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Natural models of set theories

by

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For any set x, the *power set* $P(x)$ and the *sum-set* $\bigcup x$ are, respectively, $\{y \mid y \subseteq x\}$ and $\{y \mid$ for some z, $y \in z$ and $z \in x\}$. Greek letters "α", ..., "δ", "α'", ... will be used to denote ordinals. The sets $R(\alpha)$ are defined recursively, for arbitrary α, by the condition:

$$R(\alpha) = \bigcup \{P(R(\beta)) \mid \beta < \alpha\}\ ^{(1)}.$$

If A is any set, we mean by ϵ_A the set of all ordered couples $\langle x, y\rangle$ such that $x \in A$, $y \in A$, and $x \in y$. For each $\alpha > 0$, the relational system $\langle R(\alpha), \epsilon_{R(\alpha)}\rangle$ will be denoted by $\mathfrak{A}_\alpha$.

Models of axiomatic set theories which are of the form $\mathfrak{A}_\alpha$ may be regarded as *natural models* (cf. Tarski [5]). In particular, it is common in working with Zermelo-Skolem set theory (2), with or without the axiom of infinity, to have in mind the models $\mathfrak{A}_\omega$ or $\mathfrak{A}_{\omega+\omega}$, respectively; and in connection with the Zermelo-Fraenkel-Skolem or von Neumann-Bernays theories to consider the respective models $\mathfrak{A}_\theta$ or $\mathfrak{A}_{\theta+1}$, where θ is the first strongly inaccessible ordinal greater than ω (3).

As was observed in Tarski [5], ω and $\omega+\omega$ are, respectively, the first ordinals α and β such that $\mathfrak{A}_\alpha$ is a model of Zermelo-Skolem set theory without the axiom of infinity and $\mathfrak{A}_\beta$ is a model of that theory with this axiom. It is also known (cf. Shepherdson [1]) that $\theta+1$ is the first ordinal α such that $\mathfrak{A}_\alpha$ is a model of the von Neumann-Bernays axioms. Our principal result in this paper is that, on the other hand, there are ordinals $\theta' < \theta$ such that $\mathfrak{A}_{\theta'}$, is a model of the Zermelo-Fraenkel-

(1) The sets $R(\alpha)$ are the "Stufe" of von Neumann [1].

(2) For an exposition of the various set-theories mentioned in this paragraph, see Wang, Mc Naughton [1]. A precise description of various axioms mentioned, and of the Zermelo-Fraenkel-Skolem and von Neumann-Bernays set theories will be given in § 2 and § 6, below. By the Zermelo-Skolem set theory is meant the theory obtained from the Zermelo-Fraenkel-Skolem theory by substituting Zermelo's axiom schema of *aussonderung* and the axiom of pairs for Fraenkel's axiom schema of replacement.

(3) ω is the first infinite ordinal. The notion of strongly inaccessible ordinal, which was introduced in Tarski [2], will be defined in § 6.

Skolem axioms. (Thus the assertion, which has been sometimes made, that the Zermelo-Fraenkel-Skolem axioms insure the existence of all "accessible ordinals" would seem to be unjustified.)

In establishing this fact we are led to various stronger or more general conclusions and to some other results which may be of interest, but we shall forego now any further summary of what follows [4].

§ 1. Preliminaries. The discussions in this paper will be carried out informally, but in such a way that they could easily be formalized within the axiomatic set theory of Zermelo-Fraenkel-Skolem. It is understood that the axiom of choice is not an axiom of this theory, but that the axiom of regularity (Zermelo's *Fundierungsaxiom*) is. Consequently, we assume in our discussion the informal counterpart of the latter but not the former.

By an *ordinal* we understand a set z such that, for any x and y, if $x \in y \in z$, then $x \in z$, while if $x, y \in z$, then either $x \in y$ or $x = y$ or $y \in x$. Among ordinals we identify $<$ with $\in$; thus each ordinal is the set of all smaller ordinals, and ω, the first infinite ordinal, is the set of all *natural numbers* (denoted by "i", "j", ..., "q"). The *successor* $S(x)$ of a set x is $x \cup \{x\}$. A *limit ordinal* is one not of the form $S(\beta)$. It is known (cf. Robinson [1]) that the usual properties of ordinals follow from these definitions.

To the notions defined in the introduction, we add that of the *rank* $r(x)$ of a set x; $r(x)$ is the first a such that $x \in R(S(a))$ [5].

The notions "r" and "R" are easily seen to have the following properties (cf., e. g., Bernays [1]).

LEMMA 1.1.

(.1) $a < \beta$ *if and only if* $R(a) \subseteq R(\beta)$.

(.2) $r(a) = a$.

(.3) $r(x) = \bigcup \{S(r(y)) \mid y \in x\}$.

(.4) *If* $x \in y \in R(a)$ *or* $x \subseteq y \in R(a)$, *then* $x \in R(a)$.

(.5) $x \in R(a)$ *if and only if* $r(x) < a$.

(.6) $R(S(a)) = P(R(a))$.

(.7) $R(a) = \bigcup \{R(\beta) \mid \beta < a\}$, *if* a *is a limit ordinal.*

(.8) *If* A *is any set of ordinals, then* $\bigcup \{R(a) \mid a \in A\}$ *is* $R(\beta)$, *for some* β.

(.9) $x \subseteq R(r(x))$.

(.10) $x \in R(S(r(x)))$.

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[4] We wish to thank Professor Alfred Tarski for posing problems and stimulating interest in this line of investigation. Many of our results were announced, without proof, in Montague [2], Vaught [1], and Montague, Vaught [1].

[5] That every set belongs to some $R(\beta)$, and hence to some $R(S(\beta))$, is a known consequence of the axiom of regularity. (This result is due to von Neumann and Gödel (cf. Bernays [1], p. 67-68)).

An ordinal a is said to be *confinal* with an ordinal β, if β has a subset X such that $\langle X, \epsilon_X \rangle$ is isomorphic to $\langle a, \epsilon_a \rangle$, while, for any $\gamma < \beta$, there is a $\delta \in X$ with $\gamma \leqslant \delta$. We write $x \sim y$ to mean that the sets x and y are set-theoretically equivalent (equinumerous).

We shall have occasion to speak of $\overline{\overline{x}}$, the *cardinal number* or *power* of x. The treatment of cardinal numbers in publications concerning set theory (e. g., Gödel [1], Tarski [2]) depends either on the axiom of choice or on the introduction of a special symbol and axioms concerning it. Scott, however, has recently remarked (cf. Scott [1]) that under the assumption (which we make) of the axiom of regularity, $\overline{\overline{x}}$ may be defined as follows:

$$(1) \qquad \overline{\overline{x}} = \{y \mid y \sim x \textit{ and, for any } z, \textit{ if } z \sim x, \textit{ then } r(y) \leqslant r(z)\}.$$

(The set so defined "exists", since, if a is the smallest ordinal such that for some y, $y \sim x$ and $r(y) = a$, then it coincides with the set $\{y \mid y \in R(S(a))$ and $y \sim x\}$.) It follows that

$$(2) \qquad \overline{\overline{x}} = \overline{\overline{w}} \quad \textit{if and only if} \quad x \sim w,$$

and hence that (1) (which we adopt) is an adequate definition.

In metamathematical considerations we shall employ the terminology of Tarski, Mostowski, Robinson [1]. Thus, by a *theory with standard formalization*, or simply a *standard theory*, we understand a theory formalized in the first order logic with identity. The logical constants are the symbols $\vee$, $\sim$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\bigwedge$, $\bigvee$, and $=$. The only variables are individual variables, assumed arranged in a given non-repeating sequence $v_0, \ldots, v_n, \ldots$ We assume a theory T has finitely many non-logical constants, arranged in a given order $\xi_0, \ldots, \xi_{m-1}$. (A non-logical constant is either an individual constant, an n-place predicate, or an n-place operation symbol, for some positive integer n.) Each theory T then possesses definite sets of *symbols*, *terms*, and *formulas*, as well as *constant terms* and *sentences* (i. e., terms or formulas with no free variables). In addition, a theory T has given a set of its sentences, called the *valid* sentences of T, and such that any sentence of the theory which is logically derivable from valid sentences is itself valid. A set K of sentences of a theory T is called a set of *axioms* for the theory if the valid sentences of T are exactly the logical consequences of K. An arbitrary formula φ of T is called *valid* in T, and we write

$$\vdash_T \varphi,$$

provided the *closure* of φ (i. e., the sentence $\bigwedge \mathrm{v}_{k_0} \ldots \bigwedge \mathrm{v}_{k_{n-1}} \varphi$ (6), where $k_0 < k_1 < \ldots < k_{k-1}$, and $\mathrm{v}_{k_0}, \ldots, \mathrm{v}_{k_{n-1}}$ are the free variables of φ) is valid in T.

If ψ is a formula, we denote by $\bigvee !\mathrm{v}_n \psi$ the formula $\bigvee \mathrm{v}_k \bigwedge \mathrm{v}_n(\psi \leftrightarrow \mathrm{v}_k = \mathrm{v}_n)$, where v_k is the first variable (in the sequence $\mathrm{v}_0, \mathrm{v}_1, \ldots$) distinct from v_n and not free in ψ.

If we are given a standard theory T and a formula ψ of T such that v_0 is the only free variable of ψ and $\vdash_T \bigvee !\mathrm{v}_0 \psi$, then we may form the corresponding *iota-theory* $T^{(\psi)}$ as follows: The symbols of $T^{(\psi)}$ are those of T together with the new symbol ι. The sets of terms and formulas of $T^{(\psi)}$ are defined by the same recursive conditions as in the case of a standard theory, with the added condition that $\iota \mathrm{v}_n \varphi$ is a term of $T^{(\psi)}$ whenever φ is a formula of $T^{(\psi)}$. The valid sentences of $T^{(\psi)}$ are the logical consequences of the set consisting of the valid sentences of T together with all closures of formulas of the form

(3) $$\mathrm{v}_k = \iota \mathrm{v}_n \varphi \leftrightarrow \bigwedge \mathrm{v}_n(\varphi \leftrightarrow \mathrm{v}_n = \mathrm{v}_k) \vee \big(\sim \bigvee !\mathrm{v}_n \varphi \wedge \bigwedge \mathrm{v}_0(\mathrm{v}_0 = \mathrm{v}_k \to \psi)\big)$$

where φ is a formula of $T^{(\psi)}$ and v_k is a variable distinct from v_n and v_0 and not free in φ.

It is well-known (cf., e. g., Hilbert-Bernays [1]) that with each formula φ of $T^{(\psi)}$ may be correlated a formula φ^* of T having the same free variables as φ and fulfilling the condition:

(4) $$\vdash_{T^{(\psi)}} \varphi^* \leftrightarrow \varphi .$$

Given a formula or term φ and arbitrary terms $\tau_0, \ldots, \tau_{n-1}$ we understand by

$$\varphi(\tau_0, \ldots, \tau_{n-1})$$

the formula obtained from φ by the proper simultaneous substitution of τ_0 for $\mathrm{v}_0, \ldots, \tau_{n-1}$ for v_{n-1} (7).

By an *inessential extension* of a standard theory T we mean a standard theory T' whose non-logical constants comprise, in addition to the non-logical constants $\xi_0, \ldots, \xi_{m-1}$ of T, some new individual constants $\xi_m, \xi_{m+1}, \ldots, \xi_{m+n}$, and whose valid sentences are all the logical consequences (in T') of the valid sentences of T.

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(6) Concatenation of expressions is denoted by juxtaposition.

(7) That is, let φ' be the first formula or term (in a fixed enumeration of all expressions) such that no variable free in any of $\tau_0, \ldots, \tau_{n-1}$ is bound in φ and φ' is an alphabetic variant of φ (in the sense of Quine [1] — with suitable modifications when ι is present); and let $\varphi(\tau_0, \ldots, \tau_{n-1})$ be the result of replacing simultaneously in φ' all free occurrences of v_i by τ_i, for $i < n$.

By a *realization* of a standard theory T with the list $\xi_0, \ldots, \xi_{m-1}$ of non-logical constants is meant any *(algebraic) system* $\mathfrak{A} = \langle A, X_0, \ldots, X_{m-1} \rangle$, where A is a non-empty set, and each X_i is a distinguished element, a q-ary relation among the elements of A or a q-ary operation on A, according as ξ_i is an individual constant, q-placed predicate, or q-placed operation symbol. The set A is called the *universe* of $\mathfrak{A}$. The notions of *satisfaction* of a formula φ of T and *value* of a term ψ of T in $\mathfrak{A}$ for an infinite sequence $\langle a_0, \ldots, a_n, \ldots \rangle$ of elements of A are introduced by the usual recursive conditions (cf. Tarski [1]), with the understanding that for each n, a_n is to be "assigned" to the variable v_n. If the free variables of φ or τ are at most $\mathrm{v}_0, \ldots, \mathrm{v}_{k-1}$, then we may, under the same conditions, say that $\langle a_0, \ldots, a_{k-1} \rangle$ satisfies φ in $\mathfrak{A}$, abbreviated

(5) $$\models_{\mathfrak{A}} \varphi[a_0, \ldots, a_{k-1}]$$

or speak of the value of τ in $\mathfrak{A}$ at $\langle a_0, \ldots, a_{k-1} \rangle$, abbreviated

(6) $$\tau^{\mathfrak{A}}[a_0, \ldots, a_{k-1}] .$$

By a realization of an iota-theory $T^{(\psi)}$ is means any realization of T in which $\bigvee !\mathrm{v}_0 \psi$ is satisfied. The notions of satisfaction and value for formulas and terms of $T^{(\psi)}$ are defined by the usual recursive conditions augmented by the condition

The value in $\mathfrak{A}$ of $\iota \mathrm{v}_k \varphi$ (where φ is any formula of $T^{(\psi)}$) at $\langle a_0, \ldots, a_n, \ldots \rangle$ is the unique b such that φ is satisfied in $\mathfrak{A}$ by $\langle a_0, \ldots, a_{k-1}, b, a_{k+1}, \ldots \rangle$, if there is such a b, and, if not, is the unique z such that $\models_{\mathfrak{A}} \psi[z]$.

Abbreviations (5) and (6) will also be applied when φ or ψ is a formula or term of $T^{(\psi)}$.

A sentence (of a standard or iota-theory) is said to be *true* in a realization $\mathfrak{A}$ of the theory, if it is satisfied in $\mathfrak{A}$ by any sequence of elements of $\mathfrak{A}$ (8). A realization $\mathfrak{A}$ of a theory T is called a *model* of a set Q of sentences of T if all sentences belonging to Q are true in $\mathfrak{A}$; $\mathfrak{A}$ is a *model* of T if it is a model of the set of valid sentences of T. To each realization $\mathfrak{A}$ of a theory T corresponds a particular standard theory, called $Th(\mathfrak{A})$, whose valid sentences are all sentences of T true in $\mathfrak{A}$.

An element a of a realization $\mathfrak{A}$ of a standard theory T is called *definable in* $\mathfrak{A}$ if there is a formula φ of T, whose only free variable is v_0, such that $\models_{\mathfrak{A}} \bigvee !\mathrm{v}_0 \varphi$ and $\models_{\mathfrak{A}} \varphi[a]$. We denote by $D(\mathfrak{A})$ the set of all definable elements of $\mathfrak{A}$.

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(8) Given a system $\mathfrak{A} = \langle A, \ldots \rangle$, we shall speak of "elements of $\mathfrak{A}$" to mean "elements of A". Similar remarks will apply to the phrases "$\mathfrak{A}$ is denumerable" and "$\mathfrak{A}$ is infinite".

The following facts are easily established:

LEMMA 1.2. *Suppose that* $\mathfrak{A}$ *is a model of a standard theory* T, ψ *is a formula of* T *whose only free variable is* v_0, *and* $\vdash_T \bigvee ! v_0 \psi$. *Then:*

(.1) $\mathfrak{A}$ *is also a model of* $T^{(\psi)}$.

(.2) *An element* a *is definable in* $\mathfrak{A}$ *if and only if there is a constant term* τ *of* $T^{(\psi)}$ *such that* $\tau^{(\mathfrak{A})} = a$.

§ 2. Elementary extensions.

Throughout this section we suppose that T is a fixed but arbitrary standard theory.

A realization $\mathfrak{B} = \langle B, X_0', \ldots, X_{m-1}' \rangle$ of T is called a *subsystem* of a realization $\mathfrak{A} = \langle A, X_0, \ldots, X_{m-1} \rangle$ of T (and A is called an *extension* of $\mathfrak{B}$) provided that (i) $B \subseteq A$, and, for each $i < m$, (ii) if X_i is a relation, X_i' is obtained by restricting X_i to B, (iii) if X_i is an operation, B is closed under the operation X_i', and (iv) if X_i is a distinguished element, then $X_i = X_i' \in B$.

Tarski [3] calls any realizations $\mathfrak{A}$ and $\mathfrak{B}$ of T *elementarily* (or arithmetically) *equivalent* if the same sentences (of T) are true in $\mathfrak{A}$ that are true in $\mathfrak{B}$.

A stronger notion is that of *elementary extensionality*, first introduced by Tarski (cf. Tarski-Vaught [1]). A realization $\mathfrak{A}$ of T is called an *elementary extension* of a realization $\mathfrak{B}$ of T if (i) $\mathfrak{A}$ is an extension of $\mathfrak{B}$, and (ii) given any $n \in \omega$, any formula φ of T whose free variables are at most $v_0, \ldots, v_{n-1}$, and any elements $b_0, \ldots, b_{n-1}$ of $\mathfrak{B}$,

$$(1) \qquad \models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}] \quad \text{if and only if} \quad \models_{\mathfrak{B}} \varphi[b_0, \ldots, b_{n-1}].$$

This notion was used by Tarski (cf. Tarski-Vaught [1]) in formulating and proving the following version of the Löwenheim-Skolem theorem ([9]):

THEOREM 2.1. *If* $\mathfrak{A}$ *is a realization of* T, *and the universe of* $\mathfrak{A}$ *is infinite and can be well-ordered, then* $\mathfrak{A}$ *has a denumerable subsystem of which it is an elementary extension.*

A simple property of elementary extensions is formulated in

LEMMA 2.2. *If* $\mathfrak{A}$ *is an elementary extension of* $\mathfrak{B}$, *then* $D(\mathfrak{A})$, *the set of definable elements of* $\mathfrak{A}$, *coincides with* $D(\mathfrak{B})$.

Proof. If $x \in D(\mathfrak{A})$, then there is a formula φ of the theory T (of which, we may suppose, $\mathfrak{A}$ and $\mathfrak{B}$ are realizations), whose only free variable is v_0, such that $\models_{\mathfrak{A}} \varphi[x]$ and $\models_{\mathfrak{A}} \bigvee ! v_0 \varphi$. By (1) (with $n = 0$), we have, also, $\models_{\mathfrak{B}} \bigvee ! v_0 \varphi$. Therefore, there is an element y of $\mathfrak{B}$ such that $\models_{\mathfrak{B}} \varphi[y]$. Consequently, by (1), $\models_{\mathfrak{A}} \varphi[y]$, and hence, clearly, $x = y \in D(\mathfrak{B})$. Thus, $D(\mathfrak{A}) \subseteq D(\mathfrak{B})$. The proof that $D(\mathfrak{B}) \subseteq D(\mathfrak{A})$ is similar, though even simpler.

([9]) In Tarski-Vaught [1], the axiom of choice is assumed, but the proof given establishes 2.1 as stated (cf. Mostowski [2], p. 163).

The following lemma, which we shall use in establishing our main result, is due to Tarski (cf. Tarski-Vaught [1]):

LEMMA 2.3. *Let* $\mathfrak{A} = \langle A, \ldots \rangle$ *be a realization of* T *and* $B \subseteq A$. *In order that* $\mathfrak{A}$ *have a subsystem with universe* B *of which* $\mathfrak{A}$ *is an elementary extension, it is necessary and sufficient that:*

(.1) *for any* $n \in \omega$, *any formula* φ *of* T *whose free variables are among* $v_0, \ldots, v_n$, *and any elements* $b_0, \ldots, b_{n-1}$ *of* B, *if there is an element* x *of* $\mathfrak{A}$ *such that* $\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}, x]$, *then there is also an element* y *of* B *such that* $\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}, y]$.

2.4. Remark. Suppose that a model $\mathfrak{A}$ of T is an elementary extension of a model $\mathfrak{B}$, and that ψ is a formula of T, whose only free variable is v_0, such that $\vdash_T \bigvee ! v_0 \psi(v_0)$. Then, using 1.2.1, and (4) of § 1, one easily shows that: *Condition* (1) *of* § 2 *holds for arbitrary formulas of* $T^{(\psi)}$; *moreover, if* τ *is any term of* $T^{(\psi)}$, *with at most the free variables* $v_0, \ldots, v_{n-1}$, *and* $b_0, \ldots, b_{n-1}$ *are any elements of* $\mathfrak{B}$, *then* $\tau^{\mathfrak{A}}[b_0, \ldots, b_{n-1}] = \tau^{\mathfrak{B}}[b_0, \ldots, b_{n-1}]$.

§ 3. Zermelo-Fraenkel-Skolem set theory.

To increase readability, we agree henceforth that x, y, z, u, v, w are, in order, the particular variables $v_0, \ldots, v_5$.

DEFINITION 3.1. By ZFS *(Zermelo-Fraenkel-Skolem set theory)* we understand the standard theory, whose only non-logical constant is the binary predicate ϵ, based on the axioms which are the closures of the following formulas:

Extensionality:	$\bigwedge z(z \epsilon x \leftrightarrow z \epsilon y) \rightarrow x = y$.
Union:	$\bigvee y \bigwedge z[z \epsilon y \leftrightarrow \bigvee u(z \epsilon u \wedge u \epsilon x)]$.
Power set:	$\bigvee y \bigwedge z[z \epsilon y \leftrightarrow \bigwedge u(u \epsilon z \rightarrow u \epsilon x)]$.
Infinity:	$\bigvee y\big(\bigvee z[z \epsilon y \wedge \bigwedge x \sim x \epsilon z] \wedge \bigwedge z\big[z \epsilon y \rightarrow \bigvee u(u \epsilon y \wedge \bigwedge v[v \epsilon u \leftrightarrow v \epsilon z \vee v = z])\big]\big)$.
Regularity:	$\bigvee y\, y \epsilon x \rightarrow \bigvee y[y \epsilon x \wedge \sim \bigvee z(z \epsilon y \wedge z \epsilon x)]$.
Replacement:	All formulas of the form: $\bigwedge x \bigvee ! y \varphi \rightarrow \bigvee v \bigwedge y[y \epsilon v \leftrightarrow \bigvee x(x \epsilon u \wedge \varphi)]$, where φ is a formula of ZFS in which v is not free.

Though we do not include it in the axioms of ZFS we shall frequently refer to the *axiom of choice*, whose formal version is understood to be the closure of the formula:

$$\bigwedge y(y \epsilon x \rightarrow \bigvee u(u \epsilon y)) \wedge \bigwedge y \bigwedge z(y \epsilon x \wedge z \epsilon x \wedge \sim y = z \rightarrow \sim \bigvee u(u \epsilon y \wedge u \epsilon z)) \rightarrow \bigvee v \bigwedge y(y \epsilon x \rightarrow \bigvee ! w(w \epsilon v \wedge w \epsilon y)).$$

DEFINITION 3.2. $\boldsymbol{Z}$ is the formula $\bigwedge y \sim y \in x$.

LEMMA 3.3. $\vdash_{\mathrm{ZFS}} \bigvee ! x \boldsymbol{Z}(x)$.

It is necessary to sketch the development of a small part of the theory ZFS, or rather of the more convenient iota-theory $\mathrm{ZFS}^{(\boldsymbol{Z})}$, which, by 3.3, can be formed. Because the following lemmas are well-known, they will not be proved here. For brevity, we write "$\vdash$" in place of "$\vdash_{\mathrm{ZFS}^{(\boldsymbol{Z})}}$" in the remainder of this section.

DEFINITION 3.4. $\boldsymbol{I}$ is the formula $\bigwedge z(z \in x \to z \in y)$.

DEFINITION 3.5. $\boldsymbol{S}$ is the term $\iota y \bigwedge z(z \in y \leftrightarrow z \in x \vee z = x)$.

LEMMA 3.6. $\vdash z \in \boldsymbol{S}(x) \leftrightarrow z \in x \vee z = x$.

DEFINITION 3.7. $\boldsymbol{U}$ is the term $\iota y \bigwedge z[z \in y \leftrightarrow \bigvee u(z \in u \wedge u \in x)]$.

LEMMA 3.8. $\vdash z \in \boldsymbol{U}(x) \leftrightarrow \bigvee u(z \in u \wedge u \in x)$.

DEFINITION 3.9. **Ord** is the formula $\bigwedge y \bigwedge z(y \in z \wedge z \in x \to y \in x) \wedge \bigwedge y \bigwedge z(y \in x \wedge z \in x \to y \in z \vee y = z \vee z \in y)$.

DEFINITION 3.10. $\boldsymbol{P}$ is the term $\iota y \bigwedge z\big(z \in y \leftrightarrow \boldsymbol{I}(z, x)\big)$.

LEMMA 3.11. $\vdash z \in \boldsymbol{P}(x) \leftrightarrow \boldsymbol{I}(z, x)$.

DEFINITION 3.12. $\boldsymbol{R}$ is the first term [10] whose only free variable is x such that

$$\vdash \mathbf{Ord}(x) \to \boldsymbol{R}(x) = \boldsymbol{U}\Big(\iota u \bigwedge z\big[z \in u \leftrightarrow \bigvee y\big[y \in x \wedge z = \boldsymbol{P}\big(\boldsymbol{R}(y)\big)\big]\big]\Big).$$

LEMMA 3.13. $\vdash \mathbf{Ord}(x) \to \Big(v \in \boldsymbol{R}(x) \leftrightarrow \bigvee y\big[y \in x \wedge v \in \boldsymbol{P}\big(\boldsymbol{R}(y)\big)\big]\Big)$.

DEFINITION 3.14. $\boldsymbol{r}$ is the term $\iota y\big[\mathbf{Ord}(y) \wedge \boldsymbol{I}\big(x, \boldsymbol{R}(y)\big) \wedge \sim x \in \boldsymbol{R}(y)\big]$.

LEMMA 3.15. $\vdash y = \boldsymbol{r}(x) \leftrightarrow \mathbf{Ord}(y) \wedge \boldsymbol{I}\big(x, \boldsymbol{R}(y)\big) \wedge \sim x \in \boldsymbol{R}(y)$.

DEFINITION 3.16. The terms $\boldsymbol{\Delta}_n$ are defined recursively by the conditions: $\boldsymbol{\Delta}_0 = \iota x \boldsymbol{Z}(x)$, and $\boldsymbol{\Delta}_{n+1} = \boldsymbol{S}(\boldsymbol{\Delta}_n)$.

It will be useful to note here the following easily provable generalization of the axiom schema of replacement:

LEMMA 3.17. *Let T be either* ZFS *or an inessential extension of it. Let $\tau_0, \ldots, \tau_{n-1}$ be constant terms of $T^{(\boldsymbol{Z})}$ and τ a term of $T^{(\boldsymbol{Z})}$ with at most the free variables $v_0, \ldots, v_{n-1}$. Then*

$$(.1)\quad \vdash_{T^{(\boldsymbol{Z})}} \bigvee ! v_{n+1} \bigwedge v_n \Big(v_n \in v_{n+1} \leftrightarrow \bigvee v_0 \ldots \bigvee v_{n-1} \big(v_0 \in \tau_0 \wedge \ldots \wedge v_{n-1} \in \tau_{n-1} \wedge v_n = \tau(v_0, \ldots, v_{n-1})\big)\Big).$$

By a *set theory* we shall mean any (standard) theory whose only non-logical constant is a binary predicate, say $\in$. As Tarski has remarked

[10] The existence (and a specific method of construction) of such a term $\boldsymbol{R}$ is well-known, in view of general results about the possibility of representing recursive definitions explicitly in $\mathrm{ZFS}^{(\boldsymbol{Z})}$; cf., e. g., Gödel [1].

(cf. Mostowski [1], p. 208), the theory of von Neumann-Bernays (as presented in Gödel [1], and, specifically, with the axioms of groups A, B, C, and D, but not E, given there) is easily converted into such a theory, which we shall call VNB [11].

We shall study *natural models* of set theories, i. e., models of the form $\mathfrak{A}_a$. A related concept, that of *supercomplete models*, was introduced in Shepherdson [1] [12]. A *complete* model of a set theory is a model of the form $\langle A, \in_A \rangle$, where A has the property that any member of A is a subset of A. A complete model $\langle A, \in_A \rangle$ with the additional property that any subset of a member of A is a member of A is called a *supercomplete model*. A natural model is of course supercomplete (by 1.1.4), but the converse need not be true. However, Shepherdson [1] showed that (in our terminology) the concepts "natural" and „supercomplete" coincide for models of VNB, and we will note later (at the end of § 4) that the same equivalence holds for models of ZFS.

In three places later on (5.4, 5.5, and § 7) it will become essential that our concept of the theory ZFS be given a more complete specification than that which is accomplished by 2.1 and the general discussion of theories in § 1. Let us agree, then, that the symbols $\wedge, \sim, \vee, \to, \longleftrightarrow, \bigwedge, \bigvee, =, \in, v_0, v_1, \ldots$ of the theory ZFS are, in that order, the numbers $2^1, 2^2, 2^3, 2^4, \ldots$ The expressions of ZFS are all natural numbers of the form $p_0^{k_0} \cdot p_1^{k_1} \cdot \ldots \cdot p_n^{k_n}$, where $k_0, \ldots, k_n > 0$ and $p_0, p_1, \ldots$ are the primes, in order. The concatenation of the expressions $p_0^{k_0} \cdot \ldots \cdot p_n^{k_n}$ and $p_0^{l_0} \cdot \ldots \cdot p_q^{l_q}$ is the expression $p_0^{k_0} \cdot \ldots \cdot p_n^{k_n} \cdot p_{n+1}^{l_0} \cdot \ldots \cdot p_{n+q+1}^{l_q}$. The notions of atomic terms and formulas, terms, formulas, etc., of ZFS are then understood in the ordinary way (using the so-called parenthesis free notation).

§ 4. Absoluteness of some notions.

Lemma 4.1, which follows, presents some invariance properties of the special formulas and terms defined in § 2, analogous to the notion of "absoluteness" of Gödel [1]. For models of VNB, the results of 4.1 were established in Shepherdson [1] [13].

LEMMA 4.1. *Suppose that $\mathfrak{A} = \langle A, \in_A \rangle$ is a natural model of* ZFS, *and $x, y \in A$. Then*:

[11] To be somewhat more precise, VNB is the set theory whose axioms are obtained from those of groups A, B, C, and D of Gödel [1] as follows (roughly speaking): First, reduce to only one kind of variable, in the usual way — by replacing $\bigvee X\varphi(X)$ by $\bigvee x(\mathcal{M}(x) \wedge \varphi(x))$, etc. Then, replace any occurrence of $\mathcal{C}ls(x)$ by $x = x$ and of $\mathcal{M}(x)$ by $\bigvee y(x \in y)$.

[12] Shepherdson considered only models of VNB (possibly without axiom D); for general set theories, the notion was introduced in Tarski [5].

[13] Our 4.1 follows from the ideas in Shepherdson's proofs. (However, 4.1.6 and 4.1.7 cannot be derived directly from his stated results. This is because, while any natural (or supercomplete) model of VNB gives rise in a simple way to a natural model of ZFS, the converse is not the case, as we shall see later). Therefore we shall only sketch the proof.

(.1) $\models_{\mathfrak{A}} \boldsymbol{Z}[x]$ *if and only if* $x = 0$.
(.2) $\models_{\mathfrak{A}} \boldsymbol{I}[x, y]$ *if and only if* $x \subseteq y$.
(.3) $\boldsymbol{S}^{(\mathfrak{A})}[x] = S(x)$.
(.4) $\boldsymbol{U}^{(\mathfrak{A})}[x] = \bigcup x$.
(.5) $\models_{\mathfrak{A}} \mathbf{Ord}[x]$ *if and only if x is an ordinal.*
(.6) $\boldsymbol{P}^{(\mathfrak{A})}[x] = P(x)$.
(.7) *If x is an ordinal, then* $\mathbf{R}^{(\mathfrak{A})}[x] = R(x)$.
(.8) $\boldsymbol{r}^{(\mathfrak{A})}[x] = r(x)$.
(.9) *For each* $n \in \omega$, $\Delta_n^{(\mathfrak{A})} = n$.

Proof. To show (.1), suppose that $\models_{\mathfrak{A}} \boldsymbol{Z}[x]$. Then, by 3.2, for any $y \in A$, $y \notin x$. But, by 1.1.4, for any y whatever, if $y \in x$, then $y \in A$, so that, in fact, $y \notin x$. Thus $x = 0$. In the other direction, (.1) is obvious.

Each of (.2)-(.6) is proved in a similar way, using 3.4, 3.6, 3.8, 3.9, or 3.11, and, possibly, preceding parts of 4.1. (.7) is proved by transfinite induction on x, as follows: Assume that y is an ordinal and (.7) holds for each ordinal $x < y$. Then, by 3.13 and (.5), for any $z \in A$, $z \in \boldsymbol{R}^{(\mathfrak{A})}[y]$ if and only if there is an $x \in A$ such that $x \in y$ and $z \in \boldsymbol{P}^{(\mathfrak{A})}[\boldsymbol{R}^{(\mathfrak{A})}[x]]$. By 1.1.4, (.6), and the inductive assumption, it follows that, for any z, $z \in \boldsymbol{R}^{(\mathfrak{A})}[y]$ if and only if, for some $x \in y$, $z \in P(R(x))$. Therefore, by the definition of $R(y)$, $\mathbf{R}^{(\mathfrak{A})}[y] = R(y)$, i. e., (.7) holds for y.

(.8) may now be obtained, using (.2), (.5), (.7), and 3.15; and (.9) can be derived by ordinary induction, using 3.16, (.1), and (.3).

Exactly the same arguments show that

(1) 4.1 *also holds if "natural model of* ZFS" *is replaced by "supercomplete model of* ZFS" ([14]).

From (1), we easily derive the result, already mentioned, that *any supercomplete model* $\mathfrak{A} = \langle A, \epsilon_A \rangle$ *of* ZFS *is a natural model.* Indeed, by (1) and 1.1.10, we see that, for any $x \in A$, $x \in R(S(r(x))) = \mathbf{R}^{(\mathfrak{A})}[\boldsymbol{S}^{(\mathfrak{A})}[\boldsymbol{r}^{(\mathfrak{A})}[x]]] \in A$; hence, $\mathfrak{A}$ being complete, $A = \bigcup \{R(S(r(x))) \mid x \in \mathrm{A}\}$, and so, by 1.1.8, A is of the form $R(a)$.

We may also mention here the fact that *if a natural model $\mathfrak{A}$ of a set theory is an elementary extension of a complete model $\mathfrak{B}$, then $\mathfrak{B}$ is a natural model.* This is shown by the argument just given, with Remark 2.4 used as justification, in place of (1).

([14]) By making some simple changes in the arguments, one sees that 4.1-4.9, with the exception of 4.6 and 4.7, hold for all complete models of ZFS (cf. Shepherdson [1]). (To obtain 4.8 in this case, one uses the fact that the formal counterpart of 1.1.3 is a theorem of ZFS.)

On the other hand, it is also easily seen that all of 4.1-4.9 hold in any system $\mathfrak{A}_a$, where a is a limit ordinal (cf. Scott [2]). (To make this statement correct, one must assume that the explicit definition of $\boldsymbol{R}$ (cf. 3.12) is made in the usual way.)

§ 5. Inner natural models.

The following lemma, which plays an essential role in our discussion, is an immediate consequence of proposition (II) in Tarski [4].

LEMMA 5.1. *Let T be the theory* ZFS *or an inessential extension of it. With each formula φ of $T^{(\boldsymbol{Z})}$, having at most the free variables $\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}, \mathrm{v}_n$, can be correlated a term τ of $T^{(\boldsymbol{Z})}$ with no free variables beyond $\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}$, such that:*

$$\vdash_{T^{(\boldsymbol{Z})}} \bigvee \mathrm{v}_n \varphi(\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}, \mathrm{v}_n) \to \bigvee \mathrm{v}_n \big(\varphi(\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}, \mathrm{v}_n) \wedge \mathrm{v}_n \boldsymbol{\epsilon}\, \tau(\mathrm{v}_0, \ldots, \mathrm{v}_{n-1})\big).$$

Proof. For τ may be taken the term

$$\iota \mathrm{v}_{n+1} \bigwedge \mathrm{v}_n \big(\mathrm{v}_n \epsilon \mathrm{v}_{n+1} \leftrightarrow \varphi(\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}, \mathrm{v}_n) \wedge \\ \wedge \bigwedge \mathrm{v}_{n+2} \big(\varphi(\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}, \mathrm{v}_{n+2}) \to \sim \boldsymbol{r}(\mathrm{v}_{n+2})\, \boldsymbol{\epsilon}\, \boldsymbol{r}(\mathrm{v}_n)\big)\big).$$

We omit the details of the proof, which are suggested by (the formal counterpart of) the discussion following (1) of § 1 on Scott's definition of cardinal number.

Note that τ, as just defined, has the additional property that $\vdash_{T^{(\boldsymbol{Z})}} \mathrm{v}_n \epsilon \tau \to \varphi$ (cf. Tarski [4], Scott [1]). This will play no role in our discussion — though its informal counterpart is critical for the derivation of (2) of § 1.

In theorems 5.2, 5.3, 6.8, and 6.9, below, we will see that, under certain circumstances, a natural model $\mathfrak{A}$ of ZFS is an elementary extension of a smaller natural model $\mathfrak{B}$. Speaking roughly, this may be shown as follows: In proofs of the Löwenheim-Skolem Theorem 2.1 (such as that in Tarski-Vaught [1]), the universe B of the submodel. of $\mathfrak{A}$ is built up in an infinite succession of steps, in each of which one chooses, and puts in B, an element having a certain property. Now follow the same procedure, but, at each step, let γ be the smallest ordinal such that some $x \in R(\gamma)$ has the property in question, and put all elements of $R(\gamma)$ in B.

Following this idea, proofs of 5.2, 5.3, 6.8, and 6.9 can be constructed However, we shall present, instead, a somewhat different argument, perhaps slightly shorter, and having the feature that the set B will be defined at once, explicitly, rather than by an inductive procedure.

THEOREM 5.2. *Suppose that* $\mathfrak{A} = \langle A, E \rangle$ *is a model of* ZFS. Then:

(.1) *$\mathfrak{A}$ is an elementary extension of its subsystem with universe* $B = \{x \mid$ *for some* $y, x\, E\, y$ *and* $y \in D(\mathfrak{A})\}$.

(.2) *Moreover, if a is any element of A,* $\mathfrak{A}^* = \langle A, E, a \rangle$, *and* $B^* = \{x \mid$ *for some* y, $x\, E\, y$ *and* $y \in D(\mathfrak{A}^*)\}$, *then $a \in B^*$ and $\mathfrak{A}^*$ is an elementary extension of its subsystem with universe B^*.*

Proof. To establish (.1), we shall demonstrate that the necessary and sufficient condition 2.3.1 holds. Suppose, then, that φ is a formula of ZFS whose free variables are among $v_0, \ldots, v_n$, that $b_0, \ldots, b_{n-1} \in B$, and that x is an element of $\mathfrak{A}$ such that

(1) $$\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}, x].$$

By the definition of B, there exist $d_0, \ldots, d_{n-1}$ in $D(\mathfrak{A})$ such that, for all $i < n$,

(2) $$b_i E d_i.$$

By 1.2.2, there are constant terms $\tau_0, \ldots, \tau_{n-1}$ of $\mathrm{ZFS}^{(Z)}$ such that, for all $i < n$,

(3) $$\tau_i^{(\mathfrak{A})} = d_i.$$

By 5.1, there is a term τ of $\mathrm{ZFS}^{(Z)}$ containing no free variables beyond $v_0, \ldots, v_{n-1}$ such that

(4) $$\vdash_{\mathrm{ZFS}^{(Z)}} \bigvee v_n \varphi(v_0, \ldots, v_{n-1}, v_n) \rightarrow \bigvee v_n[\varphi(v_0, \ldots, v_{n-1}, v_n) \wedge v_n \in \tau(v_0, \ldots, v_{n-1})].$$

Let

$$\tau' = \iota v_{n+1} \bigwedge v_n \big(v_n \in v_{n+1} \leftrightarrow \bigvee v_0 \ldots \bigvee v_{n-1} [v_0 \in \tau_0 \wedge \ldots \wedge v_{n-1} \in \tau_{n-1} \wedge v_n = \tau(v_0, \ldots, v_{n-1})]\big).$$

By 3.17 and (3) of § 1, it follows that

$$\vdash_{\mathrm{ZFS}^{(Z)}} v_0 \in \tau_0 \wedge \ldots \wedge v_{n-1} \in \tau_{n-1} \rightarrow \tau(v_0, \ldots, v_{n-1}) \in \tau'.$$

Hence, by (4) and 3.8, letting τ'' be the term $\boldsymbol{U}(\tau')$,

(5) $$\vdash_{\mathrm{ZFS}^{(Z)}} v_0 \in \tau_0 \wedge \ldots \wedge v_{n-1} \in \tau_{n-1} \wedge \bigvee v_n \varphi(v_0, \ldots, v_{n-1}, v_n) \rightarrow \bigvee v_n[\varphi(v_0, \ldots, v_{n-1}, v_n) \wedge v_n \in \tau''].$$

By (2), (3), and (1), the antecedent of (5) is satisfied in $\mathfrak{A}$ by $\langle b_0, \ldots, b_{n-1} \rangle$. Therefore the consequent of (5) is also satisfied by $\langle b_0, \ldots, b_{n-1} \rangle$. Thus there is an element y of $\mathfrak{A}$ such that

(6) $$\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}, y]$$

and

(7) $$y E \tau''^{(\mathfrak{A})}.$$

By 1.2.2,

$$\tau''^{(\mathfrak{A})} \in D(\mathfrak{A});$$

hence, by (7),

(8) $$y \in B.$$

From (6) and (8), we see that the condition 2.3.1 holds, demonstrating (.1) [15].

Let T be an inessential extension of ZFS, with one new constant. Then exactly the same arguments with „ZFS", "$\mathrm{ZFS}^{(Z)}$", "$\mathfrak{A}_a$", and "B" replaced by "T", "$T^{(Z)}$", "$\mathfrak{A}_a^*$", and "B^*", establishes (.2).

THEOREM 5.3. *Suppose that $\mathfrak{A}_a$ is a model of ZFS. Then:*

(.1) *The union B of all definable elements of $\mathfrak{A}_a$ is $R(\beta)$, for a certain $\beta \leqslant a$, and β is the smallest ordinal such that $\mathfrak{A}_a$ is an elementary extension of $\mathfrak{A}_\beta$.*

(.2) *If a is any element of $\mathfrak{A}_a$, then the union B^* of all elements definable in $\mathfrak{A}^* = \langle R(a), \epsilon_{R(a)}, a\rangle$ is $R(\gamma)$, for some $\gamma \leqslant a$, and $\mathfrak{A}_a$ is an elementary extension of $\mathfrak{A}_\gamma$, to which a belongs.*

(.3) *ω is confinal with each of β and γ, so that if ω is not confinal with a, then $\beta < a$ and $\gamma < a$.*

Proof. Suppose $x \in D(\mathfrak{A}_a)$. Then, by 1.2.2, there is a constant term τ of $\mathrm{ZFS}^{(Z)}$ such that x is $\tau^{(\mathfrak{A}_a)}$. By 4.1 and 1.2.2, $R(r(x)) = \big(\boldsymbol{R}(\boldsymbol{r}(\tau))\big)^{(\mathfrak{A}_a)} \in D(\mathfrak{A}_a)$. Since, for each x, $x \subseteq R(r(x))$, by 1.1.9, we see that $B = \bigcup\{x \mid x \in D(\mathfrak{A}_a)\} = \bigcup\{R(r(x)) \mid x \in D(\mathfrak{A}_a)\}$. Therefore, by 1.1.8, B is of the form $R(\beta)$. The same argument, but involving an inessential extension of ZFS, shows that B^* is of the form $R(\gamma)$.

By 5.2.1 and 5.2.2, $\mathfrak{A}_a$ is an elementary extension of $\mathfrak{A}_\beta$ and of $\mathfrak{A}_\gamma$.

If $\mathfrak{A}_a$ is an elementary extension of $\mathfrak{A}_{\bar{\beta}}$, then, by 2.2, $D(\mathfrak{A}_a) \subseteq R(\bar{\beta})$, and so, by 1.1.4 and 1.1.1, $R(\beta) \subseteq R(\bar{\beta})$ and $\beta \leqslant \bar{\beta}$, so that β is the smallest such $\bar{\beta}$.

Suppose $\delta < \beta$. Applying 1.1 and 1.2.2 we argue as follows: $\delta \in R(\beta) = B$, so that, for some constant term τ of $\mathrm{ZFS}^{(Z)}$, $\delta \in \tau^{(\mathfrak{A}_a)}$. But then $\delta = r(\delta) < r(\tau^{(\mathfrak{A}_a)}) = (\boldsymbol{r}(\tau))^{(\mathfrak{A}_a)} \in D(\mathfrak{A}_a)$. Thus, if δ is any ordinal $< \beta$, there is an ordinal $\delta' \in D(\mathfrak{A}_a)$ with $\delta < \delta' < \beta$. Since the set of terms of $\mathrm{ZFS}^{(Z)}$ and, hence, the set of ordinals definable in $\mathfrak{A}^{(a)}$ are countable, it follows that ω is confinal with β. A similar argument applies to γ, completing the proof of 5.3.

Suppose now that a is any ordinal such that $\mathfrak{A}_a$ is a model of ZFS. We denote (henceforth) by a' the least ordinal such that $\mathfrak{A}_a$ is an elementary extension of $\mathfrak{A}_{a'}$, and by a^* the least ordinal such that $\mathfrak{A}_a$ is elementarily equivalent to $\mathfrak{A}_{a^*}$. From the definition of a^* it follows that

[15] A device involving the notion of rank, anticipating that used in this proof, was employed earlier, in connection with problems concerning finite axiomatizability of set theories, in Montague [1] and [3]. (The results of the latter (unpublished) thesis will largely be reproduced in the forthcoming monograph, *The Method of Arithmetization and Some of its Applications* (North Holland), by S. Feferman and R. Montague.)

$\alpha^{*\prime} = \alpha^*$. Consequently, by 5.3, ω is confinal with α^* as well as with α'. Clearly, $\alpha^* \leqslant \alpha' \leqslant \alpha$. In 5.3, conditions have been given under which the second of these inequalities may be replaced by a strict inequality. The following theorem shows that whenever the second inequality can be sharpened, the first can also be.

THEOREM 5.4. *Suppose that $\mathfrak{A}_\alpha$ is a model of* ZFS *and $\alpha' < \alpha$. Then $\alpha^* < \alpha'$.*

Proof. From the work of Tarski [1], it is known that the notion of the truth of a sentence (of ZFS) in a relational system which is a member of the domain of discourse can be formalized adequately within ZFS. (cf. the remarks at the end of § 3). By performing this formalization explicitly (and also continuing Lemma 4.1 so as to formalize the notion "$\mathfrak{A}_\delta$"), one easily shows the following: There is a formula θ of ZFS, with the free variables v_0 and v_1, such that, if $\mathfrak{A}_\gamma$ is any model of ZFS and Y, $z \in R(\gamma)$, then $\models_{\mathfrak{A}_\gamma} \theta[Y, z]$ if and only if z is an ordinal, Y is a set of sentences of ZFS, and $\mathfrak{A}_z$ is a model of Y.

5.4 may now be proved as follows: Let X be the set of all sentences (of ZFS) true in $\mathfrak{A}_\alpha$. Then $\mathfrak{A}_{\alpha'}$ is a model of X and, hence, $\models_{\mathfrak{A}_\alpha} \theta[X, \alpha']$. Therefore, $\models_{\mathfrak{A}_\alpha} (\bigvee v_1 \theta)[X]$. But since $X \in R(\alpha')$, and $\mathfrak{A}_\alpha$ is an elementary extension of $\mathfrak{A}_{\alpha'}$, it follows that $\models_{\mathfrak{A}_{\alpha'}} (\bigvee v_1 \theta)[X]$. Consequently, there is an element $z \in R(\alpha')$ such that $\models_{\mathfrak{A}_{\alpha'}} \theta[X, z]$. But then z is an ordinal less than α' and $\mathfrak{A}_z$ is a model of X; hence, $\alpha^* < \alpha'$.

Clearly, when $\alpha^* < \alpha$, there is no theory of which $\mathfrak{A}_\alpha$ is the smallest natural model. It may be of interest that the same applies to $\mathfrak{A}_{\alpha^*}$, provided that one considers only theories fulfilling certain very weak restrictions. In fact, we have:

THEOREM 5.5. *Suppose that $\mathfrak{A}_\alpha$ is a model of* ZFS *and $\alpha^* < \alpha$* ([16]). *Let Q be any set of sentences true in $\mathfrak{A}_\alpha$, which is definable in $\mathfrak{A}_\alpha$. Then* (.1) *Q has a model $\mathfrak{A}_\beta$, where $\beta < \alpha^*$, and* (.2) *there is a sentence σ which is true in $\mathfrak{A}_\alpha$ and such that Q has a natural model smaller than any natural model of $Q \cup \{\sigma\}$.*

Proof. Suppose that (.1) were false. Then α^* would be the smallest β such that $\mathfrak{A}_\beta$ is a model of Q. Consequently (by the definition of α^*), (9) and (10) below are equivalent, for any sentence σ of ZFS:

(9) $$\models_{\mathfrak{A}_\alpha} \sigma$$

(10) *σ is true in $\mathfrak{A}_\beta$, where β is the smallest ordinal such that $\mathfrak{A}_\beta$ is a model of Q.*

([16]) This applies in particular, as we shall observe, if α is the first strongly inaccessible ordinal $> \omega$.

Now, as already remarked, it is known that, roughly speaking, the notion of the truth of sentences in systems which are members of the domain of discourse can be formalized within ZFS. Applying this fact (together with 4.1 and the hypothesis that Q is definable in the particular model $\mathfrak{A}_\alpha$ of ZFS), one easily sees that statement (10) can be formalized within $\mathrm{Th}(\mathfrak{A}_\alpha)$. We omit the precise details of this argument, but we formulate its conclusion precisely, as follows: There is a formula φ of ZFS, whose only free variable is v_0, such that, for any sentence σ of ZFS, (10) holds if and only if

(11) $$\models_{\mathfrak{A}_\alpha} \varphi(\Delta_\sigma).$$

Thus (9) and (11) are equivalent, so that:

(12) *for any sentence σ of* ZFS, $\models_{Th(\mathfrak{A}_\alpha)} \varphi(\Delta_\sigma) \leftrightarrow \sigma$.

It is a theorem of Tarski [1] that condition (12) cannot hold in any theory which is a consistent extension of ZFS (or even in certain much weaker theories). Since $Th(\mathfrak{A}_\alpha)$ is, in fact, such a consistent extension, we have arrived at a contradiction, establishing (.1).

Each of (.1) and (.2) is easily seen to imply the other. (If there were no σ as in (.2), then the smallest natural model of Q would be a model of $Th(\mathfrak{A}_\alpha)$ and, hence, would be $\mathfrak{A}_{\alpha^*}$.)

In case Q is recursive, or definable in $\mathfrak{A}_\omega$, a certain specific sentence "saying" that there exists a natural model of Q may be taken for the sentence σ of (.2), and a different proof of 5.5.2 and 5.5.1 (in which one considers the smallest natural model of Q) can be constructed, closely resembling ideas which appear in Shepherdson [1] and Mostowski [2]. In the most general case of 5.5, however, we do not know how to exhibit a specific sentence fulfilling (.2), though, as we have seen, 5.5 can still be obtained, by relying on Tarski's theorem on the impossibility of defining truth.

§ 6. Inaccessible ordinals.

DEFINITION 6.1. An ordinal α is called *weakly inaccessible* if α is *regular* (i. e. no ordinal $\beta < \alpha$ is confinal with α), and α *is of the form* ω_γ, with γ a limit number ([17]).

It was shown in Shepherdson [1], assuming the axiom of choice, that (in our terminology) the natural models of VNB are exactly the systems $\mathfrak{A}_{\alpha+1}$, where α is a strongly inaccessible ordinal $> \omega$. The notion of strongly inaccessible ordinal has often been defined only under that assumption. Lévy ([1]) has recently proposed that, in the absence of

([17]) The initial ordinals ω_α are, of course, defined inductively by the requirement: $\omega_0 = \omega$; if $\alpha \neq 0$, ω_α is the first β such that $\overline{\overline{\omega_\gamma}} < \overline{\overline{\beta}}$ whenever $\gamma < \alpha$.

the axiom of choice, a definition essentially the same as that of 6.2, below (or its equivalent in 6.3) be given — Shepherdson's result then insuring equivalence with the ordinary notion when the axiom of choice is assumed.

DEFINITION 6.2. α is called *strongly inaccessible* provided that

(1) *α is infinite, and whenever $x \in R(\alpha)$, $y \subseteq R(\alpha)$, and f is a function on x onto y, then $y \in R(\alpha)$.*

LEMMA 6.3. *α is a strongly inaccessible ordinal $> \omega$ if and only if $\mathfrak{A}_{\alpha+1}$ is a model of* VNB.

Proof. It is elementary to verify that α is a limit ordinal $> \omega$ if and only if $\mathfrak{A}_{\alpha+1}$ is a model of the axioms of VNB excluding C_4 — the VNB version of the axiom of replacement [18]. If $\mathfrak{A}_{\alpha+1}$ is, moreover, a model of axiom, C_4 then clearly α fulfills (1). Conversely, if (1) holds, one easily sees that α must be a limit ordinal and that C_4 is true in $\mathfrak{A}_{\alpha+1}$.

LEMMA 6.4. *If the axiom of choice is assumed, then* 6.2 *coincides with the usual definition of "strongly inaccessible ordinal"* (at least, as applied to infinite ordinals).

Proof. As already remarked, this result (using 6.3) is given in Shepherdson [1]. It may also be proved directly from 6.2 and with no use of metamathematical notions; the proof depends, of course, on which one selects of various definitions known to be equivalent (for infinite ordinals) on the basis of the axiom of choice (cf. Tarski [2]).

As a further justification of 6.2, we note:

LEMMA 6.5. *A strongly inaccessible ordinal is weakly inaccessible.*

Proof. One easily sees that if (1) holds for α, then α is regular and hence, of the form ω_γ. To see that $\gamma \neq \delta+1$, one uses the well-known fact (not depending on the axiom of choice) that there is a one-to-one correspondence between $\omega_{\delta+1}$ and the family of all similarity classes of well-orderings of ω_δ.

6.6. Remarks. It may also be shown that (1) is equivalent to the apparently weaker condition obtained from it by replacing "function" by "one-to-one function". (One has only to consider the set $\{X \mid$ for some $v \in y$, $X = \{u \mid u \in x$ and $f(u) = v\}\}$).

It may be of interest to note that certain natural conditions equivalent to (1) under the assumption of the axiom of choice, appear to be (in its absence) of varying strengths. Consider the conditions (2) and (3), below [19]:

[18] We speak of the particular axioms of Gödel [1] when we really should speak of their translations — in the sense of footnote 11.

[19] Cardinal numbers are denoted by small Gothic letters. The less-than relation and exponentiation among cardinal numbers are assumed defined in the ordinary way.

(2) $\begin{cases} \text{(a) } \alpha \text{ is an initial ordinal,} \\ \text{(b) whenever } \mathfrak{b} < \bar{\bar{\alpha}},\ 2^{\mathfrak{b}} < \bar{\bar{\alpha}}, \\ \text{(c) if } X \subseteq R(\alpha),\ \bar{\bar{X}} < \bar{\bar{\alpha}}, \text{ and, for each } x \in X,\ \bar{\bar{x}} < \bar{\bar{\alpha}}, \text{ then } \overline{\overline{\bigcup X}} < \bar{\bar{\alpha}}. \end{cases}$

(3) *α is infinite, and whenever $x \subseteq R(\alpha)$ and $\bar{\bar{x}} < \overline{\overline{R(\alpha)}}$, then $x \in R(\alpha)$.*

The following facts can be established: Condition (2) is equivalent to the conjunction of (1) and the condition:

(4) *every $x \in R(\alpha)$ can be well-ordered.*

Condition (3) is equivalent to the conjunction of (2) and the statement:

(5) *$R(\alpha)$ can be well-ordered.*

However, we do not see any way to derive (4) from (1), or (5) from (2).

The equivalences just stated are proved as follows:

To derive (1) and (4) from (2) one may proceed as follows (assuming (2)): First one easily proves by induction on β that $\overline{\overline{R(\beta)}} < \bar{\bar{\alpha}}$, for every $\beta < \alpha$. Hence (2) implies not only (4), but also that, for any $x \in R(\alpha)$, $\bar{\bar{x}} < \bar{\bar{\alpha}}$. Now, assuming the hypothesis of (1), one sees that $\bar{\bar{y}} < \bar{\bar{\alpha}}$, and by applying 1.1.3 and 2(c), one finds that $r(y) < \alpha$, so that $y \in R(\alpha)$, and (1) holds.

Assume now that (1) and (4) hold. It is easy to derive 2(a) from (1). Now note that *if $x \in R(\alpha)$ then $\bar{\bar{x}} < \bar{\bar{\alpha}}$.* Indeed, by (4), there is some γ with $x \sim \gamma$. But, if $\gamma \geqslant \alpha$, then from (1), one easily obtains $\alpha \in R(\alpha)$, which is absurd. Hence $x \sim \gamma < \alpha$, so, by 2(a), $\bar{\bar{x}} < \bar{\bar{\alpha}}$. Using the italicized statement, and 2(a), one sees that α is a limit ordinal and that if $\bar{\bar{\beta}} < \alpha$, then $P(\beta) \in R(\alpha)$ and $\overline{\overline{P(\beta)}} < \bar{\bar{\alpha}}$. Thus 2(b) holds. Under the hypotheses of 2(c), we obtain $X \sim \beta < \alpha$, and so, by (1), $X \in R(\alpha)$, whence $\bigcup X \in R(\alpha)$, and $\overline{\overline{\bigcup X}} < \bar{\bar{\alpha}}$. Hence 2(c) holds.

Assume (3). Taking "α" for "x" in (3), it follows that $\bar{\bar{\alpha}} = \overline{\overline{R(\alpha)}}$. (5) is an immediate consequence. From 1.1, it follows that if $x \in R(\alpha)$, then $\bar{\bar{x}} < \overline{\overline{R(\alpha)}}$. Thus, if $\beta < \alpha$, then $\bar{\bar{\beta}} \leqslant \overline{\overline{R(\beta)}} < \overline{\overline{R(\alpha)}} = \bar{\bar{\alpha}}$, so that 2(a) is valid. 2(b) and 2(c) are now easily obtained.

Finally, suppose that (2) and (5) hold. From (2), as we saw above, it follows that $\overline{\overline{R(\beta)}} < \bar{\bar{\alpha}}$, for each $\beta < \alpha$. On the other hand, from (5) it easily follows that $\overline{\overline{R(\alpha)}}$ is the least upper bound of all $\overline{\overline{R(\beta)}}$, for $\beta < \alpha$. Consequently, $\overline{\overline{R(\alpha)}} = \bar{\bar{\alpha}}$. Using this last fact, (3) is easily derived from (1) [20].

Returning now to our principal topic, we note the familiar

LEMMA 6.7. *If α is a strongly inaccessible ordinal $> \omega$, then $\mathfrak{A}_\alpha$ is a model of* ZFS.

Proof. 6.7 is a corollary of 6.3 and the well-known fact that the subsystem of any model $\langle A, E \rangle$ of VNB with universe $\{x \mid$ for some y, $xEy\}$ is a model of ZFS. Alternatively, one may easily verify 6.7 directly.

It follows at once from 6.5 and 6.1 that if α is a strongly inaccessible ordinal $> \omega$, then ω is not confinal with α. Applying 6.7 we obtain at

[20] Condition (3) appears in Mostowski [2], p. 156. Compare also p. 84-87 of Tarski [2]. In the latter, a condition is proved which is related to the fact that (3) implies (5), but much more difficult to establish.

once from 5.3 our principal result, that, if a strongly inaccessible ordinal exists, then the converse of 6.7 is false. Indeed, applying also 5.4 and 5.5, we obtain at once:

THEOREM 6.8. *Suppose there is a strongly inaccessible ordinal and denote by θ the first such ordinal. Let θ' and θ^* and β be the smallest ordinals such that $\mathfrak{A}_\theta$ is an elementary extension of $\mathfrak{A}_{\theta'}$ and $\mathfrak{A}_\theta$ is elementarily equivalent to $\mathfrak{A}_{\theta^*}$ and $\mathfrak{A}_\beta$ is a model of* ZFS, *respectively. Then:*

$$\beta < \theta^* < \theta' < \theta .$$

6.8 has the following

COROLLARY 6.9. *If α is an ordinal for which* (2) *holds, then there exists a denumerable set B such that $\mathfrak{A}_\alpha$ is an elementary extension of $\langle B, \epsilon_B\rangle$.*

Proof. Such a B is obtained by applying, in order, 6.8 and 2.1, noting that, by 6.6, (2) implies (1) and (4). (Elementary extensionality is transitive, as one easily sees from its definition (cf. Tarski-Vaught [1]).

6.10. Remarks. Corollary 6.9 is a form of the so-called Skolem paradox (cf. Skolem [1]; Tarski-Vaught [1], footnote 7). As far as we can see, the old method of proof — in which 2.1 is applied directly — could not be employed without the additional assumption that $R(\alpha)$ can be well-ordered.

On the other hand, if by the Skolem paradox, one means the statement

(6) *there exists a denumerable set B such that $\langle B, \epsilon_B\rangle$ is a model of* ZFS,

then, as Mostowski ([2], p. 163) has remarked, the Skolem paradox can be established (and without any use of our methods) on the basis of the assumption that a weakly inaccessible ordinal exists. This is accomplished by making use of the results of Gödel [1] on "constructible" sets.

In the following theorem, we shall see that the result of 6.8 that $\theta' < \theta$ cannot be avoided by adjoining to the natural model $\langle R(\alpha), \epsilon_{R(\alpha)}\rangle$ any further relations or operations.

THEOREM 6.11. *Suppose that α is a strongly inaccessible ordinal, and $\mathfrak{A}$ is any algebraic system of the form $\langle R(\alpha), \epsilon_{R(\alpha)}, X_0, \ldots, X_{m-1}\rangle$. Then the union of all definable elements of $\mathfrak{A}$ is $R(\beta)$, for a certains $\beta < \alpha$, and $\mathfrak{A}$ has a subsystem with universe $R(\beta)$ of which it is an elementary extension.*

Proof. Almost all of the statements of 6.11 may be established by direct analogues of the arguments in 5.2 and 5.3 (noting as above that ω is not confinal with α, to show that $\beta < \alpha$). The only exception is the major assertion, namely, that $\mathfrak{A}$ has a subsystem $\mathfrak{B}$ with universe $B = \bigcup D(\mathfrak{A})$ and $\mathfrak{A}$ is an elementary extension of $\mathfrak{B}$. Here a slight, but essential, modification in the argument is needed.

It again suffices to show that 2.3.1 holds. Suppose, then, that φ is a formula of the theory $Th(\mathfrak{A})$, with at most the free variables $v_0, \ldots, v_n$, $b_0, \ldots, b_{n-1} \in B$ and $\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}, x]$. Then, as in the proof of 5.2, there are constant terms $\tau_0, \ldots, \tau_{n-1}$ of $Th(\mathfrak{A})$ such that

$$b_i \in \tau_i^{(\mathfrak{A})} \quad (i = 0, \ldots, n-1) .$$

Defining τ exactly as in the proof of 5.1, one cannot now apply 5.1, but nevertheless, from the fact that α is a limit ordinal, one sees easily that:

(7) $$\vdash_{Th(\mathfrak{A})} \bigvee v_n \varphi(v_0, \ldots, v_{n-1}, v_n) \rightarrow \bigvee v_n \big(\varphi(v_0, \ldots, v_{n-1}, v_n) \wedge v_n \in \tau(v_0, \ldots, v_{n-1})\big).$$

Let τ' and τ'' be defined exactly as in the proof of 5.2.

Let $C = \tau_0^{(\mathfrak{A})} \times \ldots \times \tau_{n-1}^{(\mathfrak{A})} =$ the set of all functions x on $n = \{0, \ldots, n-1\}$ such that, for each $i < n$, $x(i) \in \tau_i^{(\mathfrak{A})}$. Since α is a limit number, it follows easily that $C \in R(\alpha)$. Since α is strongly inaccessible, we see, by 6.2, that the set

$$C' = \{z \mid \text{for some } x \in C,\ z = \tau^{(\mathfrak{A})}[x(0), \ldots, x(n-1)]\}$$

also belongs to $R(\alpha)$.

It now follows, by elementary semantical considerations, that $C' = \tau'^{(\mathfrak{A})}$, and hence, using (7), that, for some y,

$$y \in \tau''^{(\mathfrak{A})} \in D(\mathfrak{A})$$

and

$$\vdash_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}, y] .$$

Thus 2.3.1 holds.

§ 7. A different formulation. Here we shall discuss, briefly and rather roughly, a way in which the ideas involved in the procedures of § 5 and § 6 may be employed in a new, slightly different, context.

As we remarked already, the whole development of § 1-§ 6 can be translated into a collection of theorems of the formalized theory ZFS, itself. On the other hand, the new procedure would be formally carried out not in ZFS but in another theory, ZFS′. This theory has, in addition to ϵ, another non-logical constant, the binary predicate ***St*** — which might be labelled "the satisfaction predicate". The axioms of ZFS are also axioms of ZFS′. Moreover, in the schema of replacement (cf. 3.1) the formula φ is now allowed to be any formula of ZFS′ (in which v is not free). Finally, ZFS′ has one more axiom, expressing the recursion conditions for the notion of satisfaction. We will not give this axiom explicitly, but will content ourselves with indicating its informal counterpart. If we use, in the informal language, the symbol "*St*" to correspond to the formal ***St***, then the informal counterpart of the axiom in question is the statement:

If $x = \langle x_0, \ldots, x_n, \ldots\rangle$ is any (ordinary infinite) sequence, φ and ψ are any formulas of ZFS, *and $k, l \in \omega$, then:*

$x St\, v_k = v_l$ or $x St\, v_k \epsilon v_l$ if and only if $x_k = x_l$ or $x_k \epsilon x_l$, respectively;

$x St \sim\varphi$ if and only if it is not the case that $x St\varphi$; $x St\varphi \wedge \psi$ if and only if $x St\varphi$ and $x St\psi$; and similarly for $\vee$, $\rightarrow$, $\leftrightarrow$;

$x St \bigwedge v_k \varphi$ if and only if, for any sequence y, if $y_i = x_i$ for every $i \neq k$, then $y St\varphi$; and similarly for $\bigvee$.

What we now claim is this:

THEOREM 7.1. *The statements* (1) *and* (2), *which follow, are provable on the basis of the informal assumptions corresponding to* ZFS′ (or, alternatively, the formal counterparts of (1) and (2) are valid sentences of ZFS′):

(1) *There exists an ordinal β such that, for any formula φ of* ZFS *and any sequence x of members of $R(\beta)$, x satisfies φ in $\mathfrak{A}_\beta$ if and only if $x St\varphi$.*

(2) *There exists a natural model of* ZFS.

Proof. To prove (1), on the basis claimed, one proceeds (roughly speaking) by imitating either the proofs of 5.1, 5.2, and 5.3, or else that of 6.11 (with $m = 0$). All phrases of the form "x satisfies φ in $\mathfrak{A}$" or "x satisfies φ in $\mathfrak{A}_a$" in the old proofs are replaced by "$x St\varphi$", but phrases "x satisfies φ in $\mathfrak{B}$" or "x satisfies φ in $\mathfrak{A}_\beta$" are left unchanged. Of course one first must eliminate in the old proofs all defined notions whose definitions involved the notion of satisfaction. (Moreover, the imitation and replacement procedure must also be carried out for 2.3 and its proof (cf. Tarski-Vaught [1]).) The old arguments showing that $\beta < a$ are replaced by an argument (based on one of the new, informal, replacement axioms) showing that there is a set consisting of all "definable" sets.

To prove (2) one notes that it is a consequence of (1) and the statement:

(3) *If φ is any valid sentence of* ZFS, *and x is any sequence, then $x St\varphi$.*

It is a simple matter to derive (3) from our present assumptions [21]. (In doing so, however, one uses, again, one of the new (informal) replacement axioms [22]).

7.2. Remarks. In analogy with 6.9, one easily establishes that, in the system ZFS″ obtained by adding to ZFS′ the axiom of choice, the formal counterpart of the following statement is valid:

[21] One may want to derive (3) already in the proof of (1), if the proofs of 5.1, 5.2, and 5.3 are imitated.

[22] That this use cannot be avoided follows from results in Mostowski [3].

(4) *There exists a denumerable set B such that, for any formula φ of* ZFS *and any sequence x of members of B, x satisfies φ in $\langle B, \epsilon_B\rangle$ if and only if $x St\varphi$.*

Both (1) and (4) may be taken as assertions regarding the 'universe', (4) being a very strong form of the Skolem paradox.

It appears that none of (1), (2), and (4) could be established in ZFS″ without the use of our present method. To obtain (4) by the older methods one would need some form of the "axiom of choice for the universe" — say, either Gödel's "constructibility axiom" or a new non-logical constant, the "selector", and an appropriate axiom and axiom schema involving it.

It may also be remarked that, using an analogue of the idea of Mostowski mentioned in 6.10, one can establish (with or without the use of our method):

In ZFS′, *the formal counterpart of statement* (6) *or* § 6 *is valid.*

Certainly the assumptions of the informal counterpart of ZFS′ or ZFS″ are very much weaker than the assumption of the existence of an inaccessible ordinal [23]. Yet in ZFS′ such important sentences of ZFS itself as the formal counterparts of (6) of § 6 and (2) are valid [24].

In (1) and (2), above, we have stated analogues of only part of the content of 6.8. Of course, one could similarly obtain analogues of the rest of 6.8. To obtain an analogue of 6.11 with $m > 0$, one must expand ZFS′ into a certain stronger theory with non-logical constants ϵ, $\xi_0, \ldots, \xi_{m-1}$, St'. There is an interesting special case of this, in which one iterates the precedure that took us from ZFS to ZFS′, obtaining a theory with non-logical constants ϵ, St, St'. This theory, and those obtained by any finite number of further iterations, seem still to have a fairly simple intuitive content.

§ 8. A converse theorem.

Theorem 8.1, below, of this concluding section, gives a kind of converse to the preceding results (e. g., 5.3). This theorem and the ideas used in its proof were suggested to us by

[23] Assuming that ZFS′ or ZFS″ is consistent, one can show in a finitary way that the theory T — obtained by adding to ZFS the formal counterpart of the statement "there exists a weakly inaccessible ordinal" — is stronger in various precise senses. For example, one can easily show that the formal counterparts of such statements as "ZFS″ has a model of the form $\langle A, \epsilon_A, \ldots\rangle$" and "ZFS″ is consistent" are valid in T. ("*Weakly* inaccessible" suffices, in view of the idea of Mostowski quoted in § 6.10.) For precise notions corresponding to various terms used in this footnote, see Feferman [1] and Montague [3].

[24] This remark perhaps sheds a somewhat different light on the question discussed on p. 161 of Mostowski [2] (the $\mathcal{F}$ mentioned there is valid in ZFS′, as one easily sees by using the ideas here together with those in Mostowski [2]).

some recent work of Dr. Azriel Lévy, in which Theorem 6.8 is applied to the problem of showing the consistency of a new theory of sets of Ackermann [25]. (More recently still, Lévy has obtained a certain result of which 8.1 is an almost immediate corollary.)

THEOREM 8.1. *Suppose that* $\beta < \alpha$ *and that* $\mathfrak{A}_\alpha$ *is an elementary extension of* $\mathfrak{A}_\beta$. *Then* $\mathfrak{A}_\alpha$ *and* $\mathfrak{A}_\beta$ *are models of* ZFS.

Proof. From 3.1 and 1.1, one easily sees that in any system of the form $\mathfrak{A}_\delta$ the axioms of extensionality, union, and regularity hold. Moreover, the power axiom is true in $\mathfrak{A}_\delta$ if and only if δ is a limit ordinal, and the axiom of infinity holds in $\mathfrak{A}_\delta$ if and only if $\omega < \delta$.

Assuming, now, that our hypotheses hold for α and β, we note first that β must be a limit ordinal. Indeed, suppose $\beta = \gamma + 1$. Then in $\mathfrak{A}_\beta$, γ satisfies the formula $\bigwedge \mathrm{v}_1(\sim \mathrm{v}_0 \,\epsilon\, \mathrm{v}_1)$. Consequently, γ satisfies the same formula in $\mathfrak{A}_\alpha$, which is absurd, since $\gamma \in \beta \in R(\alpha)$.

It follows that the power axiom holds in $\mathfrak{A}_\beta$, and consequently, also in $\mathfrak{A}_\alpha$, as $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\beta$ are elementarily equivalent. Moreover, since we now have $\alpha > \beta \geqslant \omega$, the axiom of infinity holds in $\mathfrak{A}_\alpha$ and hence, also, in $\mathfrak{A}_\beta$.

It remains to show that the axioms of replacement are true in $\mathfrak{A}_\beta$ (and, hence, in $\mathfrak{A}_\alpha$). Clearly, it will suffice to show that: if φ is any formula of ZFS, with free variables at most $\mathrm{v}_0, \ldots, \mathrm{v}_n$, where $n \geqslant 3$, and $x_0, \ldots, x_{n-2} \in R(\beta)$, and

$$(1) \qquad \models_{\mathfrak{A}_\beta} (\bigwedge \mathrm{v}_{n-1} \bigvee ! \mathrm{v}_n \varphi)[x_0, \ldots, x_{n-2}],$$

then

$$(2) \quad \models_{\mathfrak{A}_\beta} \Big(\bigvee \mathrm{v}_{n+1} \bigwedge \mathrm{v}_n \big(\mathrm{v}_n \,\epsilon\, \mathrm{v}_{n+1} \leftrightarrow \bigvee \mathrm{v}_{n-1}(\mathrm{v}_{n-1} \,\epsilon\, \mathrm{v}_{n-2} \wedge \varphi)\big)\Big)[x_0, \ldots, x_{n-2}].$$

Suppose, then, that $x_0, \ldots, x_{n-2} \in R(\beta)$ and that (1) holds. Since $\mathfrak{A}_\alpha$ is an elementary extension of $\mathfrak{A}_\beta$, the statement (1′), obtained from (1) by replacing "$\mathfrak{A}_\beta$" by "$\mathfrak{A}_\alpha$", also holds. For the same reason, it will be enough to establish (2′), obtained in the same way from (2). Now obviously (2′) will be established if we can show that the set

$$C = \{c \mid \textit{for some } b,\ b \in x_{n-2} \textit{ and } \models_{\mathfrak{A}_\alpha} \varphi[x_0, \ldots, x_{n-2}, b, c]\}$$

is a member of $R(\alpha)$, and, hence, if we can show that $C \subseteq R(\beta)$. But suppose that $c \in C$, so that, for some b,

$$(3) \qquad b \in x_{n-2} \quad \text{and} \quad \models_{\mathfrak{A}_\alpha} \varphi[x_0, \ldots, x_{n-2}, b, c].$$

([25]) See Lévy [1]. A version of the part of Lévy [1] in question will appear soon in the Journal of Symbolic Logic under the title *An inner model for Ackermann's set theory*.

Now, by (1), there is a $c' \in R(\beta)$ such that

$$(4) \qquad \models_{\mathfrak{A}_\beta} \varphi[x_0, \ldots, x_{n-2}, b, c'].$$

Consequently (4′) (obtained as before) also holds. From (4′), (3), and (1′), it follows that $c = c'$. Thus $c \in R(\beta)$, as was to be proved.

Using Theorem 8.1 or, rather, its formal counterpart, together with Gödel's theorem on the impossibility of proving consistency, one easily sees that the formal counterpart of the statement:

there exist α and β such that $\alpha > \beta$ and $\mathfrak{A}_\alpha$ is an elementary extension of $\mathfrak{A}_\beta$,

is not a valid sentence of ZFS.

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A note on theories with selectors

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It is well known that for certain elementary theories, such as Peano's arithmetic, no gain is made by adjoining to the theory the Hilbert ε-symbol (and the associated new rules of proof). Such theories might be called "theories with built-in Hilbert ε-symbols" or, simply, "theories with selectors". Our purpose in this note is to point out that the (purely syntactical) property of being such a theory is equivalent to a certain semantical property.

Familiarity will be assumed with the introductory sections of our paper, *Natural models of set theories* (this volume, p. 219-242) (1). Specifically, what will be needed is the second part of § 1 (beginning "In metamathematical considerations..."), and § 2 of that paper.

The semantical properties we shall discuss involve the notion of the set $D(\mathfrak{A})$ of all definable elements of a realization $\mathfrak{A}$ of a theory. In addition, the following further notions are required:

DEFINITION 1. *Let* $\mathfrak{A} = \langle A, X_0, \ldots, X_{m-1}\rangle$ *be a realization of an arbitrary standard theory.*

(.1) *If* $D(\mathfrak{A})$ *is not empty, then the corresponding* submodel of definable elements, *or* $\mathfrak{D}(\mathfrak{A})$, *is the subsystem of* $\mathfrak{A}$ *with universe* $D(\mathfrak{A})$.

(.2) *If* B *is any subset of* A, *then by* $D(\mathfrak{A}, B)$ — the set of all elements of A definable in $\mathfrak{A}$ in terms of elements of B — *we mean the union of all sets* $D(\mathfrak{A}^*)$, *where* $\mathfrak{A}^* = \langle A, X_0, \ldots, X_{m-1}, b_0, \ldots, b_{n-1}\rangle$, $n \in \omega$, *and* $b_0, \ldots, b_{n-1} \in B$.

(.3) *Assuming that* $D(\mathfrak{A}, B)$ *is not empty if* B *is empty, then, by* $\mathfrak{D}(\mathfrak{A}, B)$ *we mean the subsystem of* $\mathfrak{A}$ *with universe* $D(\mathfrak{A}, B)$.

(It is obvious that $D(\mathfrak{A})$ and $D(\mathfrak{A}, B)$ are closed under any operations as required in (.1) or (.3).)

We shall actually establish three different equivalences, corresponding to various precise senses that might be given to the notion "theories with selectors". The first is:

(1) Hereinafter referred to as NM. Numbers in brackets will refer to the bibliography of NM.