. Let us denote by $\varkappa$ the length of a side of a quarter Q^2 of S^2. Then

$$\varrho_T(x, y) \leqslant \tfrac{1}{2}\pi < \varkappa \quad \text{and} \quad \varrho_T(x, y) \leqslant \varrho_\Delta(x, y) .$$

By the definition of the metric ϱ_T, it remains to prove that for every passage $x = x_0, x_1, \ldots, x_m, x_{m+1} = y$ in T from x to y, satisfying the inequality

$$(17) \qquad |x_0x_1 \ldots x_{m+1}| < \varkappa ,$$

we have

$$(18) \qquad |x_0x_1 \ldots x_{m+1}| \geqslant \varrho_\Delta(x, y) .$$

First let us consider the case when both points x, y lie on the boundary of the triangle Δ. If three successive points x_i, x_{i+1}, x_{i+2} lie in one simplex $\Delta' \in T$, then

$$\varrho_{\Delta'}(x_i, x_{i+1}) + \varrho_{\Delta'}(x_{i+1}, x_{i+2}) \geqslant \varrho_{\Delta'}(x_i, x_{i+2})$$

and, by cancelling the point x_{i+1}, we obtain from $x_0, x_1, \ldots, x_{m+1}$ another passage in T from x to y with the length $\leqslant |x_0x_1 \ldots x_{m+1}|$. It easily follows that it suffices to prove (18) in the case when no three successive points x_i, x_{i+1}, x_{i+2} belong to one simplex of T. By (17) we have $\varrho(x_i, x_{i+1}) < \varkappa$, whence two successive points x_i and x_{i+1} cannot coincide with two different vertices of Δ_i. It follows that for every $i = 0, 1, \ldots, m$, there exists in T a triangle Δ_i such that x_i and x_{i+1} belong to the boundary of Δ_i, but do not belong to the same side of Δ_i. It follows that if one of the points x_i, x_{i+1} is a vertex of Δ_i, then the other belongs to the opposite side of Δ_i and consequently $\varrho_{\Delta_i}(x_i, x_{i+1}) > \varkappa$ contrary to inequality (17). Consequently we can assume that none of the points $x_0, x_1, \ldots, x_m, x_{m+1}$ is a vertex of the triangulation T.

Without loss of generality we may assume that Δ coincides with a quarter of S^2 with vertices abc and that $x \in S^2(ab)$ and $y \in S^2(ab) \cup S^2(ac)$.

Then a and b are common vertices of triangles Δ and Δ_0. Since every triangle Δ_i is isometric with Δ, there exists for every $i = 0, 1, \ldots, m$ an isometric transformation φ_i of Δ_i onto Δ such that

$$(19) \qquad \varphi_0(a) = a , \quad \varphi_0(b) = b ,$$

$$(20) \qquad \varphi_i(x) = \varphi_{i+1}(x) \quad \text{for every} \quad x \in \Delta_i \cap \Delta_{i+1} .$$

Then the points $\varphi_0(x) = \varphi_0(x_0)$, $\varphi_0(x_1) = \varphi_1(x_1), \ldots, \varphi_{m-1}(x_m) = \varphi_m(x_m)$, $\varphi_m(x_{m+1}) = \varphi_m(y)$ lie on the boundary of Δ and

$$|x_0x_1 \ldots x_{m+1}| = \sum_{i=0}^{m} \varrho_{S^2}[\varphi_i(x), \varphi_i(x_{i+1})] .$$

Moreover, all these points are distinct from a, b, c, because, by our assumption, none of the points $x_0, x_1, \ldots, x_{m+1}$ is a vertex of T. Finally, since x_i and x_{i+1} do not belong to the same side of Δ_i, the points $\varphi_{i-1}(x_i) = \varphi_i(x_i)$ and $\varphi_i(x_{i+1})$ (where $i = 1, 2, \ldots, m$) do not belong to the same side of Δ. Now we distinguish two cases:

Case 1°. None of the points $\varphi_0(x_0), \varphi_1(x_1), \ldots, \varphi_m(x_m)$ belongs to the side $S^2(bc)$.

First let us observe that all triangles $\Delta_0, \Delta_1, \ldots, \Delta_m$ have the vertex a in common and that $\varphi_i(a) = a$ for every $i = 0, 1, \ldots, m$. By (19) it is so for $i = 0$. Suppose that for an $i < m$ it is $a \in \Delta_i$ and $\varphi_i(a) = a$. Then the side of Δ_i, opposite to the vertex a, is mapped by φ_i onto $S^2(bc)$. It follows that the point x_{i+1} is an inner point of a side L of Δ_i containing the vertex a. Then L is the common side of Δ_i and Δ_{i+1} and, by (20), we infer that

$$\varphi_{i+1}(a) = \varphi_i(a) = a .$$

Consequently

$$\varrho_\Delta(a, y) = \varrho_\Delta[\varphi_m(a), \varphi_m(y)] = \varrho_\Delta[a, \varphi_m(y)] .$$

We infer that each of the points y, $\varphi_m(y)$ coincides with one of two points b', c' lying respectively on $S^2(ab)$ and $S^2(ac)$ at the distance $\varrho_{S^2}(a, y)$ from a. But we easily see that the sum of the lengths of all spherical segments $S^2[\varphi_i(x_i)\varphi_i(x_{i+1})]$ constituting a connected graph joining on Δ the point $x = \varphi_0(x)$ with the point $\varphi_m(y)$ is not smaller than $\varrho_{S^2}(x, b')$ and also than $\varrho_{S^2}(x, c')$. Hence

$$|x_0x_1 \ldots x_{m+1}| = \sum_{i=0}^{m} \varrho_{S^2}[\varphi_i(x_i), \varphi_i(x_{i+1})] \leqslant \varrho_\Delta(x, y) .$$

Case 2°. Each side $S^2(ab), S^2(ac), S^2(bc)$ contains at least one of the points $\varphi_0(x_0), \varphi_1(x_1), \ldots, \varphi_m(x_m)$.

In this case there exist three successive indices i_0, i_0+1, i_0+2 such that $\varphi_{i_0}(x_{i_0}), \varphi_{i_0+1}(x_{i_0+1}), \varphi_{i_0+2}(x_{i_0+2})$ lie on different sides, say on $S^2(ab)$, $S^2(ac)$ and $S^2(bc)$ respectively. Then we infer by IV of No. 5 that

$$|x_0x_1 \dots x_{m+1}| = \sum_{i=0}^{m} \varrho_{S^2}(\varphi_i(x_i), \varphi_i(x_{i+1}))$$
$$\geqslant \varrho_{S^2}(\varphi_{i_0}(x_{i_0}), \varphi_{i_0}(x_{i_0+1})) + \varrho_{S^2}(\varphi_{i_0+1}(x_{i_0+1}), \varphi_{i_0+1}(x_{i_0+2})) \geqslant \varkappa,$$

contrary to inequality (17).

Passing to the case when at least one of the points x, y does not lie on the boundary of the triangle Δ, consider a passage $x = x_0, x_1, \dots, x_m, x_{m+1} = y$ in T from x to y. If all points x_i belong to Δ, then (18) is evident. If, however, not all points x_i belong to Δ, then there exist two indices i and j such that $0 \leqslant i < j \leqslant m+1$ and that x_i, x_j lie on the boundary of Δ. We may assume that x_i is the first and x_j the last point lying on the boundary of Δ. Then, for $\nu < i$ and $\nu > j$, we have $x_\nu \in \Delta$. Applying the already settled case, we infer that

$$|x_0x_1 \dots x_{m+1}| = |x_0 \dots x_i| + |x_i \dots x_j| + |x_j \dots x_{m+1}|$$
$$\geqslant \varrho_\Delta(x, x_i) + \varrho_\Delta(x_i, x_j) + \varrho_\Delta(x_j, y) \geqslant \varrho_\Delta(x, y) .$$

Thus (18) holds also in this case.

9. Passages with shortest lengths. We shall prove the following

LEMMA 2. *If $x, y \in P$ and $\varrho_T(x, y) \leqslant \frac{1}{2}\pi$ then there exists a passage $x = x_0, x_1, \dots, x_m, x_{m+1} = y$ in T from x to y such that*

$$|x_0x_1 \dots x_{m+1}| = \varrho_T(x, y) .$$

Proof. By $\varrho_T(x, y) \leqslant \frac{1}{2}\pi$ the points x, y belong to one component P_0 of P. As it has already been shown, P_0 metrized by ϱ_T is convex. Hence there exists in P_0 a metric segment $L(xy)$ with endpoints x, y. Consider the points of $L(xy)$ in their natural order from x to y. Setting $x_0 = x$, let us denote by x_1 the last point of $L(xy)$ such that there exists in T a triangle Δ_0 containing both points x_0 and x_1. Assuming that the points $x_0, x_1, \dots, x_i \in L(xy)$ are already determined and $x_i \neq y$, we denote by x_{i+1} the last point of $L(xy)$ such that there exists in T a triangle Δ_i containing both points x_i and x_{i+1}. It follows that the point $x_{i+1} \in \Delta_i$ does not belong to any triangle Δ_j with $0 \leqslant j < i$. Hence the triangles $\Delta_i \in T$ are distinct. Consequently in this manner we obtain only a finite number of points $x = x_0, x_1, \dots, x_m, x_{m+1} = y$ constituting a passage in T from x to y. Evidently for $i < j$ the point x_i precedes the point x_j on the segment $L(xy)$. Applying the lemma 1 we infer that

$$\varrho_T(x, y) = \sum_{i=0}^{m} \varrho_T(x_i, x_{i+1}) = \sum_{i=0}^{m} \varrho_{\Delta_i}(x_i, x_{i+1}) = |x_0x_1 \dots x_{m+1}| .$$

10. Sets V_c^ε. For every point $c \in P$, let us denote by A_c the sum of all simplexes of T containing c, and by B_c the sum of all other simplexes of T. Evidently

(21) $$0 < \varrho_T(c, B_c) < \pi .$$

Let ε be an arbitrarily given positive number. We denote by V_c^ε the subset of P consisting of all points x satisfying the inequality

(22) $$\varrho_T(x, c) \leqslant \tfrac{1}{4}\mathrm{Min}[\varepsilon, \varrho_T(c, B_c)] .$$

Evidently V_c^ε is a compact neighbourhood of c (in P) with diameter $\leqslant \frac{1}{2}\mathrm{Min}(\varepsilon, \pi)$. To prove the theorem (formulated in No. 7) it suffices to show that V_c^ε is strongly convex.

We begin with two following lemmas:

LEMMA 3. *The set V_c^ε is convex.*

Proof. Let $x, y \in V_c^\varepsilon$. By lemma 2, there exists a passage $x = x_0, x_1, \dots, x_m, x_{m+1} = y$ in T from x to y with the length $\varrho_T(x, y)$. It follows by (22) that

$$\varrho_T(c, x_i) \leqslant \varrho_T(c, x) + \varrho_T(c, y) \leqslant \tfrac{1}{2}\varrho_T(c, B_c) .$$

Consequently the simplex $\Delta_i \in T$, containing x_i and x_{i+1}, has c as one of its vertices. Let x_i' denote the point lying on the segment $\Delta_i(cx_i)$ at the distance $\mathrm{Min}[\frac{1}{4}\varepsilon, \frac{1}{4}\cdot\varrho_T(c, B_c), \varrho_T(x_i, c)]$ from c. Then $x_0' = x$, $x_{m+1}' = y$, $x_i' \in V_c^\varepsilon$ for $i = 0, 1, \dots, m+1$ and the points $x_0', x_1', \dots, x_{m+1}'$ constitute a passage in T from x to y. Applying I of No. 5, we conclude that the length of this passage is $\leqslant |x_0x_1 \dots x_{m+1}| = \varrho_T(x, y)$. But $x_0' = x$ and $x_{m+1}' = y$ imply that the length of this passage is $\geqslant \varrho_T(x, y)$. Hence $|x_0'x_1' \dots x_m'x_{m+1}'| = \varrho_T(x, y)$. It suffices to set

$$L = \bigcup_{i=0}^{m} \Delta_i(x_i'x_{i+1}')$$

in order to obtain a metric segment joining in V_c^ε the points x and y.

LEMMA 4. *If x, y belong to one triangle $\Delta \in T$ and $\varrho_T(x, y) \leqslant \frac{1}{2}\pi$ then the set of points $z \in P$ lying between x and y coincides with the segment $\Delta(xy)$.*

Proof. By lemma 1, it suffices to show that the supposition that a point $z \in P - \Delta$ lies between x and y leads to a contradiction. Applying the lemma 2, we easily see that there exists a metric segment $L \subset P$ with endpoints x and y such that $z \in L$. Now let us distinguish 3 cases:

1° *Both points x, y lie on one side $\Lambda = \Delta(ab)$ of Δ.* Since $\varrho_T(x, y) \leqslant \frac{1}{2}\pi$, at least one of the endpoints of Λ belongs to $P-L$. It easily follows that L contains points which do not lie on Λ, but belong to one of the triangles of T adjacent to Λ. If z' is one of such points, lying on a triangle Δ' adjacent to Λ, then z' lies between $x, y \in \Delta'$ and z' does not belong to $\Lambda(ab) = \Delta'(ab)$, which is impossible on account of lemma 1.

2° *x, y lie on disjoint sides of Δ.*

Let a, b, c be the vertices of Δ and $x \in \Delta(ac)$, $y \in \Delta(bc)$. Let M denote the set composed of a, b, c and of all points lying in the interior of any triangle of T adjacent to Δ along one of its sides. Evidently M cuts P between each point of Δ and each point of $P-(M \cup \Delta)$. Since the length of L is $\leqslant \frac{1}{2}\pi$, the side $\Delta(ab)$ is disjoint with L. It easily follows that there exists a point $z' \in L$ lying in the interior G' of some triangle Δ' adjacent to one of the sides $\Delta(ac)$, $\Delta(bc)$, say to $\Delta(ac)$. It is easy to observe that there exists a point x_1 belonging to the common part of L and of the side $\Delta'(cb')$ of Δ' different from $\Delta'(ac)$. Applying lemma 2, we obtain a passage $x = x_0, x_1, \ldots, x_m, x_{m+1} = y$ in T with the length $\varrho_T(x, y)$. Moreover we may assume that every two successive points x_i, x_{i+1} lie on the boundary of a triangle Δ_i having c as one of its vertices. Let us now denote by y' the point lying on $\Delta(ac)$ at the distance $\varrho_T(cy)$ from c. By the same argument as that used in the proof of lemma 1, case 1°, we obtain the isometries $\varphi_0, \varphi_1, \ldots, \varphi_m$ of triangles $\Delta_0 = \Delta', \Delta_1, \Delta_2, \ldots, \Delta_m$ onto the triangle Δ (which we may identify with a quarter Q^2 of the sphere S^2) in such a manner that $\varphi_{i+1}(x_i) = \varphi_i(x_i)$ and that the succesive points of the sequence $\varphi_0(x) = x, \varphi_0(x_1), \varphi_1(x_2), \ldots, \varphi_m(x_{m+1}) = \varphi_m(y)$ lie alternatively on sides $\Delta(ac)$ and $\Delta(bc)$ and point $\varphi_m(y)$ coincides with y or with y'. Since $x_1 \neq y$, we easily infer that the sum of lengths of segments $\Delta\left(\varphi_i(x_i)\varphi_i(x_{i+1})\right)$ is $> \varrho_T(x, y)$ But this sum is equal to $|x_0x_1 \ldots x_{m+1}| = \varrho_T(x, y)$. Thus we obtain a contradiction.

3° At least one of the points x, y does not lie on the boundary of Δ. In this case, let x' denote the first point of the segment L (ordered naturally from x to y) lying on the boundary of Δ, and by y' the last point of L lying on the boundary of Δ. Then z lies between x' and y', and thus the proof of impossibility is reduced to the already settled cases 1° or 2°.

11. Decomposition of metric segments into spherical segments. Angles. Let us prove the following

LEMMA 5. *If L is a metric segment on V_c^ε, then the common part of L with any triangle Δ of T either is empty or contains only one point, or is a segment.*

Proof. Let x, y denote the endpoints of L and suppose that $L \cap \Delta$ contains at least two points. Let x_1 be the first, and y_1 — the last point of the subset $L \cap \Delta$ of L (where L is naturally ordered from x to y). We infer by lemma 4 that the set of points of L lying between x_1 and y_1 coincides with the segment $\Delta(x_1 y_1)$. Hence $L \cap \Delta = \Delta(x_1 y_1)$.

The segments which are common parts of a metric segment $L \subset V_c^\varepsilon$ with single triangles of T constitute a decomposition of L into segments with disjoint interiors:

$$L = L_0 \cup L_1 \cup \ldots \cup L_m \,. \tag{23}$$

This decomposition is uniquely determined by the triangulation T. We call it the *decomposition of L corresponding to the triangulation T*. Evidently the indices i may be so fixed that $L_i = L \cap \Delta_i = \Delta_i(x_i x_{i+1})$ for $i = 0, 1, \ldots, m$.

Now let us assume that $L \subset V_c^\varepsilon - (c)$. Then the segments:

$$\Delta_i(x_i x_{i+1}) \,,\ \Delta_i(x_i c) \,,\ \Delta_i(x_{i+1} c)$$

constitute a triangle $x_i c x_{i+1}$ isometric with a spherical triangle lying on S^2. Let a_i denote the size of its angle at the point c. The number

$$a(L) = \sum_{i=0}^{m} a_i$$

will be called the *amplitude* of segment $L \subset V_c^\varepsilon$ relatively to c.

LEMMA 6. *For every metric segment $L \subset V_c^\varepsilon - (c)$ it is $0 \leqslant a(L) < \pi$.*

Proof. Applying the notation just introduced, consider an isometric transformation φ_i of the triangle $x_i c x_{i+1}$ into S^2. It is easy to observe that these isometries φ_i may be chosen successively in such a manner that φ_i and φ_{i+1} coincide on the common part of triangles $x_i c x_{i+1}$ and $x_{i+1} c x_{i+2}$, for every $i = 0, 1, \ldots, m-1$. Consequently the points $a_i = \varphi_i(x_i)$ and the point $b = \varphi_i(c)$ lie on S^2 and satisfy the premises of II, No. 5. It follows $\sum_{i=0}^{m} a_i < \pi$, because otherwise we should have

$$\sum_{i=0}^{m} \varrho_T(x_i, x_{i+1}) = \sum_{i=0}^{m} \varrho_{S^2}(a_i, a_{i+1}) > \varrho_{S^2}(a_0, b) - \varrho_{S^2}(b, a_{m+1})$$
$$= \varrho_T(x, c) + \varrho_T(c, y) \geqslant \varrho_T(x, y) \,,$$

which is impossible, because the length of L is equal to $\varrho_T(x, y) = \sum_{i=0}^{m} \varrho_T(x_i, x_{i+1})$.

12. Strong convexity of V_c. Now we pass to the proof that each of the neighbourhoods V_c^ε is strongly convex. We distinguish three following cases:

Case I. *c lies in the interior of a triangle $\Delta \in T$.*

By the lemma 1, the set V_c is isometric in this case with the spherical cap in S^2 with radius $< \frac{1}{2}\pi$. Hence it is strongly convex.

Case II. *c lies in the interior of a 1-dimensional simplex $\Lambda \in T$.* Since P has dimension 2 at each of its points, there exist in T triangles adjacent to Λ. Let $\Delta_1, \Delta_2, \ldots, \Delta_k$ be all these triangles. Then

$$V_c^\varepsilon \subset \bigcup_{i=1}^{k} \Delta_i .$$

If $x, y \in V_c^\varepsilon$, then there exist among triangles $\Delta_1, \Delta_2, \ldots, \Delta_k$ such triangles Δ_x and Δ_y, that $x \in \Delta_x$ and $y \in \Delta_y$.

If $\Delta_x = \Delta_y = \Delta$, then we infer by lemma 4 that the set of all points $z \in P$ lying between x and y coincides with the segment $\Delta(xy)$.

If $\Delta_x \neq \Delta_y$, then let us denote by Z the set of all points of P lying between x and y. By lemma 3, the set Z contains a metric segment L joining x and y in V_c^ε. For every point $z \in Z$ we have

$$\varrho_T(c, z) \leqslant \varrho_T(c, x) + \varrho_T(c, y) < \tfrac{1}{2} \cdot \varrho_T(c, B_c) .$$

Hence $Z \subset \Lambda_c$. Let us show that $Z \subset \Delta_x \cup \Delta_y$. Otherwise there exists in T a triangle $\Delta_z \neq \Delta_x, \Delta_y$ such that $z \in (\Delta_z - \Lambda) \cap Z$. Applying lemma 3 we infer that there exist two points $z_1, z_2 \in \Lambda$ such that z_1 lies between x and z, and z_2 lies between y and z. Then

$$\varrho_T(z_1, z_2) = \varrho_{\Delta_z}(z_1, z_2) < \varrho_{\Delta_z}(z_1, z) + \varrho_{\Delta_z}(z, z_2) ,$$

and consequently

$$\begin{aligned}\varrho_T(x, z_1) + \varrho_T(z_1, z_2) + \varrho_T(z_2, y) &< \varrho_T(x, z_1) + \varrho_T(z_1, z) + \varrho_T(z, z_2) + \varrho_T(z_2, y) \\ &= \varrho_T(x, z) + \varrho_T(z, y) = \varrho_T(x, y) ,\end{aligned}$$

which is impossible.

Hence $Z \subset \Delta_x \cup \Delta_y$.

Consider now two isometries φ_x and φ_y mapping Δ_x and Δ_y onto two adjacent quarters Q_x and Q_y of S^2 respectively and coincident on the segment Λ. Evidently in the segment $\varphi_x(\Lambda) = \varphi_y(\Lambda)$ there exists exactly one point such that the sum of its spherical distances from the points $\varphi_x(x)$ and $\varphi_y(y)$ is minimal. It follows that also in the segment Λ there exists exactly one point z such that the sum $\varrho_{\Delta_x}(x, z) + \varrho_{\Delta_y}(y, z)$ is minimal. We infer that $Z \cap \Lambda = (z)$. It follows by lemma 1 that $Z = \Delta_x(xz) + \Delta_y(yz)$, whence Z is a metric segment, i. e. the set V_c is strongly convex.

Case III. *c is a vertex of the triangulation T.*

We have to prove that for every two points $x, y \in V_c^\varepsilon$ the set of points $z \in V_c^\varepsilon$ lying between x and y is a metric segment. We distinguish two following possibilities:

1° *c lies between x and y.*

Since the case when x and y belong to one of the triangles of T is already settled by lemma 4, we may assume that the triangles Δ_x, Δ_y of T, containing x and y respectively, are distinct and none of the points x, y belong to $\Delta_x \cap \Delta_y$. In particular $x \neq c \neq y$. Evidently the set

$$L_0 = \Delta_x(xc) \cup \Delta_y(yc)$$

is a metric segment joining x and y in V_c^ε. Suppose that there exists a point $z \in V_c^\varepsilon - L_0$ lying between x and y. Applying lemma 3, we may find a segment $L \subset V_c^\varepsilon$ with endpoints x and y, which passes through z. Then c does not lie on L, because otherwise z would lie between x and c, or between y and c, and consequently z would belong to L_0.

Consider now the decomposition of the segment L,

$$L = \Delta_0(x_0 x_1) \cup \Delta_1(x_1 x_2) \cup \ldots \cup \Delta_m(x_m x_{m+1}) ,$$

into spherical segments with disjoint interiors, defined as in No. 11. By lemma 6, the sum of angles α_i at c in the triangles $x_i c x_{i+1}$ is $< \pi$. It easily follows that there exist isometries $\varphi_0, \varphi_1, \ldots, \varphi_m$ mapping triangles $x_i c x_{i+1}$ onto some triangles with disjoint interiors on S^2 and such that φ_i coincides with φ_{i+1} on the segment $\Delta_i(c x_{i+1})$. In particular all points $\varphi_i(c)$ coincide with a fixed point $b \in S^2$. Since $\sum_{i=0}^{m} \alpha_i < \pi$, we infer that φ_i constitute together a homeomorphic mapping φ of the sum M of all triangles $x_i c x_{i+1}$ into the sum N of triangles $\varphi_i(x_i) b \varphi_{i+1}(x_{i+1})$ lying on S^2. It follows by III of No. 5 that there exists in N a simple arc L' with length smaller than $\varrho_{S^2}(\varphi(x), b) + \varrho_{S^2}(b, \varphi(y))$ joining $\varphi(x)$ with $\varphi(y)$. It is easy to observe that the inverse homeomorphism φ^{-1} maps L' onto an arc $\varphi^{-1}(L') \subset V_c^\varepsilon$ having the same length as the arc L'. But this is impossible, because $\varrho_T(x, c) + \varrho_T(c, y) = \varrho_T(x, y)$.

2° *c does not lie between x and y.*

By lemma 3, there exists a metric segment $L \subset V_c^\varepsilon$ joining x and y. By our hypotheses, c does not belong to L. We have to show that every point z lying between x and y belongs to L.

Since the case in which x and y belong to one simplex of T is already settled, we may assume that the simplexes Δ_x and Δ_y are distinct and none of points x, y belongs to their common part. Applying lemma 5 we infer that there exists in T a system of triangles $\Delta_0 = \Delta_x, \Delta_1, \ldots, \Delta_m = \Delta_y$ with common vertex c and with disjoint interiors such that Δ_i and Δ_{i+1}

have a common side Λ_{i+1} which has with L exactly one point x_{i+1} in common and that

$$L = \Delta_0(xx_1) \cup \Delta_1(x_1x_2) \cup \ldots \cup \Delta_{m-1}(x_{m-1}x_m) \cup \Delta_m(x_m y) .$$

Let a_i denote the size of the angle at the vertex c in the spherical triangle x_icx_{i+1}. Evidently $a_i = \frac{2}{3}\pi$ for $0 < i < m$. It follows by lemma 6 that

$$(24) \qquad a_0 + (m-1)\cdot\tfrac{2}{3}\pi + a_m < \pi ,$$

whence $m = 1$ or $m = 2$. We consider the two cases separately:

1) $m = 1$. Then the triangles $\Delta_x = \Delta_0$ and $\Delta_1 = \Delta_y$ have a common side Λ_1 and inequality (24) has the form

$$(25) \qquad a_0 + a_1 < \pi .$$

Suppose that there exists a point $z \in V_c^\varepsilon - L$ lying between x and y. Applying lemma 3, we can construct a metric segment L' joining x and y in V_c and passing through z. Evidently there exists a homeomorphism φ mapping $\Delta_x \cup \Delta_y$ onto the sum of two adjacent quarters on S^2, which is an isometry on Δ_x and also on Δ_y. Since

$$\varrho_{S^2}(\varphi(x), \varphi(y)) \leqslant \varrho_{S^2}(\varphi(x), \varphi(c)) + \varrho_{S^2}(\varphi(c), \varphi(y)) = \varrho_T(x, c) + \varrho_T(y, c) \leqslant \tfrac{1}{2}\pi,$$

whence in $\varphi(\Delta_x \cup \Delta_y)$ there exists only one segment joining $\varphi(x)$ and $\varphi(y)$. It follows that $\varphi(L')$ is not contained in $\varphi(\Delta_x \cup \Delta_y)$, whence there exists a point $z' \in L' - (\Delta_x \cup \Delta_y)$. By an argument already used in this proof, we infer that there exists a triangle Δ' of T adjacent to Δ_x and Δ_y such that

$$L' = \Delta_x(xx_1') \cup \Delta'(x_1'x_2') \cup \Delta_y(x_2'y) ,$$

where x_1' and x_2' lie on the sides of Δ' common with Δ_x and Δ_y respectively. It follows that

$$a(L') = a_0' + \tfrac{2}{3}\pi + a_2' < \pi ,$$

wherein $a_0' = \frac{2}{3}\pi - a_0$ and $a_2' = \frac{2}{3}\pi - a_1$. We infer $a_0 + a_1 > \pi$, which contradicts (25).

2) $m = 2$. In this case inequality (24) has the form

$$(26) \qquad a_0 + a_2 < \tfrac{1}{3}\pi .$$

It follows that both points x, y belong to the set D which is the sum of the triangle Δ and of the interiors of triangles Δ_x and Δ_y. Evidently there exists a homeomorphism mapping D onto a subset of the sphere S^2. By quite an elementary argument we see that in the set $\varphi(D)$ there exists only one metric segment joining the points $\varphi(x)$ and $\varphi(y)$. Consequently every metric segment $L' \subset V_c^\varepsilon$ different from L and joining x and y contains at least one point which does not belong to D.

Suppose now that there exists a point $z \in V_c^\varepsilon - L$ lying between x and y. It follows that there exists a segment $L' \subset V_c^\varepsilon$ joining x with y and passing through z. By an argument already used in this proof, we infer that either

$$\text{(a)} \qquad L' \subset \Delta_x \cup \Delta_y$$

or there exists a triangle Δ', adjacent to Δ_x and also to Δ_y, such that

$$\text{(b)} \qquad L' \subset \Delta_x \cup \Delta' \cup \Delta_y .$$

Let us investigate both cases, (a) and (b). Since L' is not contained in D, we easily infer that in case (a) the segment L' contains a point z' lying on the side of Δ_x or of Δ_y not contained in Λ_1 and consequently the metric segment L' decomposes into two metric segments L_1 and L_2 with common endpoint z'. We easily infer by (26) that

$$a(L') = a(L_1) + a(L_2) > \pi ,$$

which is impossible in view of lemma 6.

In case (b) the triangle Δ' contains a side Λ', which lies on $\Delta_x \cup \Delta_y$ but does not lie on Δ_z. Without loss of generality, we may assume that $\Lambda' \neq \Delta_x$. Let z' be a point of $\Lambda' \cap L'$. We easily see that then $L' = L'(xz') + L'(z'y)$ and we have, by (26),

$$a(L') = a[L'(xz')] + a[L'(z'y)] \geqslant \tfrac{1}{3}\pi + \tfrac{2}{3}\pi = \pi ,$$

which is impossible in view of lemma 6.

Thus the proof of the main theorem is complete.

Remark. Since the set V_c^ε coincides with a (closed) ball (in the space P metrized by ϱ_T) with centre c and radius $\frac{1}{4}\cdot \mathrm{Min}(\varepsilon, \varrho_T(c, B_c))$, we infer that the polytope P metrized by ϱ_T satisfies the following condition: *For every $c \in P$ there exists a positive number ε such that each closed ball in P with centre c and radius $\leqslant \varepsilon$ is strongly convex.*

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