Fixed point theorems for connectivity maps

by

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Introduction. Some problems of topology may be solved only through the study of non-continuous functions. Such is, perhaps, the question: Does every continuous function of an acyclic plane continuum into itself have a fixed point? But the investigation of non-continuous functions for their own sake also has interest. Connectivity maps are a species of such functions; their importance till now lies in Hamilton's Theorem: *A connectivity map of an n-cell into itself has a fixed point* [1]. His proof makes use of another kind of non-continuous function, the peripherally continuous function; but I believe that his proof contains a gap (see section 3 of this paper).

In order to extend Hamilton's Theorem and method of proof, it is necessary for me to introduce still other sorts of non-continuous functions, the almost continuous and the polyhedrally almost continuous functions; this assortment of functions is worthy of study because of the scope of the theorems of this paper. A start is made here to the study of the abstract properties of these functions, to the understanding of their differences and similarities. There are many easily asked questions about these functions, which appear to be very difficult to answer; some of these are collected in section 6. One of the most provocative is a problem about the topology of the unit interval.

1. Connectivity maps. The *graph* of a function $f\colon X\to Y$ is the subset of $X\times Y$ consisting of the points $(x, f(x))$; this set will be symbolized $\Gamma(f)$. For $C\subset X$, the function $f|C\colon C\to Y$ is defined to be the restriction of f to C. Hence $\Gamma(f|C)\subset\Gamma(f)$.

DEFINITION 1. If X and Y are topological spaces, $f\colon X\to Y$ a function, then that f is a *connectivity map* means that for any connected $C\subset X$, $\Gamma(f|C)$ is connected.

DEFINITION 2. If X and Y are topological spaces, $f\colon X\to Y$ a function, then that f is a *local connectivity map* means that there is a covering of X by open sets $\{U_a\}$ such that $f|U_a$ is a connectivity map.

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Obviously, every connectivity map is a local connectivity map; I do not know under what conditions the converse is true. It is also clear that every continuous map is a connectivity map, and that the restriction of a connectivity map $X \to Y$ to a subset $Q \subset X$ is a connectivity map.

We recall for future reference Hamilton's Theorems 1 and 2.

HAMILTON'S THEOREM 1. *If f is a connectivity map of a T_1 space A into a T_1 space B; $p \in A$, V, U open in A, B, resp., containing p, $f(p)$, resp., then every nondegenerate connected subset of A containing p contains a point $q \in V$, such that $q \neq p$, $f(q) \in U$.*

HAMILTON'S THEOREM 2. *If f is a connectivity map of a T_1 space A into a T_1 space B, and if $C \subset B$ is a closed subset of B, then each component of $f^{-1}(C)$ is closed.*

(A *T_1 space* is a space in which each subset of finite cardinality is a closed set. Hamilton stated these propositions only about Hausdorff spaces; but his proof applies equally well to T_1 spaces.)

2. Polyhedral almost continuity.

DEFINITION 3. By a *polyhedron* P, I mean a finite simplicial complex K together with a geometric realization $|K|$ of it. A *subpolyhedron* Q is then a subcomplex L of a subdivision of K, and the geometric realization $|L|$ which is in a canonical way identified with a subset of $|K|$. A *star-neighborhood* of a point $x \in |K|$ is the realization of the open star of a vertex of some subdivision of K, such that this realization contains x. The *Cartesian product* $P \times Q$ of the polyhedra $P = (K, |K|)$, $Q = (L, |L|)$ is given by the product $K \times L$ of their respective complexes (as defined in [2], p. 67) and a geometric realization $|K \times L|$ which is identified in a canonical way with $|K| \times |L|$, so that the projections $|K \times L| \to |K|$, $|K \times L| \to |L|$ are induced by simplicial maps $K \times L \to K$, $K \times L \to L$; and so that furthermore if the *diagonal* $\Delta \subset |K| \times |K|$ is defined as the set of points (p, p), then Δ is the geometric realization of a simplicial complex D which is isomorphic to K, and (D, Δ) is a subpolyhedron of $P \times P$.

Henceforth, for obvious reasons there will be a systematic confusion among the polyhedron $P = (K, |K|)$, the simplicial complex K, and the geometric realization $|K|$.

DEFINITION 4. If P is a polyhedron, then that a subset N is a *polyhedral open set*, means that $P - N$ is a subpolyhedron of P.

DEFINITION 5. If $f\colon P \to Q$ is a function of one polyhedron into another, then that f is *polyhedrally almost continuous* means that for every polyhedral open set $N \subset P \times Q$, if $\Gamma(f) \subset N$ then there exists a continuous function $g\colon P \to Q$ such that $\Gamma(g) \subset N$.

The relation with fixed point theory is provided by this theorem:

THEOREM 1. *Let N be a polyhedral open set in $P \times P$. Suppose that every continuous $g\colon P \to P$ for which $\Gamma(g) \subset N$, has a fixed point. Then every polyhedrally almost continuous $f\colon P \to P$ for which $\Gamma(f) \subset N$, has a fixed point.*

Proof. Suppose to the contrary that $f\colon P \to P$ is polyhedrally almost continuous, $\Gamma(f) \subset N$, and f has no fixed point. Then $\Gamma(f) \subset N - \Delta$; now. $P \times P - (N - \Delta) = (P \times P - N) \cup \Delta$, both $P \times P - N$ and Δ are subpolyhedra of $P \times P$, and the union of two subpolyhedra is again a subpolyhedron. Thus $N - \Delta$ is a polyhedral open set. Hence there exists a continuous $g\colon P \to P$ such that $\Gamma(g) \subset N - \Delta$; but this contradicts the hypothesis that if $\Gamma(g) \subset N$, g must have a fixed point.

A proposition which is important for the main theorems is now proved.

THEOREM 2. *Let $f\colon I \to P$ be a connectivity map, $I = [0, 1]$, P an arbitrary polyhedron. Let N be a polyhedral open set containing $\Gamma(f)$. Then there exists a continuous $g\colon I \to P$, $\Gamma(g) \subset N$, $g(0) = f(0)$, $g(1) = f(1)$.*

Proof. Suppose $I \times P$ is triangulated so that $I \times P - N$ is a subcomplex, and the projection $\pi\colon I \times P \to I$ is a simplicial map, where the vertices of I in its triangulation are $0 = v_0 < v_1 < \ldots < v_n = 1$. Define $g(v_i) = f(v_i)$. The problem now is to extend g to the intervals $[v_i, v_{i+1}]$; this is done as follows: $\pi^{-1}[v_i, v_{i+1}] \cap \bar{N}$ is a subcomplex of $I \times P$, and according to [3], Lemma 2, $\pi^{-1}(v_i, v_{i+1}) \cap N$ is homeomorphic to $(v_i, v_{i+1}) \times \times (\pi^{-1}(v_i + v_{i+1}/2) \cap N)$ with π as the projection on the first factor. Let Q_i be the component of $\pi^{-1}(v_i, v_{i+1}) \cap N$ containing $\Gamma(f|(v_i, v_{i+1}))$; such exists since f is a connectivity map. $\bar{Q}_i$ is a polyhedron and contains $\Gamma(f \mid [v_i, v_{i+1}])$ for the same reason. It follows now from the consideration that Q_i is homeomorphic to $(v_i, v_{i+1}) \times R_i$ where R_i is a component of $\pi^{-1}(v_i + v_{i+1}/2) \cap N$, that there exists an arc $A(t)$, $v_i \leqslant t \leqslant v_{i+1}$ in $\bar{Q}_i$, which except for its end points lies in Q_i, such that $A(v_i) = (v_i, f(v_i))$, $A(v_{i+1}) = (v_{i+1}, f(v_{i+1}))$, and such that $\pi(A(t)) = t$. The arcs $A(t)$, $v_i \leqslant t \leqslant v_{i+1}$ can then be pieced together to make one arc $A(t)$, $0 \leqslant t \leqslant 1$, which is the graph of a continuous function $g\colon I \to P$, with the desired properties.

COROLLARY. *Let P be a 1-dimensional polyhedron; Q an arbitrary polyhedron. Then every connectivity map $f\colon P \to Q$ is polyhedrally almost continuous.*

Proof. Let N be a polyhedral open set in $P \times Q$, such that $\Gamma(f) \subset N$. Let J be a polyhedron consisting of a number of disjoint segments and points and $\varphi\colon J \to P$ a simplicial map onto P such that $\varphi^{-1}(p)$ consists of only one point unless p is a vertex of P. Then it is easy to see that

$f\varphi$: $J \to Q$ is a connectivity map; thus we may apply Theorem 2 to each 1-simplex of J, and define on each vertex v of J the function $g(v) = f\varphi(v)$, extending this to a continuous function g on J to Q such that $\Gamma(g) \subset N' \subset J \times Q$, where N' is the polyhedral open set of all points (j, q) such that $(\varphi(j), q) \in N$. Then $g\varphi^{-1}$: $P \to Q$ is single-valued and continuous, and $\Gamma(g\varphi^{-1}) \subset N$, thus showing that f is polyhedrally almost continuous.

This Corollary, although a particular case of a later theorem, was injected in order to give you a taste of the methods to come.

Note that the ideas of this section are much concerned with simplicial things, and not with truly topological things. I do not known whether this can be avoided; I conjecture that the Corollary is still true when in the phrase "polyhedrally almost continuous", the word "polyhedrally" is deleted. (See next section for definition.)

3. Almost continuity and peripheral continuity.

DEFINITION 6. If f: $X \to Y$ is a function of topological spaces, then that f is *almost continuous* means that for any open set $N \subset X \times Y$, if $\Gamma(f) \subset N$, there exists a continuous function g: $X \to Y$ such that $\Gamma(g) \subset N$.

As in section 2 we have a fixed point theorem:

THEOREM 3. *If N is an open set of $X \times X$, where X is a Hausdorff space, and if every continuous function g: $X \to X$ whose graph lies in N has a fixed point, then for every almost continuous f: $X \to X$, if $\Gamma(f) \subset N$, f has a fixed point.*

Proof. Suppose on the contrary that f: $X \to X$ is almost continuous, $\Gamma(f) \subset N$, f has no fixed point. Let $\Delta = \{(x, y) \in X \times X \mid x = y\}$; since X is a Hausdorff space, Δ is closed in $X \times X$. Thus $\Gamma(f) \subset N - \Delta$, and $N - \Delta$ is open in $X \times X$; hence there is a continuous g: $X \to X$ with $\Gamma(g) \subset N - \Delta$; which contradicts the hypothesis.

We now recall the definition of peripheral continuity due to Hamilton.

DEFINITION 7. If f: $X \to Y$ is a function of topological spaces, then that f is *peripherally continuous* means that for each $x \in X$, each open $V \subset X$ for which $x \in V$, each open $U \subset Y$ for which $f(x) \in U$, there exists a neighborhood N of x, $x \in N \subset V$, such that $\Gamma(f | \operatorname{bd} N) \subset V \times U$.

Hamilton purported to show that a connectivity map of an n-cell, $n \geqslant 2$, into itself, is peripherally continuous. However, his proof contains a gap (see remark after proof of theorem below). A generalization of this theorem, which is of great importance to this paper, is proved below after an auxilliary definition.

DEFINITION 8. That a polyhedron P is lpc *(locally peripherally connected)* means that for each $p \in P$ there exist arbitrarily small neighborhoods N of p such that $\operatorname{bd} N$ is connected. This is clearly equivalent to saying that the local 1-dimensional homology groups are 0.

THEOREM 4. *If f: $P \to Y$ is a local connectivity map of the* lpc *polyhedron P into a regular Hausdorff space Y, then f is peripherally continuous.*

The proof will be preceded by a Lemma, similar in spirit to Hamilton's Theorems 1 and 2; this Lemma provides the bridge over the gap in Hamilton's proof.

LEMMA. *Let X be a compact Hausdorff space such that each point $x \in X$ has arbitrarily small neighborhoods N such that $X - N$ is connected. Let f: $X \to Y$ be a connectivity map, where Y is a T_1 space; and let C be a closed subset of Y. Then each component of $f^{-1}(C)$ is closed, by Hamilton's Theorem 2. If $\{Q_a\}$ is a family of subcontinua of $f^{-1}(C)$, indexed on a directed set $\{a\}$ and if $\operatorname{Lim}_a Q_a$ exists and is a nondegenerate continuum Q, then $Q \subset f^{-1}(C)$.*

(That $\operatorname{Lim}_a Q_a = Q$ means that for any neighborhood U of the diagonal Δ of $X \times X$, there exists an a such that for all β if $\beta > a$ then $U[Q_\beta] \supset Q$ and $U[Q] \supset Q_\beta$. Here, $U[x] = \{y \mid (x, y) \in U\}$, and $U[\Delta] = \bigcup\{U[x] \mid x \in A\}$. See [4], Chapters 2 and 6.)

Proof of Lemma. Suppose to the contrary $x \in Q$, $x \notin f^{-1}(C)$, and $w \in Q$, $w \neq x$. Then there exists a set $\{w_a\}$, $w_a \in Q_a$, $\operatorname{Lim}_a w_a = w$, and a neighborhood N of x not containing any w_a, for $a >$ some β. By hypothesis, we can suppose that $X - N$ is connected. Then $(X - N) \cup \bigcup_{a > \beta} Q_a \cup \{x\}$ is a connected set D; but the open set $N \times (Y - C)$ contains just the one point $(x, f(x))$ of $\Gamma(f|D)$; hence $\Gamma(f|D)$ cannot be connected, which is a contradiction.

Proof of Theorem 4. According to Definition 7, we must consider points $x \in P$, neighborhoods V of x and U of $f(x)$ and find a neighborhood N of x, $N \subset V$, $\Gamma(f | \operatorname{bd} N) \subset V \times U$. Since this is a local matter, we can assume simply that f is a connectivity map; since the space P is an lpc polyhedron, we assume that V is a star-neighborhood of x and that $\operatorname{bd} V$ is connected. Let V^* be a star-neighborhood and U^* a neighborhood of x and $f(x)$ resp. such that $\overline{V^*} \subset V$, $\overline{U^*} \subset U$. Then consider the set $D = f^{-1}(\overline{U^*}) \cap \overline{V^*}$, and the family of its components $\{D_a\}$, each of which, by Hamilton's Theorem 2, is closed. Define $D_a \prec D_\beta$ if D_β separates D_a and $\operatorname{bd} V$ in $\overline{V}$; this is a patrial ordering relation. Let $\{D_{a\nu}\}$ be a linearly ordered subset, where the indices $\{\nu\}$ are themselves linearly ordered, and $\mu \prec \nu$ if and only if $D_{a\mu} \prec D_{a\nu}$. Let $L = \bigcap_\nu \operatorname{Cl}[\bigcup_{\mu \succ \nu} D_{a\mu}]$; L is a subcontinuum of $\overline{V^*}$, and clearly L separates $D_{a\nu}$ from $\operatorname{bd} V$ in $\overline{V}$; omitting the trivial case when $\{D_{a\nu}\}$ contains only one element, since L separates points in $V - L$ from $D_{a\nu}$, L is nondegenerate; otherwise P would not be lpc at the point L. Finally $L = \operatorname{Lim}_\nu D_{\alpha\nu}$, the limit as ν increases; for the fact that every neighborhood of L contains all the $D_{\alpha\mu}$ for $\mu \succcurlyeq$ some ν follows easily from the compactness of $\overline{V^*}$ and L. It must only be shown that for each $\varepsilon > 0$ there is ν such that for all $\mu \succcurlyeq \nu$ the ε-neighborhood of $D_{\alpha\mu}$ contains L; if this were false, from the compactness of L there would exist $x \in L$ and a neighborhood N of x in V (we can assume that N is connected, since V is locally connected) such that $D_{\alpha\mu} \cap N = \emptyset$ for $\mu \in$ some cofinal subset $\{\mu\} \subset \{\nu\}$; but this contradicts the fact, obvious from the definitions of D_α and L, that any connected set that intersects L and $D_{\alpha\nu}$ must intersect each $D_{\alpha\mu}$ for $\mu \succcurlyeq \nu$. Hence $\operatorname{Lim}_\nu D_{\alpha\nu}$ is contained D by the Lemma, for every component of P is such a space X as occurs in the Lemma. If $D_\beta \supset \operatorname{Lim}_\nu D_{\alpha\nu}$, then it is clear that $D_{\alpha\nu} \precsim D_\beta$. Therefore every linearly ordered set of D_α's has an upper bound; so by Zorn's Lemma, *every* D_α *is* $\precsim$ *a maximal* D_β.

Let $\{D_\beta\}$ be then the set of maximal elements of the set $\{D_\alpha, \precsim\}$. Let $E_\beta = \{v \in V \mid D_\beta \text{ separates } v \text{ from } \operatorname{bd} V\}$. Let $\widetilde{D}_\beta = D_\beta \cup E_\beta$. And let $\widetilde{D} = \bigcup_\beta \widetilde{D}_\beta$. Then E_β is open; D is contained in $\widetilde{D}$. And finally, *the components of* $\widetilde{D}$ *are just the sets* $\widetilde{D}_\beta$: For, since $\widetilde{D}_\beta$ is connected, each component of $\widetilde{D}$ is of the form $\bigcup_\gamma \widetilde{D}_{\beta\gamma}$. Now $\bigcup_\gamma D_{\beta\gamma}$ is not connected unless there is only one $D_{\beta\gamma}$ involved; supposing there is more than one $D_{\beta\gamma}$ involved, there is a partition of $\{\gamma\} = \{\gamma_1\} \cup \{\gamma_2\}$ such that $\bigcup_{\gamma_1} D_{\beta\gamma_1}$ and $\bigcup_{\gamma_2} D_{\beta\gamma_2}$ are disjoint open-closed sets in $\bigcup_\gamma D_{\beta\gamma}$, which cover $\bigcup_\gamma D_{\beta\gamma}$. Then I assert that $\bigcup_{\gamma_1} \widetilde{D}_{\beta\gamma_1}$, $\bigcup_{\gamma_2} \widetilde{D}_{\beta\gamma_2}$ is a disjoint open-closed partition of $\bigcup_\gamma \widetilde{D}_{\beta\gamma}$; disjoint, since $\widetilde{D}_{\beta_1} \cap \widetilde{D}_{\beta_2} = \emptyset$ if $D_{\beta_1} \neq D_{\beta_2}$; open-closed, since, for example, each point of $D_{\beta\gamma_1}$ has a connected neighborhood disjoint from $\bigcup_{\gamma_2} D_{\beta\gamma_2}$ and hence disjoint from $\bigcup_{\gamma_2} \widetilde{D}_{\beta\gamma_2}$, and any point of $E_{\beta\gamma_1}$ lies in the interior of $\widetilde{D}_{\beta\gamma_1}$; so any point of $\widetilde{D}_{\beta\gamma_1}$ has a neighborhood disjoint from $\bigcup_{\gamma_2} D_{\beta\gamma_2}$. Therefore the components of $\widetilde{D}$ are just the sets $\widetilde{D}_\beta$.

This implies that $\overline{V} - \widetilde{D}$ is connected; for if not, since $\overline{V}$ is unicoherent, some component of $\widetilde{D}$ must separate $\overline{V}$; but since $\operatorname{bd} V$ is connected, any such component $\widetilde{D}_\beta$ does not separate $\overline{V}$. Hence, from Hamilton's Theorem 1, it follows that $x \in \operatorname{int} \widetilde{D}$; otherwise $(\overline{V} - \widetilde{D}) \cup \{x\}$ is a connected set for which the only point in $\Gamma\big(f \mid (\overline{V} - \widetilde{D}) \cup \{x\}\big) \cap (V^* \times U^*)$ is $\big(x, f(x)\big)$. Since V is locally connected, there is a component $\widetilde{D}_\beta$ of $\widetilde{D}$ such that $x \in \operatorname{int} \widetilde{D}_\beta$. And $\operatorname{bd}(\operatorname{int} \widetilde{D}_\beta) \subset D_\beta \subset D$, so that $\Gamma\big(f \mid \operatorname{bd}(\operatorname{int} \widetilde{D}_\beta)\big) \subset V \times U$. We have thus found the neighborhood $N = \operatorname{int} \widetilde{D}_\beta$, for which we were seeking.

Remark. Hamilton's proof of this theorem (in a restricted case) is sketchy. He seems to assume that if C is a set in I^n (for example, $n = 2$) each of whose components is closed and $C \cap \operatorname{bd} I^n = \emptyset$, then if we fill up C by adding all points x which C separates from $\operatorname{bd} I^n$, thereby getting

a set $\widetilde{C}$, then any pair of points separated by $\widetilde{C}$ is separated by the filled up $\widetilde{C}_\alpha$ of some component C_α of C. Thas this is false is shown by the following example: C is the subset of a large disk in the complex z-plane, consisting of part of the real line $\{\Im z = 0,\ |z| \leqslant 4\}$, semicircles

$$A_n = \left\{\left|z - \frac{1}{n} i\right| = 1 + \frac{1}{n},\ \Im z \geqslant \frac{1}{n}\right\}, \qquad n = 1, 2, \ldots,$$
$$B_n = \left\{\left|z - \frac{1}{n} i\right| = 4 - \frac{1}{n},\ \Im z \geqslant \frac{1}{n}\right\},$$

and pairs of line intervals

$$I_n = \left\{1 + \frac{1}{n} \leqslant |\Re z| \leqslant 4 - \frac{1}{n},\ \Im z = \frac{1}{n}\right\}.$$

The points $\pm \frac{1}{2} i$ are separated by $\widetilde{C}$, but not by any set $\widetilde{C}_\alpha$.

This error is rectified here by the Lemma; the set C which Hamilton had to consider was of the form $f^{-1}(Q)$, where f is a connectivity map and Q is a closed set. The property of C given by the Lemma and the argument which followed in the proof of Theorem 4, show that the assertion of Hamilton is in this particular case correct.

I also remark that a peripherally continuous function $f\colon P \to X$, where P is lpc, has the following property:

If $x \in P$, $x \in V$ *open* $\subset P$, $f(x) \in U$ *open* $\subset X$, *then there exists a connected neighborhood* N *of* x, $N \subset V$, *such that* $\operatorname{bd} N$ *is connected, and* $\Gamma(f \mid \operatorname{bd} N) \subset V \times U$.

For we may assume that V is a star-neighborhood, hence unicoherent, and $\operatorname{bd} V$ is connected; then for any neighborhood M of x, $\overline{M} \subset V$, for which $\Gamma(f \mid \operatorname{bd} M) \subset V \times U$, $\operatorname{bd} M$ separates the two connected sets $\operatorname{bd} V$ and x; hence by the standard theorems about unicoherence [5], p. 51, th. 4.12, there is a closed connected subset $C \subset \operatorname{bd} M$ such that C separates $\operatorname{bd} V$ and x. If $\widetilde{C} = C \cup \{y \in V \mid C \text{ separates } \operatorname{bd} V \text{ and } y\}$, then let N = component of $\operatorname{int} \widetilde{C}$ containinig x; again by the standard theorems $\operatorname{bd} N \subset C$ and $\operatorname{bd} N$ is connected. Since $\operatorname{bd} N \subset C \subset M$, it follows that $\Gamma(f \mid \operatorname{bd} N) \subset V \times U$.

4. The main theorems. We proceed to a very important theorem which will show that on lpc polyhedra, peripherally continuous functions are almost continuous.

In a metric space X, $\varrho(x, y)$ will denote the distance between x and y; $N(x; \varepsilon) = \{y \in X \mid \varrho(x, y) < \varepsilon\}$. E^{k+1} denotes the unit ball in Euclidean $(k+1)$-space, and S^k is its boundary k-sphere.

DEFINITION 9. That a metric space X is *uniformly locally n-connected* means: For any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in X$, any integer k, $0 \leqslant k \leqslant n$, and any continuous $\varphi\colon S^k \to N(x;\delta)$, there is an extension of φ to a continuous $\varphi'\colon E^{k+1} \to N(x;\varepsilon)$.

DEFINITION 10. In such a space X, we shall use the following notation:

$$\delta(\varepsilon) = \sup\{\delta > 0 \mid \text{for all } x \in X, \text{ all integers } k,\ 0 \leqslant k \leqslant n, \text{ and for all continuous } \varphi\colon S^k \to N(x;\delta), \text{ there exists an extension of } \varphi,\ \varphi'\colon E^{k+1} \to N(x;\varepsilon)\},$$

$$\delta_0(\varepsilon) = \varepsilon;$$

$$\delta_1(\varepsilon) = \delta(\tfrac{1}{5}\varepsilon);$$

$$\delta_{p+1}(\varepsilon) = \delta\big(\tfrac{1}{5}\delta_p(\varepsilon)\big).$$

The following propositions are easy to prove:

(A) *For any* $\varepsilon > 0$, $\delta_p(\varepsilon) > 0$.

(B) $\delta_{p+1}(\varepsilon) < \delta_p(\varepsilon)$.

(C) *If* $\varepsilon \leqslant \varepsilon'$, *then* $\delta_p(\varepsilon) \leqslant \delta_p(\varepsilon')$.

These ideas will be useful in the proof of the next theorem, for which in the applications in this paper, X will be a polyhedron.

THEOREM 5. *Let P be an* lpc *polyhedron of dimension n, X a uniformly locally $(n-1)$-connected metric space, $f\colon P \to X$ a peripherally continuous function. Then f is almost continuous. In fact if $v_1, \ldots, v_q$, are a finite number of points of P, and W is an open set in $P \times X$ for which $\Gamma(f) \subset W$, then there exists a continuous $g\colon P \to X$ such that $\Gamma(g) \subset W$ and $g(v_i) = f(v_i)$, $i = 1, \ldots, q$.*

Proof. For each $p \in P$, let $\varepsilon_1(p)$, $\varepsilon_2(p)$ be two positive numbers such that $N\big(p;\varepsilon_1(p)\big) \times N\big(f(p);\varepsilon_2(p)\big) \subset W$. There is a connected neighborhood of p, $U(p)$, with connected boundary, such that $\operatorname{diam}\big(U(p)\big) < \varepsilon_1(p)$, and $f\big(\operatorname{bd} U(p)\big) \subset N\Big(f(p);\ \tfrac{1}{5}\delta_n\big(\varepsilon_2(p)\big)\Big)$. Initially choose $\varepsilon_1(p) < \varepsilon$, where ε has the property that if two open connected sets with connected boundary, $C_1, C_2 \subset P$ are such that $\operatorname{diam} C_i < \varepsilon$, $C_1 \cap C_2 \neq \emptyset$, $C_1 - C_2 \neq \emptyset \neq C_2 - C_1$, then $\operatorname{bd} C_1 \cap \operatorname{bd} C_2 \neq \emptyset$; and also let ε satisfy $\varrho(v_i, v_j) > \varepsilon$, $i, j = 1, \ldots, q$, $i \neq j$.

Now the family of open sets $\{U(p)\}$ covers P; since P is compact, there can be chosen a finite subcovering $\{U(p_1), \ldots, U(p_m)\}$, so that $v_1, \ldots, v_q$ occur among the set $p_i, \ldots, p_m$, and so that no member of this covering is contained in any other member. Let $\eta > 0$ be less than the Lebesgue number of the covering $\{U(p_i), \ldots, U(p_m)\}$ and less than $\varrho\big(p_i, P - U(p_i)\big)$, $i = 1, \ldots, m$. Finally, let P be triangulated with mesh $< \tfrac{1}{2}\eta$ and so that each p_i is a vertex of the triangulation; call these vertices $a_1, \ldots, a_r$.

To each vertex a_j assign one of the points p_i, called $p(a_j)$, such that the closed star of a_j (in this particular triangulation) is contained in $U\big(p(a_j)\big)$. Do this in such a way that $p(p_j) = p_j$.

On the 0-skeleton of P, $P^0 = \{a_1, \ldots, a_r\}$, we define:

$$g(a_j) = f\big(p(a_j)\big).$$

Suppose that a continuous function g has thus been defined on the k-skeleton P^k, with the following property:

If Δ is a $(k+1)$-simplex whose boundary is $\Delta^{\cdot} \subset P^k$, then for any vertex a_i of Δ such that $\varepsilon_2\big(p(a_i)\big) = \max\big\{\varepsilon_2\big(p(a_j)\big) \mid a_j \text{ is a vertex of } \Delta\big\}$, it is true that $g(\Delta^{\cdot}) \subset N\Big(f\big(p(a_i)\big);\ \delta_{n-k}\Big(\varepsilon_2\big(p(a_i)\big)\Big)\Big)$.

g will be extended continuosly to the $(k+1)$-skeleton P^{k+1} so that same condition is satisfied, with k replaced by $k+1$.

First check that the condition is satisfied by the definition of g for $k = 0$. It must be shown that if $\Delta = (a_i, a_j)$ is a 1-simplex and $\varepsilon_2\big(p(a_j)\big) \geqslant \varepsilon_2\big(p(a_i)\big)$, then $g(\Delta^{\cdot}) \subset N\Big(f\big(p(a_j)\big);\ \delta_n\Big(\varepsilon_2\big(p(a_j)\big)\Big)\Big)$. This is clearly true if $p(a_i) = p(a_j)$; if $p(a_i) \neq p(a_j)$, noting that $U\big(p(a_i)\big) \cap U\big(p(a_j)\big) \supset \Delta$, it follows that there is a point $c \in \operatorname{bd} U\big(p(a_i)\big) \cap \operatorname{bd} U\big(p(a_j)\big)$; therefore:

$$\varrho\big(g(a_i), f(c)\big) = \varrho\Big(f\big(p(a_i)\big), f(c)\Big) < \tfrac{1}{5}\delta_n\Big(\varepsilon_2\big(p(a_i)\big)\Big)$$
$$\leqslant \tfrac{1}{5}\delta_n\Big(\varepsilon_2\big(p(a_j)\big)\Big) > \varrho\Big(f\big(p(a_j)\big), f(c)\Big) = \varrho\big(g(a_j), f(c)\big).$$

Hence, $\varrho\Big(f\big(p(a_j)\big), g(a_i)\Big) < \tfrac{2}{5}\delta_n\Big(\varepsilon_2\big(p(a_j)\big)\Big)$, so that $g(a_i) \in N\Big(f\big(p(a_j)\big);\ \tfrac{2}{5}\delta_n\Big(\varepsilon_2\big(p(a_j)\big)\Big)\Big)$. This is proof that the condition is satisfied for $k = 0$.

Assume g defined on P^k satisfying the given condition. It follows from the definition of δ_{n-k}, that if Δ is a $(k+1)$-simplex and a_i is a vertex of Δ for which $\varepsilon_2\big(p(a_i)\big)$ is maximal among the vertices of Δ, then there exists an extension of $g \mid \Delta^{\cdot}$ into Δ such that $g(\Delta) \subset N\Big(f\big(p(a_i)\big);\ \tfrac{1}{5}\delta_{n-k-1}\Big(\varepsilon_2\big(p(a_i)\big)\Big)\Big)$. Let a_j be another vertex of Δ for which $\varepsilon_2\big(p(a_j)\big)$ is maximal; if $p(a_i) = p(a_j)$ we obviously have $g(\Delta) \subset N\Big(f\big(p(a_j)\big);\ \tfrac{1}{5}\delta_{n-k-1}\Big(\varepsilon_2\big(p(a_j)\big)\Big)\Big)$. If $p(a_i) \neq p(a_j)$, then $U\big(p(a_i)\big) \cap U\big(p(a_j)\big) \supset \Delta \neq \emptyset$; hence there is a point $c \in \operatorname{bd} U\big(p(a_i)\big) \cap \operatorname{bd} U\big(p(a_j)\big)$ and

$$\varrho\big(g(a_i), f(c)\big) = \varrho\Big(f\big(p(a_i)\big), f(c)\Big) < \tfrac{1}{5}\delta_n\Big(\varepsilon_2\big(p(a_i)\big)\Big)$$
$$\leqslant \tfrac{1}{5}\delta_{n-k-1}\Big(\varepsilon_2\big(p(a_i)\big)\Big) = \tfrac{1}{5}\delta_{n-k-1}\Big(\varepsilon_2\big(p(a_j)\big)\Big)$$
$$\geqslant \tfrac{1}{5}\delta_n\Big(\varepsilon_2\big(p(a_j)\big)\Big) > \varrho\Big(f\big(p(a_j)\big), f(c)\Big).$$

Hence $\varrho\big(g(a_i), f(p(a_j))\big) < \frac{2}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_i))\big) = \frac{2}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_j))\big)$, and since $g(\Delta) \subset N\Big(f(p(a_i));\ \frac{1}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_i))\big)\Big)$, it follows that $g(\Delta) \subset N\Big(f(p(a_j));\ \frac{3}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_j))\big)\Big)$. Suppose then that g is extended to each $(k+1)$-simplex Δ in this manner; then for each $(k+1)$-simplex Δ and vertex a_i of Δ for which $\varepsilon_2(p(a_i))$ is maximal among the vertices of Δ,

$$g(\Delta) \subset N\Big(f(p(a_i));\ \tfrac{3}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_i))\big)\Big).$$

Let Δ^* be a $(k+2)$-simplex and a_i a vertex for which $\varepsilon_2(p(a_i))$ is maximal among the vertices of Δ^*. Then for each of the $(k+1)$-simplexes in the boundary of $\Delta^*, \Delta_1, \ldots, \Delta_{k+2}$, of which a_i is a vertex, $g(\Delta_j) \subset N\Big(f(p(a_i));\ \frac{3}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_i))\big)\Big)$. Let a_S be a vertex of the remaining $(k+1)$-simplex Δ_0, for which $\varepsilon_2(p(a_S))$ is maximal among the vertices of Δ_0; if $p(a_i) = p(a_S)$, it follows that $g(\Delta_0) \subset N\Big(f(p(a_i));\ \frac{3}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_i))\big)\Big)$. But if $p(a_i) \neq p(a_S)$, then $U\big(p(a_i)\big) \cap U\big(p(a_S)\big) \supset \Delta^* \neq \emptyset$; hence there is $c \in \operatorname{bd} U\big(p(a_i)\big) \cap \operatorname{bd} U\big(p(a_S)\big)$. Then:

$$\begin{aligned}\varrho\big(f(p(a_i)), c\big) &< \tfrac{1}{5}\delta_n\big(\varepsilon_2(p(a_i))\big) \leqslant \tfrac{1}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_i))\big)\\ &\geqslant \tfrac{1}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_S))\big) \geqslant \tfrac{1}{5}\delta_n\big(\varepsilon_2(p(a_S))\big) > \varrho\big(f(p(a_S)), c\big).\end{aligned}$$

Furthermore, $g(\Delta_0) \subset N\Big(f(p(a_S));\ \frac{3}{5}\delta_{n-k-1}\big(\varepsilon_2(p(a_S))\big)\Big)$; hence, $g(\Delta_0) \subset N\Big(f(p(a_i));\ \delta_{n-k-1}\big(\varepsilon_2(p(a_i))\big)\Big)$. So for any $(k+2)$-simplex Δ^*, $g(\Delta^*) \subset N\Big(f(p(a_i));\ \delta_{n-k-1}\big(\varepsilon_2(p(a_i))\big)\Big)$, for any vertex a_i of Δ^*, for which $\varepsilon_2(p(a_i))$ is maximal. This is just the condition required of g on the $(k+1)$-skeleton of P.

This procedure can be carried through right up to and including the n-skeleton of P, $P^n = P$. Note that for an n-simplex Δ of P and vertex a_i such that $\varepsilon_2(p(a_i)) = \max\big\{\varepsilon_2(p(a_j)) \mid a_j \text{ is a vertex of } \Delta\big\}$, the result is that $g(\Delta) \subset N\big(f(p(a_i));\ \frac{3}{5}\varepsilon_2(p(a_i))\big)$. It is also easy to see that $\Delta \subset U\big(p(a_i)\big)$. Hence, if we recall the definitions of ε_2 and U, we find that $\Gamma(g|\Delta) \subset W$. In the case of simplexes of less dimension, the same result holds. Thus, $\Gamma(g) \subset W$. And by construction, $g(p_i) = f(p_i)$, and so in particular $g(v_i) = f(v_i)$. Therefore the theorem is proved.

Corollary 1. *If P is a polyhedron of dimension n which is at each point of dimension $\geqslant 2$, and f: $P \to X$ is a connectivity map, where X is uniformly locally $(n-1)$-connected, then f is almost continuous.*

Proof. One can construct an n-dimensional lpc polyhedron P' and a simplicial map φ: $P' \to P$ onto P, such that except for a finite number of points $v_1, \ldots, v_q$ of P, φ^{-1} of a point is just one point of P', and such that $\varphi^{-1}(v_i)$ consists of a finite number of points.

Then it is clear that $f\varphi$: $P' \to X$ is a local connectivity map (I do not know whether this is necessarily a connectivity map). By Theorem 4, $f\varphi$ is peripherally continuous. Now let W be any open subset of $P \times X$, such that $\Gamma(f) \subset W$; let φ_*: $P' \times X \to P \times X$ be defined by $\varphi_*(p', x) = \big(\varphi(p'), x\big)$. Then $\varphi_*^{-1}(W)$ is an open subset of $P' \times X$ such that $\Gamma(f\varphi) \subset \varphi_*^{-1}(W)$. By Theorem 5, there exists a continuous g: $P' \to X$, such that $\Gamma(g) \subset \varphi_*^{-1}(W)$ and such that for any of the points v' in $\varphi^{-1}(v_i)$, $i = 1, \ldots, q$, $g(v') = f\varphi(v')$. Therefore $g\varphi^{-1}$: $P \to X$ is single-valued, and in fact continuous; furthermore, $\Gamma(g\varphi^{-1}) \subset W$. Therefore f is almost continuous.

Corollary 2. *If P and Q are polyhedra, f: $P \to Q$ a connectivity map, then f is polyhedrally almost continuous.*

Proof. Let P' be a polyhedron whose components are points, segments, and lpc polyhedra; and φ: $P' \to P$ onto P be a simplicial map such that for all points $p \in P$, $\varphi^{-1}(p)$ is just one point, except for certain points $v_1, \ldots, v_q$, for which $\varphi^{-1}(v_i)$ is a finite set consisting of endpoints of segments and certain points in the lpc polyhedra components of P'.

Then $f\varphi$: $P' \to Q$ is continuous on the point components of P', a connectivity map on the segment components of P', and local connectivity map on the lpc components of P'. Let W be any polyhedral open subset of $P \times Q$ with $\Gamma(f) \subset W$; let φ_*: $P' \times Q \to P \times Q$ be defined: $\varphi_*(p', q) = \big(\varphi(p'), q\big)$. Then $\varphi_*^{-1}(W)$ is a polyhedral open set in $P' \times Q$, $\Gamma(f\varphi) \subset \varphi_*^{-1}(W)$. Applying Theorem 4 and Theorems 2 and 5, and the fact that the polyhedron Q is uniformly locally n-connected for any n, we obtain, as in the proof of Corollary 1, a continuous function g: $P' \to Q$, such that $\Gamma(g) \subset \varphi_*^{-1}(W)$, and $g\varphi^{-1}$: $P \to Q$ is single-valued and continuous, and $\Gamma(g\varphi^{-1}) \subset W$. Therefore, f is polyhedrally almost continuous.

Combining Corollary 1 and Theorem 3, we obtain the first of the theorems below. Combining Corollary 2 and Theorem 1, we obtain the second of these theorems.

Theorem 6. *Let P be a polyhedron of dimension $\geqslant 2$ at each point, N an open set in $P \times P$; if every continuous function g: $P \to P$ whose graph lies in N has a fixed point, then any connectivity map f: $P \to P$ whose graph lies in N has a fixed point.*

Theorem 7. *If P is an arbitrary polyhedron, N a polyhedral open set in $P \times P$, and if for any continuous g: $P \to P$, if $\Gamma(g) \subset N$, g has a fixed point; then, for any connectivity map f: $P \to P$, if $\Gamma(f) \subset N$, f has a fixed point.*

So, for example, taking $N = P\times P$, we discover that P has for every connectivity map some fixed point, if and only if it has for every continuous function some fixed point.

5. Further study of almost continuity. Now, as an anticlimax, we shall try to learn certain nice facts about almost continuous functions, in the hope that these facts may be useful in future study. Connectivity maps do not, in general, satisfy a proposition similar to Proposition 4 below, and for this reason, the study of connectivity maps is difficult.

PROPOSITION 1. *Let $f\colon X\to Y$ be almost continuous, $g\colon Y\to Z$ continuous, where X, Y, Z, are topological spaces. Then $gf\colon X\to Z$ is almost continuous.*

Proof. Let N be an open set of $X\times Z$, such that $\Gamma(gf)\subset N$. Let $g_*\colon X\times Y\to X\times Z$ be defined as $g_*(x, y) = (x, g(y))$. Then $g_*^{-1}(N)$ is an open set of $X\times Y$ and $\Gamma(f)\subset g_*^{-1}(N)$. Then since f is almost continuous, there is a continuous $F\colon X\to Y$ such that $\Gamma(F)\subset g_*^{-1}(N)$. Hence $gF\colon X\to Z$ is continuous and $\Gamma(gF)\subset N$.

PROPOSITION 2. *Let $f\colon X\to Y$ be almost continuous, C a closed subset of X. Then $f|C\colon C\to Y$ is almost continuous.*

Proof. Let N be an open subset of $C\times Y$ such that $\Gamma(f|C)\subset N$. Then there is an open set N' in $X\times Y$ such that $N = N'\cap(C\times Y)$. The set $N'\cup[(X-C)\times Y]$ is then open in $X\times Y$ and $\Gamma(f)\subset N'\cup[(X-C)\times Y]$. Hence there is a continuous $F\colon X\to Y$, $\Gamma(F)\subset N'\cup[(X-C)\times Y]$. $F|C$ is then continuous: $C\to Y$ and $\Gamma(F|C)\subset N$.

PROPOSITION 3. *If $X\times Y$ is a completely normal Hausdorff space, where X and Y are topological spaces and X is connected, and if $f\colon X\to Y$ is almost continuous, then $\Gamma(f)$ is connected.*

(Z is completely normal if whenever $(A\cap\bar{B})\cup(\bar{A}\cap B) = \emptyset$, there exist disjoint open sets G, H, with $A\subset G$, $B\subset H$.)

Proof. Suppose, on the contrary, that $\Gamma(f)$ is not connected. Then there are open sets A, B in $X\times Y$, such that $A\cap B = \emptyset$, $\Gamma(f)\subset A\cup B$, $A\cap\Gamma(f)\neq\emptyset\neq B\cap\Gamma(f)$. Let $a, b\in X$ be points such that $(a, f(a))\in A$, $(b, f(b))\in B$; let $\pi\colon X\times Y\to X$ be the projection, $(x, y)\to x$. Let $A' = A-\pi^{-1}(b)$, $B' = B-\pi^{-1}(a)$. Then $A'\cup B'$ is open in $X\times Y$, and $\Gamma(f)\subset A'\cup B'$. Thus there exists a continuous $F\colon X\to Y$ such that $\Gamma(F)\subset A'\cup B'$. But A' and B' are open sets, disjoint, and $(a, F(a))\in A'$, $(b, F(b))\in B'$. Thus $\Gamma(F)$ could not be connected; however, since F is continuous, $\Gamma(F)$ is homeomorphic to X which is connected. Therefore $\Gamma(F)$ is connected.

From Propositions 2 and 3, we get a Corollary.

COROLLARY. *If $X\times Y$ is a completely normal Hausdorff space, and $f\colon X\to Y$ is almost continuous, and C is a closed connected subset of X, then $\Gamma(f|C)$ is connected.*

PROPOSITION 4. *Let X be a compact Hausdorff space, Y a Hausdorff space, Z a topological space. Suppose that $f\colon X\to Y$ is continuous, and that $g\colon Y\to Z$ is almost continuous. Then $gf\colon X\to Z$ is almost continuous.*

Proof. First note that if Y is replaced by $f(X)$, then $f\colon X\to f(X)$ is continuous; $g|f(X)\colon f(X)\to Z$ is almost continuous, by Proposition 2, since $f(X)$ is a closed subset of Y. Thus we can assume that f maps X onto Y. Let N be an open set of $X\times Z$ such that $\Gamma(gf)\subset N$; let $f_*\colon X\times Z\to Y\times Z$ be defined by $f_*(x, z) = (f(x), z)$. Then $f_*(\Gamma(gf)) = \Gamma(g)$. Now, for any $y\in Y$, $f^{-1}(y)$ is a compact subset of X; for any $x\in f^{-1}(y)$ let N_x be an open set of X containing x, and M_x be an open set in Z containing $gf(x) = g(y)$, such that $N_x\times M_x\subset N$; a finite number, say $N_1, \ldots, N_k$, of these N_x cover $f^{-1}(y)$; let $M_1, \ldots, M_k$ be the corresponding M_x. Then let $U_y = Y-f(X-\bigcup_{i=1}^{k}N_i)$; and let $W_y = U_y\times[\bigcap_{i=1}^{k}M_i]$. Then U_y is an open subset of Y containing y; hence W_y is an open subset of $Y\times Z$ containing $(y, g(y))$. Note that $f_*^{-1}(W_y)\subset N$. Let $W = \bigcup_{y\in Y}W_y$. Then W is an open subset of $Y\times Z$, and $\Gamma(g)\subset W$. Therefore there is a continuous $G\colon Y\to Z$ such that $\Gamma(G)\subset W$. Then $Gf\colon X\to Z$ is continuous. $f_*(\Gamma(Gf)) = \Gamma(G)\subset W$. Hence $\Gamma(Gf)\subset f_*^{-1}(W)\subset N$. Therefore gf is almost continuous.

6. Questions. One of the important questions left unsolved is, under what conditions a connectivity map of the unit interval I into a space is almost continuous. The first block of questions relates to this.

1. Is a connectivity map $f\colon I\to X$, where X is a uniformly locally 0-connected metric space, almost continuous?

2. If one considers I embedded in $I\times I$ as $I\times 0$, can a connectivity map $I\to X$ be extended to a connectivity map $I\times I\to X$? Supposing, for example, that X is uniformly locally 1-connected? If $\pi\colon I\times I\to I\times 0$ is the projection, and $f\colon I\to X$ is a connectivity map, then, as any simple, non-continuous example will show, $f\pi\colon I\times I\to X$ is *not* a connectivity map.

3. If $\mathcal{T}$ is a topology on $I = [0, 1]$, let $\mathcal{T}_l$ be the topology generated by the open sets of $\mathcal{T}$ and by left-closed intervals $[a, b)$; let $\mathcal{T}_r$ be the topology generated by the open sets of $\mathcal{T}$ and the right-closed intervals. Suppose that $\mathcal{T}$ is a connected topology (i. e., I, under this topology, is connected) and that every open interval (a, b) is open in $\mathcal{T}$ (i. e., $\mathcal{T}$ is finer than the ordinary topology); let L and R be subsets of I, $L\cup R = I$,

$0 \in L$, $1 \in R$, L open in $\mathcal{T}_l$, R open in $\mathcal{T}_r$. Is it then necessarily true that $L \cap R \neq \emptyset$?

4. In $I_1 \times I_2$ let C be the Cantor set described as follows (Figure 1). C_1 consists of two L-shaped closed components, so placed that the graph of every continuous function $I_1 \to I_2$ must intersect C_1. C_{n+1} is included in the interior of C_n and consists of a number of hook-shaped closed components, so placed that the graph of any continuous function $I_1 \to I_2$ must intersect C_{n+1}. The maximum diameter of a component of C_n approaches 0 as $n \to \infty$. $\bigcap_n C_n = C$. Then clearly the graph of any continuous function $I_1 \to I_2$ must intersect C. Does there exist a connectivity map: $I_1 \to I_2$ whose graph does not intersect C?

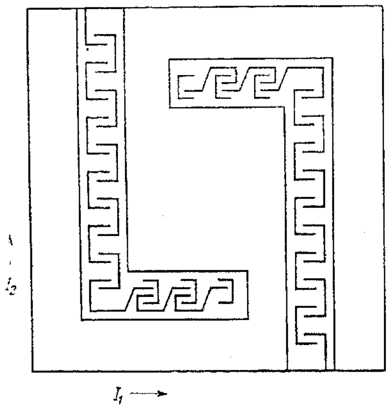

Fig. 1

The next group of questions concerns technical and esthetic matters.

5. Under what conditions is a local connectivity map $X \to Y$ a connectivity map? This is, perhaps, related to the question posed by Hamilton: When is a peripherally continuous function a connectivity map?

6. Is it possible to prove that a connectivity map $f\colon P \to Q$, of polyhedra, is polyhedrally almost continuous, by using elementary methods similar to those used in the case $\dim P = 1$?

7. To what extent are the theorems of this paper valid if the spaces concerned are not polyhedra? E. g., if they are ANR's, or quasi-complexes?

8. If $f\colon X \to Y$ is a 1-1 function onto, and both f and f^{-1} are connectivity maps (or almost continuous functions), then what is the relation, if any, between the homology and cohomology groups of X and Y? Can some sort of „homology" or „cohomology" theory be devised so that this relation is isomorphism? It is obvious that for connectivity maps this relation is not isomorphism for the singular homology theory, and the following example shows that the relation is not isomorphism for the Čech homology theory. (See [2] for the definitions of these homology theories.)

Let X = the circle, parametrized as the real numbers mod 1. Let $f\colon X \to X$ be defined thus: $f(r) = 1/r \bmod 1$, where $0 < r \leqslant 1$ describes

the points of X. Let Y be the graph of f and f^* the induced map $X \to Y$. Then f^* satisfies the conditions of the problem. We can now try to compute the integral cohomology group $H^2(Y)$ as the group of homotopy classes of maps $Y \to X$. It seems clear then that $H^1(Y)$ contains a subgroup $Z \oplus Z$ (Z = infinite cyclic group) which corresponds to the group $H^1(X \times X)$ restricted to $Y \subset X \times X$. Hence $H^1(Y)$ is not isomorphic to $H^1(X)$. On the other hand, the homology group $H_1(Y)$, computed by the inverse limit method, seems to be 0 (however, I have not carried out this computation with complete rigor).

A related question is: Do X and Y, satisfying the conditions of the above problem, have the same dimension (in some sense of the word)?

9. Under what conditions is it true that if $f\colon X \to Y$ is almost continuous and $g\colon Y \to Z$ is almost continuous, then the composed map $gf\colon X \to Z$ is almost continuous?

Finally, there is a question, suggested to me by Professor K. Borsuk, which is related to the problem, mentioned in the Introduction, of showing that an acyclic plane continuum has the fixed point property.

10. Let the acyclic continuum C be contained in the interior of the 2-cell D. Is there then an almost continuous function $f\colon D \to D$, such that $f(D) = C$ and such that $f\big(f(d)\big) = f(d)$? The affirmative answer to this question (even if the phrase "almost continuous" is replaced by "polyhedrally almost continuous") would imply, by Proposition 1 and Theorem 3 and the Brouwer Fixed-Point Theorem for D, that every such acyclic plane continuum C has the fixed point property.

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