что сама θ суммируется с некоторой степенью более низкой, чем вытекающая из теоремы знака.

Мы не пытались здесь провести полное доказательство для показателя θ, ограниченного своим максимумом θ ≤ θ. Думается, что теорема будет верна и в самой широкой формулировке.

Наконец, в приложениях очень интересно было бы установить теорему аналогичную теореме о сложных функциях полезную для построения теории неравносильных гиперболических уравнений. Эта теорема легко устанавливается для метрики Rn, но можно ли перенести её на норму Lp и при каких условиях, остается неясным.

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Цитированная литература


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On a metrization of polytopes

by

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1. Convex spaces. Let X be a metric space and let $g(x, y)$ denote the distance between two points $x, y \in X$. The point $z \in X$ is said to lie between $x$ and $y$ provided that

$$g(x, y) = g(x, z) + g(z, y).$$

The point $z \in X$ is said to be a centre of the pair $x, y$ provided that

$$g(x, z) = g(y, z) = \frac{1}{2} g(x, y).$$

Evidently every centre of the pair $x, y$ lies between $x$ and $y$.

A space X is said to be convex (Menger [9], p. 81) provided that for each two distinct points $x, y$ of it there exist a point $z \in X$ different from $x$ and $y$ which lies between $x$ and $y$. It was proved by Menger [9], p. 89, see also Aronszajn [1] that in complete convex spaces $X$ each two points $x, y \in X$ are joined by a metric segment, i.e. by a subset of $X$ isometric with the real interval of length $g(x, y)$. We shall denote metric segments by the letter $L$ with a convenient index. The existence of metric segments with given endpoints is also ensured if $X$ is complete and for every pair of points $x, y \in X$ there exists in $X$ at least one centre.

2. Strongly convex spaces. By a strongly convex space we understand a space $X$ in which for every two distinct points $x, y \in X$ the set of all points $z \in X$ lying between $x$ and $y$ is a metric segment. We shall denote this segment by $X(x, y)$. For complete spaces (in particular for compacta), strong convexity is equivalent to the condition that every pair of points $x, y \in X$ has exactly one centre.

We easily see that for a strongly convex space $X$ there exists, for each pair of points $x, y \in X$ and every $0 < t < 1$, exactly one point $z \in X$ such that

$$g(x, z) = t \cdot g(x, y) \quad \text{and} \quad g(z, y) = (1-t) \cdot g(x, y).$$

Setting $z = \varphi_t(y)$ we obtain a function of three arguments $x, y, t$, its values being points of $X$. One easily sees that if $X$ is a compactum,
then $\varphi_{\alpha}(t)$ depends continuously on the triple $x, y, t$. Hence, in this case, if we fix $y = y_0$ and set

$$
\varphi_t(x) = \varphi_{\alpha}(t)
$$

for every $x \in X$ and $0 < t \leq 1$,

then we obtain a family $\{\varphi_t\}$ of continuous mappings $y$ of $X$ into itself, depending continuously on $t$ and satisfying, for every $x \in X$, the conditions:

$$
\varphi_0(x) = x, \quad \varphi_1(x) = y_0.
$$

It means that the family $\{\varphi_t\}$ constitutes a homotopy contracting the compactum $X$ in itself to the point $y_0$. Consequently every strongly convex compactum is contractible to a point.

Let us mention that it is not every compactum contractible to a point that may be metrized in a strongly convex manner. Moreover, it has been shown by K. Siebner and K. Kocinčki [4] that among compact polytopes of dimension 2, there exists one that is contractible to a point but cannot be metrized in a strongly convex manner.

3. Local strong convexity. The notion of strong convexity may be localized in many ways. Let us formulate some conditions, each of which constitutes some manner of local strong convexity:

**Condition 1.** For every point $x \in X$, there exists a neighbourhood $U$ such that for each pair of points $x, y \in U$, there exists exactly one centre $z \in X$.

**Condition 2.** For every point $x \in X$, there exists $X$ a strongly convex neighbourhood.

**Condition 3.** For every point $x \in X$ and every neighbourhood $U$ of $x$, there exists $X$ a strongly convex neighbourhood $V \subset U$.

Evidently each of those conditions implies the preceding one. Moreover it is easy to see that for compacta, condition 3 is equivalent to the following

**Condition 3'.** For every $x \in X$ and every $r > 0$, there exists a strongly convex and compact neighbourhood of $x$ with diameter $< r$.

For our aims, it is convenient to adopt the following definition of locally strong convexity:

**Definition.** A space $X$ is said to be locally strongly convex if it satisfies condition 3.

4. Some elementary properties of the $n$-sphere. By the $n$-sphere ($n \geq 1$) we understand the set $S^n$ of all points of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ at distance 1 from the origin. In this note we always assume that $S^n$ is metrized by the spherical metric $\varphi_{\alpha}$ assigning to every two points $x = (x_1, x_2, \ldots, x_{n+1}), y = (y_1, y_2, \ldots, y_{n+1}) \in S^n$, the distance

$$
\varphi_{\alpha}(x, y) = \arccos(x_1y_1 + x_2y_2 + \cdots + x_{n+1}y_{n+1}).
$$

Let us observe that

1. $S^n$ is compact and convex but not strongly convex.
2. The metric segments in $S^n$ coincide with the area of great circles not longer than the hemicycle.
3. $S^n$ is locally strongly convex.

More exactly, for each pair of points $x, y \in S^n$ with $\varphi_{\alpha}(x, y) < \pi$, the set of points lying between $x$ and $y$ coincides with the smaller of two arcs determined by $x$ and $y$ on the great circle on $S^n$ passing through $x$ and $y$. Hence for $\varphi_{\alpha}(x, y) < \pi$, there exists in $S^n$ exactly one metric segment with endpoints $x$ and $y$. We shall denote this segment by $S^n(x, y)$ and call it a spherical segment with endpoints $x$ and $y$. In order to see that condition 3, characterizing locally strong convexity, is satisfied, let us remark that for every point $x \in S^n$ and every $0 < \eta < \frac{\pi}{2}$, the locus of points $y \in S^n$ satisfying the inequality $\varphi_{\alpha}(x, y) < \eta$ is a compact, strongly convex neighbourhood of $x$ in $S^n$ with diameter $2\eta$.

Let $a_1, a_2, \ldots, a_{n+1}$ be the vertices of a regular $(n+1)$-dimensional simplex inscribed into $S^n$. The projections of the $k$-sides of this simplex from the centre of $S^n$ on $S^n$ constitute $k$-dimensional regular spherical simplex on $S^n$. Each of those spherical simplices is strongly convex. In particular we obtain in this manner a decomposition of $S^n$ into $n+2$ isometric regular $(n+1)$-dimensional spherical simplices with disjoint interiors.

5. Some elementary properties of the 2-sphere. In the 2-sphere $S^2$ every three points $x, y, z$ of $S^2$ lying in a hemisphere (in particular every three points with mutual distances less than $\frac{\pi}{4}$) determine a spherical triangle with vertices $x, y, z$ which may degenerate to a segment) defined as the projection of the segments from $x, y, z$ onto $S^2$ of the usual triangle in $\mathbb{R}^2$ with vertices $x, y, z$. In particular the spherical triangles on $S^2$, being projections of the sides of a 3-dimensional regular simplex inscribed in $S^2$, constitute a decomposition of $S^2$ into four regular isometric spherical triangles, called quarters of $S^2$. It is clear that the size of each of the angles of a quarter of $S^2$ is equal to $\frac{\pi}{2}$. Moreover, let us observe that the length of each side of a quarter is $> \frac{\pi}{2}$.

In this note we shall need the following elementary properties of $S^2$:

1. Let $a_1, a_2, a_3$ be points of $S^2$ such that

$$
0 < \varphi_{\alpha}(a_i, a_i) \leq \frac{\pi}{2} \quad \text{for} \quad i = 1, 2.
$$

(3)
Given a positive number \( \varepsilon \), let us denote by \( a_i \), for \( i = 1, 2 \), the point lying on the spherical segment \( S(a_0, a_i) \) at the distance \( \min \{ \varepsilon, \varrho_{gb}(a_0, a_i) \} \) from \( a_0 \). Then
\[
\varrho_{gb}(a_i, \varepsilon a_i) \leq \varrho_{gb}(a_0, a_i) .
\]
Moreover, if
\[
a_0 \text{ does not lie between } a_i \text{ and } a_i \varepsilon .
\]
then
\[
a_0 \text{ does not lie between } a_i \text{ and } a_i \varepsilon .
\]

II. Let \( a_0, a_1, ..., a_m, a_{m+1} \) be points of \( S^n \) lying at a distance \( < \frac{1}{2} \pi \) from a point \( b \in S^n \) and such that for every \( i = 0, 1, ..., m \) the point \( b \) does not lie between \( a_i \) and \( a_{i+1} \). Let \( a_i \) denote the size of the angle at the vertex \( b \) in the spherical triangle \( a_0 a_i a_{i+1} \). Hence \( 0 \leq a_i < \pi \). If \( \sum a_i \geq \pi \), then
\[
\sum_{i=0}^{m} \varrho_{gb}(a_i, a_{i+1}) > \varrho_{gb}(a_0, a_i) + \varrho_{gb}(b, a_{m+1}) .
\]

Proof. It is evident that for every \( i = 0, 1, ..., m \) there exists an isometric transformation \( \varphi \) of the spherical triangle \( a_0 a_i a_{i+1} \) onto a spherical triangle \( a_0 b a_{i+1} \) satisfying the following conditions:
\[
\varphi(a_i) = a_i ; \quad \varphi(b) = b ; \quad \varphi(a_{i+1}) = a_{i+1} .
\]

(9) If \( i < j \leq m \) and if \( a_r = 0 \), for every \( i < r < j \), then the interiors of the spherical triangles \( a_0 a_i a_{i+1} \) and \( a_0 b a_{i+1} \) are disjoint.

Evidently \( a_i \) is equal to the size of the angle at the vertex \( b \) in the spherical triangle \( a_0 a_i a_{i+1} \). It follows, by the inequality \( \sum a_i \geq \pi \) and by (9), that for some index \( i_r \leq m \), there exists a point \( a_r^{'} \) lying between two successive points \( a_{i_r}, a_{i_r+1} \) such that
\[
(10) \quad b \text{ lies between } a_i^{'} \text{ and } a_i , \text{ but does not lie between } a_i^{'} \text{ and } a_i .
\]
We infer by (10) that
\[
\sum_{i=0}^{m} \varrho_{gb}(a_i^{'}, a_{i+1}) + \varrho_{gb}(a_i^{'}, a_i) > \varrho_{gb}(a_i, b) + \varrho_{gb}(b, a_i^{'}). 
\]
Moreover, we have
\[
\sum_{i=0}^{m} \varrho_{gb}(a_i^{'}, a_{i+1}) + \varrho_{gb}(a_i, a_{i+1}) + \varrho_{gb}(b, a_i^{'}) > \varrho_{gb}(b, a_{m+1}) .
\]
for every two points \(x, y \in \Delta\). Evidently \(\varrho_\Delta\) constitutes a distance-function for \(\Delta\) and it does not depend on the choice of the vertices \(a_0, a_1, \ldots, a_n\) of the regular spherical simplex \(Q^n\). Moreover, let us remark that if \(\Delta_1, \Delta_2\) are two simplexes of \(\Delta\) and the points \(x, y \in P\) belong to the common part of them, then

\[ \varrho_{\Delta}(x, y) = \varrho_{\Delta}(x, y) \]

Using the metric \(\varrho_{\Delta}\), we now introduce a metric \(\varrho_T\) for the whole polytope \(P\). Suppose first that \(P\) is connected. Then for every two points \(x, y \in P\) there exists a finite sequence of points

\[ x = x_0, x_1, x_2, \ldots, x_{n+1} = y \]

such that every two successive points \(x_i, x_{i+1}\) belong to one simplex \(\Delta_i \in T\). A sequence \(x_0, x_1, \ldots, x_{n+1}\) of this sort will be called a passage in the triangulation \(T\) from \(x\) to \(y\). The number

\[ |x_0 x_1 \ldots x_{n+1}| = \sum_{i=0}^{n} \varrho_{\Delta}(x_i, x_{i+1}) \]

will be called the length of the passage \(x_0, x_1, \ldots, x_{n+1}\).

Let us denote by \(\varrho_T(x, y)\) the lower bound of the lengths of all passages in \(T\) from \(x\) to \(y\). Evidently \(\varrho_T\) is a distance-function for the connected polytope \(P\).

Let us show that \(P\) metrized by \(\varrho_T\) is a convex space. It suffices to show, that for each pair of points \(x, y \in P\) there exists a centre. In order to do it, let us observe that if \(x_0, x_1, \ldots, x_{n+1}\) is a passage in \(T\) from \(x\) to \(y\), then there exists on one of the metric segments \(\Delta(x_0, x_{n+1}) \subset \Delta\) a point \(z\) (called the centre of the passage \(x_0, x_1, \ldots, x_{n+1}\)) such that

\[ |x_0 x_1 \ldots x_{n+1}| = |x_0 x_{n+1}| = \frac{1}{\lambda} \cdot |x_0 z \ldots x_{n+1}|. \]

By the definition of the distance-function \(\varrho_T\), for every \(v = 1, 2, \ldots\) there exists a passage \(x_0 x_{n+1} \ldots x_{n+v}\) in \(T\) from \(x\) to \(y\) satisfying the inequality

\[ \varrho_T(x, y) = |x_0 x_{n+1} \ldots x_{n+v}| < \varrho_T(x, y) + 1/v. \]

Let \(z\) denote the centre of this passage. One easily sees that

\[ \frac{1}{2} \cdot \varrho_T(x, y) < \varrho_T(x, z) \leq \frac{1}{2} \cdot \varrho_T(x, y) + 1/v. \]

Since \(P\) is compact, the sequence \((z)\) contains a subsequence convergent to a point \(z \in P\). We infer by (16) that

\[ \varrho_T(x, z) = \varrho_T(z, y) = \frac{1}{2} \cdot \varrho_T(x, y), \]

whence \(z\) is a centre of the pair \(x, y\).

Now, suppose that the polytope \(P\) is not connected and let \(P_1, P_2, \ldots, P_k\) be the components of \(P\). The simplexes of the triangulation \(T\) of \(P\) lying on \(P_i\) constitute a triangulation \(T_i\) of \(P_i\). We introduce in \(P_i\) the metric \(\varrho_{T_i}\) (using always the same regular spherical simplex \(Q^n\), where \(n = \dim P_i\)). By this metric every component \(P_i\) has a finite diameter \(d_i\). Let \(d\) denote the greatest of the numbers \(d_1, d_2, \ldots, d_k\). Setting

\[ \varrho_P(x, y) = \varrho_{T_i}(x, y), \quad \text{if} \quad x, y \in P_i, \quad \text{for} \quad i = 1, 2, \ldots, k \]

and

\[ \varrho_P(x, y) = d + 1 \quad \text{if} \quad x, y \text{ belong to distinct components of } P, \]

we obtain a distance-function \(\varrho_P\) for the whole polytope \(P\). We call it the spherical metric corresponding to the triangulation \(T\).

7. Main theorem. The purpose of this note is to prove the following theorem.

\textbf{Theorem.} Every polytope of dimension \(\leq 2\), metrized by the spherical metric corresponding to one of its triangulations, is locally strongly convex.

The problem whether the analogous statement holds also for polytopes of dimension \(> 2\) remains open (comp. [2], p. 106 problem 6).

Let us observe that the proof of the theorem may be reduced to the case in which the polytope has dimension 2 at each of its points. Let \(T\) be a triangulation of a polytope \(P\) of dimension \(\leq 2\) and let \(P_1\) denote the sum of all 2-dimensional simplexes (triangles) of \(T\), and \(P_1\) the closure of the set \(P - P_1\). Then

\[ P = P_1 \cup P_1, \]

where \(P_1\) is a polytope of dimension \(\leq 1\), and \(P_2\) a polytope which has dimension 2 at each of its points. Evidently the triangulation \(T\) contains a triangulation \(T_1\) of the polytope \(P_1\) and a triangulation \(T_2\) of the polytope \(P_2\).

Let us assume that \(P_1\) metrized by the distance-function \(\varrho_{T_1}\) is locally strongly convex. Moreover we easily see that for every point \(x_0 \in P_2\) there exists in \(P_2\) a neighbourhood \(U_{\epsilon_0}\) such that for each pair of points \(x, y \in U_{\epsilon_0}\) we have \(\varrho_{T_2}(x, y) = \varrho_{T_1}(x, y)\). Now it is evident that for each point \(x \in P_2 - (P_1 \cap P_2)\) there exists a compact, strongly convex neighbourhood with an arbitrarily small diameter. If however \(x \in P_1 \cap P_2\) then, for every \(\epsilon > 0\), there exists a compact strongly convex neighbourhood \(V_{\epsilon}\) of \(x\) in \(P_1\) with diameter \(\leq \frac{1}{2} \epsilon\). It is clear that \(V_{\epsilon}\) consists of a finite number of metric segments, with diameters \(\leq \frac{1}{2} \epsilon\), having \(x\) as the common endpoint. We can replace each of these segments by a subsegment containing \(x\) (as one of its endpoints) and having a diameter so small that the common part of it with \(P_1\) contains only the point \(x\). One easily sees
that the sum of all segments thus obtained and of the set $V^*_k$ is a compact strongly convex neighbourhood of $x$ in $P$ and the diameter of this neighbourhood is $< \alpha$.

Thus we infer that in the proof of the main theorem we can restrict ourselves to the case when the polytope $P$ has dimension 2 at every point. Hence in the sequel, we shall always assume that the polytope $P$ is 2-dimensional at every point.

8. Spherical metric $\varrho$ on individual triangles of $T$. We shall prove the following

**Lemma 1.** If the points $x, y$ belong to one triangle $A$ of a triangulation $T$ of $P$ and $\varrho_{A}(x, y) < \frac{1}{2} \pi$ then $\varrho_{A}(x, y) = \varrho_{A}(x, y)$.

**Proof.** Let us denote by $\alpha$ the length of a side of a quarter $Q' \setminus S$. Then

$$\varrho_{A}(x, y) < \frac{1}{2} \pi < \alpha \quad \text{and} \quad \varrho_{A}(x, y) = \varrho_{A}(x, y).$$

By the definition of the metric $\varrho_{A}$, it remains to prove that for every passage $x = x_1, x_2, ..., x_n, x_{n+1} = y$ in $T$ from $x$ to $y$, satisfying the inequality

$$(17) \quad |x_{i-1}; x_i; x_{i+1}| < \alpha,$$

we have

$$(18) \quad |x_{i-1}; x_i; x_{i+1}| \geq \varrho_{A}(x, y).$$

First let us consider the case when both points $x, y$ lie on the boundary of the triangle $A$. If three successive points $x_1, x_2, x_3$ belong in one simplex $A \in T$, then

$$\varrho_{A}(x_1; x_2) + \varrho_{A}(x_2; x_3) \geq \varrho_{A}(x_1, x_3)$$

and, by cancelling the point $x_3$, we obtain from $x_1, x_2, ..., x_{n+1}$ another passage in $T$ from $x$ to $y$ with the length $|x_{i-1}; x_i; x_{i+1}|$. It easily follows that it suffices to prove (18) in the case in which exactly three successive points $x_1, x_2, x_3$ belong to one simplex of $T$. By (17) we have $\varrho_{A}(x_1, x_3) < \alpha$, from two successive points $x_i$ and $x_{i+1}$ cannot coincide with two different vertices of $A$. It follows that for every $i = 0, 1, ..., m$ there exists in $T$ a triangle $A_i$ such that $x_i$ and $x_{i+1}$ belong to the boundary of $A_i$, but do not belong to the same side of $A_i$. It follows that if one of the points $x_i, x_{i+1}$ is a vertex of $A_i$, then the other belongs to the opposite side of $A_i$ and consequently $\varrho_{A}(x_i; x_{i+1}) > \alpha$ contrary to inequality (17). Consequently we can assume that none of the points $x_0, x_1, ..., x_{m+1}$ is a vertex of the triangulation $T$.

Without loss of generality we may assume that $A$ coincides with a quarter of $S$ with vertices $abc$ and that $x \in S(ab)$ and $y \in S(ab) \cup S(ac)$.

Then $a$ and $b$ are common vertices of triangles $A$ and $A_i$. Since every triangle $A_i$ is isometric with $A$, there exists for every $i = 0, 1, ..., m$ an isometric transformation $q_i$ of $A_i$ onto $A$ such that

$$(19) \quad q_i(a) = a, \quad q_i(b) = b,$$

$$(20) \quad q_i(x) = q_i(x_i) \quad \text{for every} \quad x \in A_i \setminus A_{i+1}.$$

Then the points $q_i(x) = q_i(x_i), q_i(x_2) = q_i(x_2), ..., q_{m-i}(x_m) = q_{m-i}(x_{m-i}), q_{m-i}(x_{m-i+1}) = q_{m-i}(x_{m-i+1})$.

Moreover, all these points are distinct from $a, b, c$, because, by our assumption, none of the points $a, b, c$ is a vertex of $T$. Finally, since $x_i$ and $x_{i+1}$ do not belong to the same side of $A_i$, the points $q_i(x_i) = x_i(x), q_i(x_{i+1}) = x_i(x_{i+1})$ (where $i = 1, 2, ..., m$) do not belong to the same side of $A$. Now we distinguish two cases:

Case 1: None of the points $q_i(x_i), q_i(x_2), ..., q_i(x_{m-i})$ belongs to the side $S(ab)$.

First let us observe that all triangles $A_1, A_2, ..., A_m$ have the vertex $a$ in common and that $q_i(a) = a$ for every $i = 0, 1, ..., m$. By (19) it is so for $i = 0$. Suppose that for an $i < m$ it is $a \in A_i$, and $q_i(a) = a$. Then the side of $A_i$ opposite to the vertex $a$, is mapped by $q_i$ onto $S(ab)$. It follows that the point $x_{i+1}$ is an inner point of a side $L$ of $A_i$ containing the vertex $a$. Then $L$ is the common side of $A_1$ and $A_{i+1}$ and, by (20), we infer that

$$q_{i+1}(a) = q_i(a).$$

Consequently

$$\varrho_{A}(a, y) = \varrho_{A}(q_i(a), q_i(y)) = \varrho_{A}(a, q_i(y)).$$

We infer that each of the points $y$, $q_i(y)$ coincides with one of two points $b', c'$ lying respectively on $S(ab)$ and $S(ac)$ at the distance $\varrho_{A}(a, y)$ from $a$. But we easily see that the sum of the lengths of all spherical segments $S(q_i(a); q_i(x_{i+1})$ constituting a connected graph joining on $A$ the point $x_i(x)$ with the point $q_i(y)$ is not smaller than $\varrho_{A}(a, b')$ and also than $\varrho_{A}(a, c')$. Hence

$$(19) \quad |x_{i-1}; x_i; x_{i+1}| = \sum_{i=0}^{m-i} \varrho_{A}(x_i; x_{i+1}) \leq \varrho_{A}(a, y).$$

Case 2: Each side $S(ab), S(ac)$ contains at least one of the points $q_i(x_i), q_i(x_2), ..., q_i(x_{m-i})$. 
In this case there exist three successive indices \( i, i + 1, \ldots, i + 2 \) such that \( \varphi_k(\alpha_{i}), \varphi_k(\alpha_{i+1}), \varphi_k(\alpha_{i+2}) \) lie on different sides, say on \( S_k(\alpha) \), \( S_k(\alpha_{i+k}) \) and \( S_k(\alpha_{i+2}) \) respectively. Then we infer by IV of No. 5 that
\[
|\alpha_{i} \alpha_{i+1} \ldots \alpha_{i+2}| = \sum_{i=0}^{m} g_{\alpha}(\varphi_{i}(\alpha_{i}), \varphi_{i}(\alpha_{i+1})) \geq g_{\alpha}(\varphi_{i}(\alpha_{i}), \varphi_{i}(\alpha_{i+1})), \quad \varphi_{i+1}(\alpha_{i+1}), \varphi_{i+1}(\alpha_{i+2}) \geq \pi,
\]
contrary to inequality (17).

Passing to the case when at least one of the points \( x, y \) does not lie on the boundary of the triangle \( \Delta \), consider a passage \( x = x_{0}, x_{1}, \ldots, x_{m}, x_{m+1} = y \) in \( T \) from \( x \) to \( y \). If all points \( x_{i} \) belong to \( \Delta \), then (18) is evident. If, however, not all points \( x_{i} \) belong to \( \Delta \), then there exist two indices \( i \) and \( j \) such that \( 0 \leq i < j \leq m+1 \) and that \( x_{i}, x_{j} \) lie on the boundary of \( \Delta \). We may assume that \( x_{i} \) is the first and \( x_{j} \) the last point lying on the boundary of \( \Delta \). Then, for \( \nu < i \) and \( \nu > j \), we have \( x_{\nu} \in \Delta \). Applying the already settled case, we infer that
\[
|\alpha_{i} \alpha_{i+1} \ldots \alpha_{j}| = |\alpha_{i} \alpha_{i+1}| + |\alpha_{j} \alpha_{j+1}| + |\alpha_{i} \alpha_{j+1}| \geq g_{\alpha}(\varphi_{i}(\alpha_{i}), \varphi_{i}(\alpha_{i+1})) + g_{\alpha}(\varphi_{j}(\alpha_{j}), \varphi_{j}(\alpha_{j+1})), \quad |\alpha_{i} \alpha_{j}| \geq g_{\alpha}(\varphi_{i}(\alpha_{i}), \varphi_{i}(\alpha_{j})).
\]

Thus (18) holds also in this case.

9. Passages with shortest lengths. We shall prove the following

**Lemma 2.** If \( x, y \in P \) and \( g_{\alpha}(x, y) \leq \frac{1}{\pi} \) then there exists a passage \( x = x_{0}, x_{1}, \ldots, x_{m}, x_{m+1} = y \) in \( T \) from \( x \) to \( y \) such that
\[
|\alpha_{0} \alpha_{1} \ldots \alpha_{m+1}| = g_{\alpha}(x, y).
\]

Proof. By \( g_{\alpha}(x, y) \leq \frac{1}{\pi} \) the points \( x, y \) belong to one component \( P \) of \( P \). As it has already been shown, \( P \) metrized by \( g_{\alpha} \) is convex. Hence there exists in \( P \) a metric segment \( L(xy) \) with endpoints \( x, y \). Consider the points of \( L(xy) \) in their natural order from \( x \) to \( y \). Setting \( x_{0} = x, \) let us denote by \( x_{i} \) the last point of \( L(xy) \) such that there exists in \( T \) a triangle \( \Delta_{i} \) containing both points \( x_{i} \) and \( x_{i+1} \). Assuming that the points \( x_{0}, x_{1}, \ldots, x_{i} \in L(xy) \) are already determined and \( x_{i} \neq y \), we denote by \( x_{i+1} \), the last point of \( L(xy) \) such that there exists in \( T \) a triangle \( \Delta_{i+1} \) containing both points \( x_{i} \) and \( x_{i+1} \). It follows that the point \( x_{i+1} \not\in \Delta_{i} \) does not belong to any triangle \( \Delta_{i+j} \) with \( 0 \leq j < i \). Hence the triangles \( \Delta_{i} \in T \) are distinct. Consequently in this manner we obtain only a finite number of points \( x = x_{0}, x_{1}, \ldots, x_{m}, x_{m+1} = y \) constituting a passage in \( T \) from \( x \) to \( y \). Evidently for \( i < j \) the point \( x_{i} \) precedes the point \( x_{i,j} \) on the segment \( L(xy) \). Applying the lemma 1 we infer that
\[
g_{\alpha}(x, y) = \sum_{i=0}^{m} g_{\alpha}(\varphi_{i}(\alpha_{i}), \varphi_{i}(\alpha_{i+1})) = \sum_{i=0}^{m} g_{\alpha}(\varphi_{i}(\alpha_{i}), \varphi_{i}(\alpha_{i+1})) = |\alpha_{0} \alpha_{1} \ldots \alpha_{m+1}|.
\]

10. Sets \( V_{\alpha} \). For every point \( e \in P \), let us denote by \( A \) the sum of all simplexes of \( P \) containing \( c \), and by \( B \) the sum of all other simplexes of \( P \). Evidently (21)
\[
0 < g_{\alpha}(c, B) < \pi.
\]

Let \( \varepsilon \) be an arbitrarily given positive number. We denote by \( V_{\alpha}^{\varepsilon} \) the subset of \( P \) consisting of all points \( x \) satisfying the inequality (22)
\[
g_{\alpha}(x, e) < \varepsilon \min_{e}(e, g_{\alpha}(e, B)).
\]

Evidently \( V_{\alpha}^{\varepsilon} \) is a compact neighbourhood of \( c \) (in \( P \)) with diameter \( \leq \frac{1}{\pi} \min_{e}(e, g_{\alpha}(e, B)) \). To prove the theorem (formulated in No. 7) it suffices to show that \( V_{\alpha}^{\varepsilon} \) is strongly convex.

We begin with two following lemmas:

**Lemma 3.** The set \( V_{\alpha}^{\varepsilon} \) is convex.

Proof. Let \( x, y \in V_{\alpha}^{\varepsilon} \). By lemma 2, there exists a passage \( x = x_{0}, x_{1}, \ldots, x_{m}, x_{m+1} = y \) in \( T \) from \( x \) to \( y \) with the length \( g_{\alpha}(x, y) \). It follows by (22) that
\[
g_{\alpha}(x, e) \leq g_{\alpha}(z, x) + g_{\alpha}(z, y) \leq \frac{1}{\pi} g_{\alpha}(c, B). \]

Consequently the simplex \( \Delta_{i} \in \Delta \), containing \( x_{i} \) and \( x_{i+1} \), has \( c \) as one of its vertices. Let \( x_{i} \) denote the point lying on the segment \( \Delta_{i}(x_{i+1}) \) at the distance \( \min_{e}(e, x_{i}, c) \) from \( x_{i}, \Delta_{i}(x_{i+1}), \) \( \min_{e}(e, x_{i}, c) \) from \( c \). Then \( x_{0} = x, x_{m+1} = y, \quad e \in V_{\alpha}^{\varepsilon} \) for \( i = 0, 1, \ldots, m+1 \) and the points \( x_{0}, x_{1}, \ldots, x_{m+1} \) constitute a passage in \( T \) from \( x \) to \( y \). Applying I of No. 5, we conclude that the length of this passage is \( \leq \sum_{i=0}^{m} g_{\alpha}(x_{i}, x_{i+1}) = g_{\alpha}(x, y) \). But \( x_{0} = x \) and \( x_{m+1} = y \) imply that the length of this passage is \( \geq g_{\alpha}(x, y) \). Hence \( |\alpha_{0} \alpha_{1} \ldots \alpha_{m+1}| = g_{\alpha}(x, y) \). It suffices to set
\[
L = \sum_{i=0}^{m} g_{\alpha}(x_{i}, x_{i+1})
\]
in order to obtain a metric segment joining \( V_{\alpha}^{\varepsilon} \) the points \( x \) and \( y \).

**Lemma 4.** If \( x, y \in V_{\alpha}^{\varepsilon} \) belong to one triangle \( \Delta \in \Delta \) and \( g_{\alpha}(x, y) \leq \frac{1}{\pi} \) then \( e \in V_{\alpha}^{\varepsilon} \) lying between \( x \) and \( y \) coincides with the segment \( \Delta(xy) \).

Proof. By lemma 1, it suffices to show that the assumption that a point \( e \in P \) lies between \( x \) and \( y \) leads to a contradiction. Applying
the lemma 2, we easily see that there exists a metric segment $L \subset P$ with endpoints $x$ and $y$ such that $x \in A$. Now let us distinguish 3 cases:

1° Both points $x, y$ lie on one side $A = A(ab)$ of $A$. Since $\varrho(x, y) \leq 1$, at least one of the endpoints of $A$ belongs to $P \setminus L$. It easily follows that $L$ contains points which do not lie on $A$, but belong to one of the triangles of $T$ adjacent to $A$. If $x'$ is one of such points lying on a triangle $A'$ adjacent to $A$, then $x'$ lies between $x, y$ and $x'$ does not belong to $A(ab) = A'(ab)$, which is impossible on account of lemma 1.

2° $x, y$ lie on disjoint sides of $A$.

Let $a, b, c$ be the vertices of $A$ and $x \in A(ab), y \in A(bc)$. Let $M$ denote the set composed of $a, b, c$ and of all points lying in the interior of any triangle of $T$ adjacent to $A$ along one of its sides. Evidently $M$ cuts $P$ between each point of $A$ and each point of $P \setminus (M \cup A)$. Since the length of $L$ is $\leq \frac{1}{2}$, the side $A(ab)$ is disjoint with $L$. It easily follows that there exists a point $x' \in L$ lying in the interior $O$ of some triangle $A'$ adjacent to one of the sides $A(ab), A(bc)$, say to $A(ab)$. It is easy to observe that there exists a point $a$ belonging to the common part of $A$ and of the side $A'(ab)$ of $A'$ different from $A(ab)$. Applying lemma 2, we obtain a passage $x = a_1, a_2, \ldots, a_n, a_{n+1} = y$ in $T$ with the length $\varrho(x, y)$. Moreover we may assume that every two successive points $a_i, a_{i+1}$ lie on the boundary of a triangle $A_i$ having as one of its vertices. Let us now denote by $y'$ the point lying on $A(ab)$ at the distance $\varrho(x, c)$ from $x$. By the same argument as that used in the proof of lemma 1, case 2°, we obtain the isometries $\varphi_1, \varphi_2, \ldots, \varphi_m$ of triangles $A_1 = A', A_1, A_2, \ldots, A_m$ onto the triangle $A$ (which we may identify with a quarter $Q$ of the sphere $S^3$) in such a manner that $\varphi_1(x) = a, \varphi_1(c) = b$ and that the successive points of the sequence $\varphi_1(x) = x, \varphi_1(a), \varphi_1(c), \ldots, \varphi_1(a_{m+1}) = \varphi_1(y)$ lie alternatively on sides $A(ab)$ and $A(bc)$ and point $\varphi_1(y)$ coincides with $x$ or $y'$. Since $a_2 \neq y$, we easily infer that the sum of lengths of segments of $A_1(a_1, a_{i+1})$ is $> \varrho(x, y)$. But this sum is equal to $\varrho(a_1, a_n) = \varrho(x, c)$. Thus we obtain a contradiction.

3° At least one of the points $x, y$ does not lie on the boundary of $A$.

In this case, let $x'$ denote the first point of the segment $L$ ordered naturally from $x$ to $y$, and by $y'$ the last point of $L$ lying on the boundary of $A$. Then $x'$ lies between $x$ and $y'$, and thus the proof of impossibility is reduced to the already settled cases 1° or 2°.

11. Decomposition of metric segments into spherical segments. Angles. Let us prove the following

**Lemma 5.** If $L$ is a metric segment on $V^*_e$, then the common part of $L$ with any triangle $A$ of $T$ either is empty or contains only one point, or is a segment.

**Proof.** Let $x, y$ denote the endpoints of $L$ and suppose that $L \cap A$ contains at least two points. Let $a_1$ be the first, and $a_2$ the last point of the subset $L \cap A$ of $L$ (where $L$ is naturally ordered from $x$ to $y$). We infer by lemma 4 that the set of points of $L$ lying between $a_1$ and $a_2$ coincides with the segment $A(a_1, a_2)$. Hence $L \setminus A = A(a_1, a_2)$.

The segments which are common parts of a metric segment $L \subset V^*_e$ with single triangles of $T$ constitute a decomposition of $L$ into segments with disjoint interiors:

\[
L = L_0 \cup L_1 \cup \cdots \cup L_n.
\]

This decomposition is uniquely determined by the triangulation $T$. We call it the decomposition of $L$ corresponding to the triangulation $T$. Evidently the indices $i$ may be so fixed that $L_i = L \cap A_i = A_i(a_{i+1}, a_i)$ for $i = 0, 1, \ldots, m$. Now let us assume that $L \subset V^*_e \cap (e)$. Then the segments:

\[
A_i(a_{i+1}, a_i), \quad A_j(a_{j+1}, a_j), \quad A_k(a_{k+1}, a_k)
\]

constitute a triangle $A_{i+1}A_iA_j$ isometric with a spherical triangle lying on $S^3$. Let $a_i$ denote the size of its angle at the point $c$. The number

\[
\alpha(L) = \sum_{i=0}^n a_i
\]

will be called the amplitude of segment $L \subset V^*_e$ relatively to $c$.

**Lemma 6.** For every metric segment $L \subset V^*_e \cap (e)$ it is $0 \leq \alpha(L) < \pi$.

**Proof.** Applying the notation just introduced, consider an isometrical transformation of the triangle $A_{i+1}A_iA_j$ onto $S^3$. It is easy to observe that these isometries $\varphi_i$ may be chosen successively in such a manner that $\varphi_1$ and $\varphi_m$ coincide on the common part of triangles $A_{i+1}A_iA(j)$ and $A_{i+1}A_iA(j)$, for every $i = 0, 1, \ldots, m - 1$. Consequently the points $a_i = \varphi(i)$ and the point $b = \varphi(e)$ lie on $S^3$ and satisfy the premises of $H$. No. 5.

It follows that $\sum_{i=0}^m a_i < \pi$, because otherwise we should have

\[
\sum_{i=0}^m \varrho(a_i, a_{i+1}) = \sum_{i=0}^m \varrho(a_i, a_{i+1}) > \varrho(b, a_{m+1}) = \varrho(x, c) + \varrho(y, d) > \varrho(x, y),
\]

which is impossible, because the length of $L$ is equal to $\varrho(x, y)$.
Case III. \( c \) is a vertex of the triangulation \( T \).

We have to prove that for every two points \( x, y \in V_\varepsilon \) the set of points \( z \in V_\varepsilon \) lying between \( x \) and \( y \) is a metric segment. We distinguish two following possibilities:

1°. \( c \) lies between \( x \) and \( y \).

Since the case when \( x \) and \( y \) belong to one of the triangles of \( T \) is already settled by lemma 4, we may assume that the triangles \( A_x, A_y \) of \( T \), containing \( x \) and \( y \) respectively, are distinct and none of the points \( x, y \) belong to \( A_x \cap A_y \). In particular \( x \neq c \neq y \). Evidently the set

\[ L_c = A_x(x) \cup A_y(y) \]

is a metric segment joining \( x \) and \( y \) in \( V_\varepsilon \). Suppose that there exists a point \( z \in V_\varepsilon - L_c \) lying between \( x \) and \( y \). Applying lemma 3, we may find a segment \( L \subseteq V_\varepsilon \) with endpoints \( x \) and \( y \), which passes through \( z \). Then \( c \) does not lie on \( L \), because otherwise \( z \) would lie between \( x \) and \( c \) or between \( y \) and \( c \), and consequently it would belong to \( L_c \).

Consider now the decomposition of the segment \( L \),

\[ L = A_x(x, \varepsilon) \cup A_y(y, \varepsilon) \cup \cdots \cup A_m(\varepsilon, \varepsilon) \]

into spherical segments with disjoint interiors, defined as in No. 11. By lemma 6, the sum of angles \( a_t \) at \( e \) in the triangles \( \sigma(\varepsilon, a_t) \) is \( \pi \). It easily follows that there exist isometries \( \varphi_1, \varphi_2, \ldots, \varphi_m \) mapping triangles \( \sigma(\varepsilon, a_t) \) onto some triangles with disjoint interiors on \( S^2 \) and such that \( \varphi_t \) coincides with \( \varphi_{t+1} \) on the segment \( A_t(\varepsilon, \varepsilon) \). In particular all points \( \varphi_t(e) \) coincide with a fixed point \( b \in S^2 \). Since \( \sum a_t < \pi \), we infer that \( \varphi_t \) constitute together a homeomorphic mapping \( \varphi \) of the sum \( M \) of all triangles \( \sigma(\varepsilon, a_t) \) into the sum \( N \) of triangles \( \varphi_t(\sigma(\varepsilon, a_t)) \) lying on \( S^2 \). It follows by III of No. 5 that there exists in \( X \) a simple arc \( L' \) with length smaller than

\[ \varphi_1(x) + \varphi_b(b, \varepsilon(x)) \] joining \( x \) and \( y \). It is easy to observe that the inverse homeomorphism \( \varphi^{-1} \) maps \( L' \) onto an arc \( \varphi^{-1}(L') \subseteq V_\varepsilon \) having the same length as the arc \( L' \). But this is impossible, because

\[ \varphi(x, \varepsilon) + \varphi(y, \varepsilon) = \varphi(x, y) \]

2°. \( c \) does not lie between \( x \) and \( y \).

By lemma 3, there exists a metric segment \( L \subseteq V_\varepsilon \) joining \( x \) and \( y \). By our hypotheses, \( c \) does not belong to \( L \). We have to show that every point \( z \) lying between \( x \) and \( y \) belongs to \( L \).

Since the case in which \( x \) and \( y \) belong to one simplex of \( T \) is already settled, we may assume that the simplices \( A_x \) and \( A_y \) are distinct and none of points \( x, y \) belongs to their common part. Applying lemma 5 we infer that there exists in \( T \) a system of triangles \( A_x = A_1, A_2, \ldots, A_m = A_y \) with common vertex \( c \) and with disjoint interiors such that \( A_1 \) and \( A_m \)
have a common side $A_{i+1}$ which has with $L$ exactly one point $s_{i+1}$ in common and that

$$L = A_1(s_{x_1}) \cup A_2(s_{y_2}) \cup \ldots \cup A_{m-1}(s_{x_{m-1}}) \cup A_m(s_{x_m}y).$$

Let $a_i$ denote the size of the angle at the vertex $c$ in the spherical triangle $s_{x_{i+1}}$. Evidently $a_i = \frac{i}{m}$ for $0 < i < m$. It follows by lemma 6

$$(24) \quad a_0 + (m-1)\frac{m}{2} + a_m < \pi,$$

whence $m = 1$ or $m = 2$. We consider the two cases separately:

1) $m = 1$. Then the triangles $A_1 = A_2$ and $A_1 = A_2$ have a common side $A_1$ and inequality $(24)$ has the form

$$(25) \quad a_0 + a_1 < \pi.$$

Suppose that there exists a point $z \in V_1 \cap L$ lying between $x$ and $y$. Applying lemma 3, we can construct a metric segment $L'$ joining $x$ and $y$ in $V_1$ and passing through $z$. Evidently there exists a homeomorphism $\varphi$ mapping $A_2 \cup A_3$ into the sum of two adjacent quarters on $S^\pi$, which is an isometry on $A_2$ and also on $A_3$. Since

$$\varphi(x)(x), \varphi(y) \leq \varphi(x)(x), \varphi(y) + \alpha_{xy}(\varphi(x), \varphi(y)) = \varphi(y)(x, y) + \alpha_{xy}(y, x) \leq \frac{1}{2} \pi,$$

where in $\varphi(A_2 \cup A_3)$ there exists only one segment joining $\varphi(x)$ and $\varphi(y)$. It follows that $\varphi(L')$ is not contained in $\varphi(A_2 \cup A_3)$, whence there exists a point $z' \in L' \cap (A_1 \cup A_2)$.

By an argument already used in this proof, we infer that there exists a triangle $A'$ of $L'$ adjacent to $A_2$ and $A_3$, such that

$$L' = A_2(s_{x_2}) \cup A_3(s_{y_3}) \cup A_1(s_{x_1}),$$

where $s_{x_2}$ and $s_{y_3}$ lie on the sides of $A'$ common with $A_2$ and $A_3$ respectively. It follows that

$$a(L') = a_0 + a_1 < \pi,$$

whence $a_0 = \frac{1}{2} \pi - a_0$ and $a_1 = \frac{3}{2} \pi - a_1$. We infer $a_0 + a_1 > \pi$, which contradicts $(25)$.

2) $m = 2$. In this case inequality $(24)$ has the form

$$(26) \quad a_0 + a_1 < \frac{3}{2} \pi.$$

It follows that both points $x, y$ belong to the set $D$ which is the sum of the triangle $A$ and of the interiors of triangles $A_2$ and $A_3$. Evidently there exists a homeomorphism mapping $D$ onto a subset of the sphere $S^\pi$. By the same argument we see that in the set $\varphi(L')$ there exists only one metric segment joining the points $\varphi(x)$ and $\varphi(y)$. Consequently every metric segment $L' \cap V_1$ different from $L$ and joining $x$ and $y$ contains at least one point which does not belong to $D$.

Suppose now that there exists a point $z \in V_1 \cap L$ lying between $x$ and $y$. It follows that there exists a segment $L' \cap V_1$ joining $x$ with $y$ and passing through $z$. By an argument already used in this proof, we infer that either

(a) $L' \subset A_2 \cup A_3$ or there exists a triangle $A'$, adjacent to $A_2$ and also to $A_3$, such that

(b) $L' \subset A_2 \cup A' \subset A_3$.

Let us investigate both cases, (a) and (b). Since $L'$ is not contained in $D$, we easily infer that in case (a) the segment $L'$ contains a point $z'$ lying on the side of $A_2$ or of $A_3$, not contained in $A_1$, and consequently the metric segment $L'$ decomposes into two metric segments $L_1$ and $L_2$ with common endpoint $z'$. We easily infer by $(26)$ that

$$a(L') = a(L_1) + a(L_2) > \pi,$$

which is impossible in view of lemma 6.

In case (b) the triangle $A'$ contains a side $A''$, which lies on $A_2 \cup A_3$, but does not lie on $A_1$. Without loss of generality, we may assume that $A'' \neq A_2$. Let $z'$ be a point of $A'' \cap L'$. We easily see that then $L' = L'(x_2') + L'(y_2')$ and we have, by $(26)$,

$$a(L') = a[L'(x_2')] + a[L'(y_2')] > \frac{1}{2} \pi + \frac{1}{2} \pi = \pi,$$

which is impossible in view of lemma 6.

Thus the proof of the main theorem is complete.

Remark. Since the set $V_1$ coincides with a (closed) ball (in the space $P$ metrized by $\varphi_0$) with centre $e$ and radius $\frac{1}{2} \min\{1, \varphi_0(e, B_0)\}$, we infer that the polytope $P$ metrized by $\varphi_0$ satisfies the following condition:

For every $c \in P$ there exists a positive number $\epsilon$ such that each closed ball in $P$ with centre $e$ and radius $\leq \epsilon$ is strongly convex.

References


