A note on theories with selectors

by

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It is well known that for certain elementary theories, such as Peano's arithmetic, no gain is made by adjoining to the theory the Hilbert ε-symbol (and the associated new rules of proof). Such theories might be called "theories with built-in Hilbert ε-symbols" or, simply, "theories with selectors". Our purpose in this note is to point out that the (purely syntactical) property of being such a theory is equivalent to a certain semantical property.

Familiarity will be assumed with the introductory sections of our paper, *Natural models of set theories* (this volume, p. 219-242) (1). Specifically, what will be needed is the second part of § 1 (beginning "In metamathematical considerations..."), and § 2 of that paper.

The semantical properties we shall discuss involve the notion of the set $D(\mathfrak{A})$ of all definable elements of a realization $\mathfrak{A}$ of a theory. In addition, the following further notions are required:

DEFINITION 1. *Let* $\mathfrak{A} = \langle A, X_0, \ldots, X_{m-1}\rangle$ *be a realization of an arbitrary standard theory.*

(.1) *If* $D(\mathfrak{A})$ *is not empty, then the corresponding* submodel of definable elements, *or* $\mathfrak{D}(\mathfrak{A})$, *is the subsystem of* $\mathfrak{A}$ *with universe* $D(\mathfrak{A})$.

(.2) *If* B *is any subset of* A, *then by* $D(\mathfrak{A}, B)$ — the set of all elements of A definable in $\mathfrak{A}$ in terms of elements of B — *we mean the union of all sets* $D(\mathfrak{A}^*)$, *where* $\mathfrak{A}^* = \langle A, X_0, \ldots, X_{m-1}, b_0, \ldots, b_{n-1}\rangle$, $n \in \omega$, *and* $b_0, \ldots, b_{n-1} \in B$.

(.3) *Assuming that* $D(\mathfrak{A}, B)$ *is not empty if* B *is empty, then, by* $\mathfrak{D}(\mathfrak{A}, B)$ *we mean the subsystem of* $\mathfrak{A}$ *with universe* $D(\mathfrak{A}, B)$.

(It is obvious that $D(\mathfrak{A})$ and $D(\mathfrak{A}, B)$ are closed under any operations as required in (.1) or (.3).)

We shall actually establish three different equivalences, corresponding to various precise senses that might be given to the notion "theories with selectors". The first is:

(1) Hereinafter referred to as NM. Numbers in brackets will refer to the bibliography of NM.

THEOREM 2. *If T is any standard theory, then the following two conditions are equivalent:*

(i) *for each formula φ of T, whose only free variable is v_0, there is a formula φ' of T, whose only free variable is v_0, such that*

$$\vdash_T \bigvee v_0 \varphi(v_0) \to \bigvee ! v_0 [\varphi'(v_0) \wedge \varphi(v_0)] ;$$

(ii) *for any model $\mathfrak{A}$ of T, $D(\mathfrak{A})$ is non-empty and $\mathfrak{A}$ is an elementary extension of $\mathfrak{D}(\mathfrak{A})$* [2].

Proof. First let us assume that (i) holds and $\mathfrak{A} = \langle A, \ldots \rangle$ is a model of T. Clearly, $D(\mathfrak{A}) \neq 0$. To prove that $\mathfrak{A}$ is an elementary extension of $\mathfrak{D}(\mathfrak{A})$, we will establish that the condition 2.3.1 of NM holds. Indeed, let ψ be a formula of T whose free variables are among $v_0, \ldots, v_n$, and let $d_0, \ldots, d_{n-1}$ be elements of $D(\mathfrak{A})$ such that for some $a \in A$,

(1) $$\models_{\mathfrak{A}} \psi[d_0, \ldots, d_{n-1}, a] .$$

For each $i < n$, let χ_i be a formula of T whose only free variable is v_0 such that

(2) $$\models_{\mathfrak{A}} \bigvee ! v_0 \chi_i$$

and

(3) $$\models_{\mathfrak{A}} \chi_i[d_i] .$$

Let φ be the formula $\bigvee v_1 \ldots \bigvee v_n [\chi_0(v_1) \wedge \ldots \wedge \chi_{n-1}(v_n) \wedge \psi(v_1, \ldots, v_n, v_0)]$. By (1) and (3),

(4) $$\models_{\mathfrak{A}} \bigvee v_0 \varphi(v_0) .$$

By (i), there is a formula φ' whose only free variable is v_0 such that $\models_{\mathfrak{A}} \bigvee v_0 \varphi(v_0) \to \bigvee ! v_0 [\varphi'(v_0) \wedge \varphi(v_0)]$; hence, by (4),

$$\models_{\mathfrak{A}} \bigvee ! v_0 [\varphi'(v_0) \wedge \varphi(v_0)] .$$

Thus there is an element d of $D(\mathfrak{A})$ such that

$$\models_{\mathfrak{A}} \varphi[d] .$$

Therefore, by (2), (3), and the definition of φ,

$$\models_{\mathfrak{A}} \psi[d_0, \ldots, d_{n-1}, d] .$$

This completes the proof that $\mathfrak{A}$ is an elementary extension of $\mathfrak{D}(\mathfrak{A})$.

[2] The fact that (i) implies (ii) was already observed by Tarski a number of years ago. At least in special cases, it is probably known by a number of people. The fact that, actually, equivalence holds seems, however, not to have been noticed previously.

Now let us assume (ii) and, contrary to (i), that φ is a formula of T whose only free variable is v_0 such that

(5) for each formula χ of T whose only free variable is v_0, it is not the case that $\vdash_T \bigvee v_0 \varphi \to \bigvee ! v_0 (\chi \wedge \varphi)$.

Let V be the set of valid sentences of T, and let $\psi_0, \ldots, \psi_n, \ldots$ be all the formulas of T whose only free variable is v_0. We shall show first that

(6) $V \cup \{\bigvee v_0 \varphi, \sim \bigvee ! v_0 (\psi_0 \wedge \varphi), \ldots, \sim \bigvee ! v_0 (\psi_n \wedge \varphi), \ldots\}$ is consistent.

For assume otherwise. Then for some $n \in \omega$,

(7) $$\vdash_T \bigvee v_0 \varphi \to \bigvee ! v_0 (\psi_0 \wedge \varphi) \vee \ldots \vee \bigvee ! v_0 (\psi_n \wedge \varphi) .$$

Let χ be the formula

$$[\bigvee ! v_0 (\psi_0 \wedge \varphi) \wedge \psi_0] \vee [\sim \bigvee ! v_0 (\psi_0 \wedge \varphi) \wedge \bigvee ! v_0 (\psi_1 \wedge \varphi) \wedge \psi_1] \vee \ldots \vee [\sim \bigvee ! v_0 (\psi_0 \wedge \varphi) \wedge \ldots \wedge \sim \bigvee ! v_0 (\psi_{n-1} \wedge \varphi) \wedge \bigvee ! v_0 (\psi_n \wedge \varphi) \wedge \psi_n] .$$

Then the following assertions can be shown to hold on logical grounds alone:

$$\vdash_T \bigvee ! v_0 (\psi_0 \wedge \varphi) \to \bigvee ! v_0 (\chi \wedge \varphi) ,$$

$$\vdash_T \sim \bigvee ! v_0 (\psi_0 \wedge \varphi) \wedge \bigvee ! v_0 (\psi_1 \wedge \varphi) \to \bigvee ! v_0 (\chi \wedge \varphi) .$$

. . . .

$$\vdash_T \sim \bigvee ! v_0 (\psi_0 \wedge \varphi) \wedge \ldots \wedge \sim \bigvee ! v_0 (\psi_{n-1} \wedge \varphi) \wedge \bigvee ! v_0 (\psi_n \wedge \varphi) \to \bigvee ! v_0 (\chi \wedge \varphi) .$$

From these assertions, together with (7), it follows that

$$\vdash_T \bigvee v_0 \varphi \to \bigvee ! v_0 (\chi \wedge \varphi) ;$$

But this contradicts (5), and (6) is established.

From (6) and Gödel's completeness theorem, it follows that

(8) $V \cup \{\bigvee v_0 \varphi, \sim \bigvee ! v_0 (\psi_0 \wedge \varphi), \ldots, \sim \bigvee ! v_0 (\psi_n \wedge \varphi), \ldots\}$ has a model $\mathfrak{A}$.

Since $\models_{\mathfrak{A}} \bigvee v_0 \varphi(v_0)$, there is, by (ii) and the fact that $\mathfrak{A}$ is a model of T, an element d of $D(\mathfrak{A})$ such that

(9) $$\models_{\mathfrak{A}} \varphi[d] .$$

Since $d \in D(\mathfrak{A})$, there is an $n \in \omega$ such that

(10) $$\models_{\mathfrak{A}} \bigvee ! v_0 \psi_n$$

and

(11) $$\models_{\mathfrak{A}} \psi_n[d] .$$

By (9), (10) and (11), we have:

$$\models_{\mathfrak{A}} \bigvee ! v_0(\psi_n \wedge \varphi) .$$

But, by (8),

$$\models_{\mathfrak{A}} \sim \bigvee ! v_0(\psi_n \wedge \varphi) .$$

Thus we have arrived at contradiction, and the theorem is proved.

Examples of theories satisfying condition (i) of 8.2 are well-known. We may mention, for instance, Peano's arithmetic (the theory ***P*** of Tarski, Mostowski, Robinson [1]), and the theory which is the result of adding to ZFS a version of Gödel's Axiom of Constructibility (cf. Gödel [1]). Indeed, both of these theories satisfy a stronger condition, namely

(i″) *For each formula φ of T, having the free variable v_0 (and, possibly, others), there is a formula φ' of T, having the same free variables as φ, and such that*

$$\vdash_T \bigvee v_0 \varphi \to \bigvee ! v_0(\varphi' \wedge \varphi) .$$

Condition (i″) amounts to the requirement that T satisfy both (i) and the following condition:

(i′) *For each formula φ of T, having the free variable v_0 and at least one other, there is a formula φ' of T, having the same free variables as φ, and such that*

$$\vdash_T \bigvee v_0 \varphi \to \bigvee ! v_0(\varphi' \wedge \varphi) .$$

Theorem 2 showed that condition (i) is equivalent to a simple model-theoretical property. Theorems 3 and 4, below, show that the same applies to each of (i′) and (i″).

THEOREM 3. *For any standard theory T, condition* (i′) *above is equivalent to the following condition:*

(ii′) *For any model $\mathfrak{A} = \langle A, \ldots \rangle$ of T and any non-empty set $B \subseteq A$, $\mathfrak{A}$ is an elementary extension of $\mathfrak{D}(\mathfrak{A}, B)$.*

Proof. From the elementary syntactical properties of inessential extensions of theories (cf. e. g. Tarski, Mostowski, Robinson [1], p. 16-17) it follows easily that

(12) *T satisfies condition* (i′) *if and only if every inessential extension of T satisfies condition* (i).

Applying Theorem 2 to each inessential extension of T we see that (i′) holds for T if and only if

(13) *If $\mathfrak{A} = \langle A, \ldots \rangle$ is any model of T and B is any non-empty finite subset of A, then A is an elementary extension of $\mathfrak{D}(\mathfrak{A}, B)$.*

To complete the proof of Theorem 3, it is sufficient to show that (13) is equivalent to condition (ii′). Clearly, (ii′) implies (13). Assume (13), then, and let $\mathfrak{A} = \langle A, \ldots \rangle$ be a model of T and B a non-empty subset of A. To show that $\mathfrak{A}$ is an elementary extension of $\mathfrak{D}(\mathfrak{A}, B)$, we shall demonstrate that 2.3.1 of NM holds. Let $b_0, \ldots, b_{n-1} \in D(\mathfrak{A}, B)$, and assume that φ is a formula of T such that $\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}, a]$. Clearly there is a non-empty finite set $B' \subseteq B$ such that $b_0, \ldots, b_{n-1} \in D(\mathfrak{A}, B')$. By (13) and 2.3 of NM, there is a $y \in D(\mathfrak{A}, B')$ such that $\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}, y]$. But clearly $D(\mathfrak{A}, B') \subseteq D(\mathfrak{A}, B)$, and 2.3.1 of NM is established.

From Theorems 2 and 3 follows immediately

THEOREM 4. *Condition* (i″) *is equivalent to*

(ii″) *For any model $\mathfrak{A}$ of T and any subset B of the universe of $\mathfrak{A}$, $\mathfrak{D}(\mathfrak{A}, B)$ is not empty and $\mathfrak{A}$ is an elementary extension of $\mathfrak{D}(\mathfrak{A}, B)$.*

We have already mentioned two examples of theories satisfying (i″). A familiar theory which satisfies (i′) but not (i) is the theory of the system formed by the set of all integers (positive, negative, or zero), together with their usual ordering. A theory satisfying (i) but not (i′) can also be constructed.

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