# On clusters in proximity spaces *

by

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**1. Introduction.** The topology in a metric space is determined by stating which points are close to each given set, a point $x$ being close to a set $B$ if the distance between $x$ and $B$ is zero. A continuous mapping is just a function which preserves proximity between points and sets: $fx$ is close to $fB$ whenever $x$ is close to $B$. In 1922 K. Kuratowski [3] had abstracted the proximity relation "$x$ is close to $B$" by axiomatically characterizing the set $\bar{B}$ of all points close to $B$.

Now the uniform topology in a metric space is determined by stating which sets are close to each given set, a set $A$ being close to a set $B$ if the distance between $A$ and $B$ is zero. A uniformly continuous mapping is just a function which preserves proximity between sets: $fA$ is close to $fB$ whenever $A$ is close to $B$. (See [17].) This immediately suggests abstracting uniform topology by axiomatizing the proximity relation "$A$ is close to $B$" as a binary relation on subsets of a set $X$.

Strangely enough, this remained undone until 1952 when V. A. Efremovich [1] introduced a set of axioms characterizing proximity relations and thus launched the theory of proximity spaces. This theory is an elegant generalization of uniform topology in metric spaces, yet is more specific than the theory of uniform structures. (See [6].)

The compactification of proximity spaces was first treated by Yu. M. Smirnov [8]. Smirnov's treatment involves constructions using transfinite induction. In this paper we introduce an alternative approach to the compactification of a proximity space based on the simple concept of a "cluster", which is extrinsically just the class of all sets close to some fixed point. We avoid transfinite induction by using the axiom of choice in the following form: Given a class of elements, every subclass having a property of finite character is contained in some maximal subclass having that property [16].

**2. Proximity spaces.** Let $X$ be an abstract set. A *point* is a subset of $X$ having no proper subsets. A *proximity relation* in $X$ is a binary

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relation "$A$ is close to $B$" on subsets of $X$ such that the following axioms hold:

(1) *If $A$ is close to $B$, then $B$ is close to $A$.*

(2) *$A$ is close to $X$ if and only if $A$ is non-empty.*

(3) *$A \cup B$ is close to $C$ if and only if either $A$ or $B$ is close to $C$.*

(4) *If for every $E$, either $A$ is close to $E$ or $B$ is close to $X-E$, then $A$ is close to $B$.*

(5) *If a point $x$ is close to a point $y$, then $x = y$.*

A set $X$ with a proximity relation in it is called a *proximity space*. We say $A$ is *remote from* $B$ if $A$ is not close to $B$.

We list without proof some basic properties of proximity spaces. (See [1].)

(i) *If $A$ is a subset of $C$, $B$ a subset of $D$, and $A$ is close to $B$, then $C$ is close to $D$.*

(ii) *If $A$ intersects $B$, then $A$ is close to $B$.*

(iii) *The empty set is remote from every subset of $X$.*

(iv) *If there exists a point $x$ which is close to both $A$ and $B$, then $A$ is close to $B$.*

(v) *Every proximity space is a completely regular Hausdorff space if we define the closure of $A$ to be the union of all points close to $A$.*

(vi) *$A$ is close to $B$ if and only if the closure of $A$ is close to the closure of $B$.*

(vii) *Every compact Hausdorff space is a proximity space if we define $A$ to be close to $B$ whenever $\bar{A}$ intersects $\bar{B}$.*

(viii) *Every metric space is a proximity space if we define $A$ to be close to $B$ whenever the distance between $A$ and $B$ is zero.*

Motivated by (v), P. S. Alexandrov posed the question: Which topological spaces admit a proximity relation compatible with the given topology? Smirnov [8] showed that the answer is the one suggested by (vii): A topological space admits a proximity relation if and only if it is a subspace of a compact Hausdorff space. Our version of this result is Theorem 4 below.

**3. Clusters.** A *cluster* $\boldsymbol{c}$ from a proximity space $X$ is a class of subsets of $X$ satisfying the following three conditions:

(a) *If $A$ and $B$ belong to $\boldsymbol{c}$, then $A$ is close to $B$.*

(b) *If $A$ is close to every $C$ in $\boldsymbol{c}$, then $A$ belongs to $\boldsymbol{c}$.*

(c) *If $A \cup B$ belongs to $\boldsymbol{c}$, then either $A$ or $B$ belongs to $\boldsymbol{c}$.*

It is readily seen that the class $\boldsymbol{x}$ of all sets close to a point $x$ is a cluster. Clearly, if a point $x$ belongs to a cluster $\boldsymbol{x}$, then $\boldsymbol{x}$ is just the class of all sets close to $x$. Axiom (5) means every cluster possesses at most one point. As we shall see (in Theorem 2 below), the crux of the compactification problem is that there may exist clusters which possess no points.



By (b), $X$ belongs to every cluster from $X$. So (c) implies that for arbitrary $E$ either $E$ or $X-E$ belongs to a given cluster. We now establish some deeper properties of clusters.

THEOREM 1. *If $A$ is close to $B$, there exists a cluster to which both $A$ and $B$ belong.*

Proof. Let $A_0$ be close to $B$. By the axiom of choice $A_0$ belongs to a maximal class $\boldsymbol{a}$ of sets such that every finite intersection of sets in $\boldsymbol{a}$ is close to $B$. Define $\boldsymbol{c}$ to be the class of all sets close to every set in $\boldsymbol{a}$. Clearly, both $A_0$ and $B$ belong to $\boldsymbol{c}$. We need only show that $\boldsymbol{c}$ is a cluster.

Given $P$ and $Q$ in $\boldsymbol{c}$, suppose $P$ is remote from $Q$. Then by (4) there exists $E$ remote from $P$ with $X-E$ remote from $Q$. Thus for every $A$ in $\boldsymbol{a}$, $A \cap E$ is remote from $P$ and $A-E$ is remote from $Q$. Hence, neither $A \cap E$ nor $A-E$ belongs to $\boldsymbol{a}$. Thus there exist $A_1$ and $A_2$ in $\boldsymbol{a}$ with $A_1 \cap E$ and $A_2 - E$ remote from $B$. Let $A = A_1 \cap A_2$. Then $A$ belongs to $\boldsymbol{a}$ and both $A \cap E$ and $A-E$ are remote from $B$. By (3) this implies $A$ is remote from $B$, contradicting the definition of $\boldsymbol{a}$. So $P$ must be close to $Q$. Hence (a) holds for $\boldsymbol{c}$.

Since $\boldsymbol{a}$ is a subclass of $\boldsymbol{c}$, (b) follows from the definition of $\boldsymbol{c}$.

Given $P \cup Q$ in $\boldsymbol{c}$ with $P$ not in $\boldsymbol{c}$, $P \cup Q$ is close to every $A$ in $\boldsymbol{a}$ and $P$ is remote from some $A_1$ in $\boldsymbol{a}$. Thus $A \cap A_1$ is close to $P \cup Q$ and remote from $P$ for every $A$ in $\boldsymbol{a}$. By (3), $Q$ is close to $A \cap A_1$. By (i), $Q$ is close to $A$. Since this holds for every $A$ in $\boldsymbol{a}$, $Q$ belongs to $\boldsymbol{c}$. Hence (c) is satisfied.

THEOREM 2. *A proximity space $X$ is compact if and only if every cluster from $X$ possesses a point.*

Proof. Let $X$ be compact and $\boldsymbol{c}$ be a cluster from $X$. Suppose that no point belongs to $\boldsymbol{c}$. Then by (b), every point $x$ is remote from some set in $\boldsymbol{c}$. By (4), each $x$ has a neighborhood $E_x$ which does not belong to $\boldsymbol{c}$. Since $X$ is compact, finitely many of these neighborhoods cover $X$. Since each $E_x$ does not belong to $\boldsymbol{c}$, (c) implies no finite union of the $E_x$'s can belong to $\boldsymbol{c}$. Thus $X$ cannot belong to $\boldsymbol{c}$, a contradiction. So some point $x$ must belong to $\boldsymbol{c}$.

Conversely, let every cluster have a point. Let $\boldsymbol{b}$ be a class of closed sets with finite intersections non-empty. To show that $X$ is compact we have only to prove that the intersection of all sets in $\boldsymbol{b}$ is non-empty. Using the axiom of choice we may assume that $\boldsymbol{b}$ is a maximal class of closed sets with finite intersections non-empty. Define the class $\boldsymbol{c}$ to consist of all sets whose closures belong to $\boldsymbol{b}$. Since $\boldsymbol{b}$ is maximal, $E$ belongs to $\boldsymbol{c}$ if and only if $\bar{E}$ intersects every set in $\boldsymbol{b}$. Since, by hypothesis, every cluster has a point, Theorem 1 gives the converse of (iv). So two sets are close if and only if their closures intersect. Thus $E$ belongs to $\boldsymbol{c}$ if and only if $E$ is close to every set in $\boldsymbol{b}$. We contend that $\boldsymbol{c}$ is a cluster.

Given $A$ and $B$ in $\boldsymbol{c}$, $\bar{A}$ and $\bar{B}$ belong to $\boldsymbol{b}$. Hence, $\bar{A}$ intersects $\bar{B}$. By (iv), $A$ is close to $B$. So (a) holds.

Since $\boldsymbol{b}$ is a subclass of $\boldsymbol{c}$, (b) is trivial.

Given $P\cup Q$ in $\boldsymbol{c}$ with $P$ not in $\boldsymbol{c}$, we have $\bar{P}\cup\bar{Q}$ in $\boldsymbol{b}$ with $\bar{P}$ not in $\boldsymbol{b}$. By the latter condition $\bar{P}$ fails to intersect some $B_1$ in $\boldsymbol{b}$. Now for every $B$ in $\boldsymbol{b}$, $B\cap B_1$ belongs to $\boldsymbol{b}$. Hence, $\bar{P}\cup\bar{Q}$ intersects $B\cap B_1$. Since $\bar{P}$ is disjoint from $B\cap B_1$, $\bar{Q}$ intersects $B\cap B_1$. Thus $\bar{Q}$ intersects every $B$ in $\boldsymbol{b}$. So $Q$ belongs to $\boldsymbol{c}$. Therefore (c) holds and $\boldsymbol{c}$ is a cluster.

By hypothesis there exists a point $x$ belonging to $\boldsymbol{c}$. By (a), $x$ is close to every set in $\boldsymbol{c}$. Since $\boldsymbol{b}$ is a subclass of $\boldsymbol{c}$, $x$ is close to every set in $\boldsymbol{b}$. Since the sets belonging to $\boldsymbol{b}$ are closed, $x$ is contained in every set in $\boldsymbol{b}$. This completes the proof of the theorem.

An immediate result of Theorem 1, Theorem 2, and (iv) is the following corollary which is proved directly in [1].

COROLLARY. *In a compact proximity space two sets are close if and only if their closures intersect.*

THEOREM 3. *If $X$ is a subspace of a proximity space $Y$, every cluster $\boldsymbol{b}$ from $X$ is part of a unique cluster $\boldsymbol{c}$ from $Y$. $\boldsymbol{c}$ consists of all subsets of $Y$ which are close to every set in $\boldsymbol{b}$. Every cluster $\boldsymbol{c}$ from $Y$ to which $X$ belongs contains a unique cluster $\boldsymbol{b}$ from $X$. $\boldsymbol{b}$ consists of all subsets of $X$ which belong to $\boldsymbol{c}$.*

Proof. Given a cluster $\boldsymbol{b}$ from $X$ let $\boldsymbol{c}$ consist of all subsets of $Y$ which are close to every set in $\boldsymbol{b}$. By (a), $\boldsymbol{b}$ is a subclass of $\boldsymbol{c}$. We must show that $\boldsymbol{c}$ is a cluster from $Y$.

Let $A$ be remote from $B$. Then by (4), there exists $E$ remote from $A$ with $Y-E$ remote from $B$. Given $C$ in $\boldsymbol{b}$, (c) implies either $C\cap E$ or $C-E$ belongs to $\boldsymbol{b}$. In the former case $A$, being remote from $C\cap E$, cannot belong to $\boldsymbol{c}$. In the latter case $B$, being remote from $C-E$, cannot belong to $\boldsymbol{c}$. So $\boldsymbol{c}$ satisfies (a).

Since $\boldsymbol{b}$ is a subclass of $\boldsymbol{c}$, (b) is satisfied.

Let $A\cup B$ belong to $\boldsymbol{c}$ and $A$ not belong to $\boldsymbol{c}$. Then there exists $D$ in $\boldsymbol{b}$ remote from $A$. By (4) there exists $E$ remote from $A$ with $Y-E$ remote from $D$. So for every $C$ in $\boldsymbol{b}$ we have $C-E$ remote from $D$. By (a), $C-E$ cannot belong to $\boldsymbol{b}$. So (c) implies $C\cap E$ belongs to $\boldsymbol{b}$. Hence, $A\cup B$ is close to $C\cap E$. But, since $A$ is remote from $C\cap E$, (3) implies that $B$ is close to $C\cap E$, hence close to $C$. So $B$ belongs to $\boldsymbol{c}$, giving (c). Thus $\boldsymbol{c}$ is a cluster.



By (b) any cluster $\boldsymbol{d}$ containing $\boldsymbol{b}$ must contain $\boldsymbol{c}$. By (a) every set belonging to $\boldsymbol{d}$ is close to every set in $\boldsymbol{c}$, hence must belong to $\boldsymbol{c}$, by (b). Thus $\boldsymbol{c}$ contains $\boldsymbol{d}$. So $\boldsymbol{c}=\boldsymbol{d}$, which makes $\boldsymbol{c}$ unique.

Let $\boldsymbol{c}$ be a cluster from $Y$ such that $X$ belongs to $\boldsymbol{c}$. Let $\boldsymbol{b}$ be the class of all subsets of $X$ which belong to $\boldsymbol{c}$. That $\boldsymbol{b}$ satisfies (a) and (c) is a trivial consequence of the corresponding properties of $\boldsymbol{c}$.

To prove $\boldsymbol{b}$ satisfies (b) let $A$ be a subset of $X$ that does not belong to $\boldsymbol{c}$. We must show that $A$ is remote from some subset of $X$ which belongs to $\boldsymbol{c}$. Since $A$ is not in $\boldsymbol{c}$ there exists, by (b), $C$ in $\boldsymbol{c}$ with $C$ remote from $A$. By (4) there exists $E$ remote from $A$ with $Y-E$ remote from $C$. So $Y-E$ cannot belong to $\boldsymbol{c}$. Hence $X-E$ does not belong to $\boldsymbol{c}$. Since, by hypothesis, $X$ belongs to $\boldsymbol{c}$, $X\cap E$ belongs to $\boldsymbol{c}$. Since $E$ is remote from $A$, $X\cap E$ is remote from $A$. So $\boldsymbol{b}$ satisfies (b). Thus $\boldsymbol{b}$ is a cluster from $X$.

Uniqueness is a trivial consequence of (a) and (b), which completes the proof of the theorem.

**4. The compactification of $X$.** In section 3 we noted that every point $x$ determines a cluster $\boldsymbol{x}$ consisting of all sets close to $x$. For $A$ any subset of $X$ let $\boldsymbol{A}$ be the set of all clusters $\boldsymbol{x}$ determined by points $x$ in $A$. Let $\bar{\boldsymbol{A}}$ be the set of all clusters to which $A$ belongs. Clearly $\bar{\boldsymbol{A}}$ contains $\boldsymbol{A}$ and $\bar{\boldsymbol{X}}$ is the set of all clusters from $X$.

By (5) the correspondence between $X$ and $\boldsymbol{X}$, induced by the identification of $x$ with the cluster $\boldsymbol{x}$ determined by $x$, is one-to-one.

A subset $A$ of $X$ *absorbs* a subset $\boldsymbol{P}$ of $\bar{\boldsymbol{X}}$ if $A$ belongs to every cluster in $\boldsymbol{P}$, that is, if $\bar{\boldsymbol{A}}$ contains $\boldsymbol{P}$. We define a proximity relation in $\bar{\boldsymbol{X}}$ as follows: $\boldsymbol{P}$ is close to $\boldsymbol{Q}$ if and only if $A$ is close to $B$ in $X$ whenever $A$ absorbs $\boldsymbol{P}$ and $B$ absorbs $\boldsymbol{Q}$. We must show that this satisfies the proximity axioms.

Axiom (1), symmetry, is a trivial consequence of symmetry for the proximity relation in $X$.

Setting $\boldsymbol{Q}=\bar{\boldsymbol{X}}$ above, we have $\boldsymbol{P}$ close to $\bar{\boldsymbol{X}}$ if and only if $A$ is close to $B$ whenever $A$ absorbs $\boldsymbol{P}$ and $B$ is dense in $X$. Let $\boldsymbol{P}$ be non-empty, $A$ absorb $\boldsymbol{P}$, and $B$ be dense in $X$. Then $A$ is non-empty, so $A$ is close to $X$, by (2). Since $B$ is dense in $X$, $A$ is close to $B$, by (i) and (iv). Hence, $\boldsymbol{P}$ is close to $\bar{\boldsymbol{X}}$.

Conversely, let $\boldsymbol{P}$ be empty. Let $A$ be empty and $B$ be $X$. Then $A$ is remote from $B$ by (2). Hence, $\boldsymbol{P}$ is remote from $\bar{\boldsymbol{X}}$. So (2) holds in $\bar{\boldsymbol{X}}$.

To prove (3) let $\boldsymbol{P}\cup\boldsymbol{Q}$ be close to $\boldsymbol{R}$ and $\boldsymbol{P}$ be remote from $\boldsymbol{R}$ in $\bar{\boldsymbol{X}}$. Let $B$ absorb $\boldsymbol{Q}$ and $C$ absorb $\boldsymbol{R}$. We must show that $\boldsymbol{Q}$ is close to $\boldsymbol{R}$ by proving $B$ is close to $C$. Since $\boldsymbol{P}$ is remote from $\boldsymbol{R}$, there exists $A$ absorbing $\boldsymbol{P}$ and $D$ absorbing $\boldsymbol{R}$ with $A$ remote from $D$. By (4) there exists $E$

remote from $A$ with $X-E$ remote from $D$. The latter condition implies $C-E$ is remote from $D$. Since $D$ belongs to every cluster in $\boldsymbol{R}$, $C-E$ belongs to no cluster in $\boldsymbol{R}$, by (a). But $C$ belongs to every cluster in $\boldsymbol{R}$. Hence by (c), $C\cap E$ belongs to every cluster in $\boldsymbol{R}$. That is, $C\cap E$ absorbs $\boldsymbol{R}$. Since by (c), $\overline{A\cup B}=\bar{A}\cup\bar{B}$, $A\cup B$ absorbs $\boldsymbol{P}\cup\boldsymbol{Q}$. Thus, since $\boldsymbol{P}\cup\boldsymbol{Q}$ is close to $\boldsymbol{R}$, $A\cup B$ is close to $C\cap E$. But since $A$ is remote from $C$, $A$ is remote from $C\cap E$, by (i). Hence $B$ is close to $C\cap E$. So $B$ is close to $C$, by (i).

Conversely, let $\boldsymbol{Q}$ be close to $\boldsymbol{R}$. Let $D$ absorb $\boldsymbol{P}\cup\boldsymbol{Q}$ and $C$ absorb $\boldsymbol{R}$. Then $D$ absorbs $\boldsymbol{Q}$. So $D$ is close to $C$. Hence $\boldsymbol{P}\cup\boldsymbol{Q}$ is close to $\boldsymbol{R}$. Thus (3) holds in $\overline{\boldsymbol{X}}$.

To prove (4) let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be remote subsets of $\overline{\boldsymbol{X}}$. Then there exist remote subsets $A$ and $B$ of $X$ such that $A$ absorbs $\boldsymbol{P}$ and $B$ absorbs $\boldsymbol{Q}$. By (4) there exists $E$ remote from $A$ with $X-E$ remote from $B$. The latter condition implies $X-E$ belongs to no cluster in $\boldsymbol{Q}$. Hence $E$ absorbs $\boldsymbol{Q}$. Let $\boldsymbol{R}=\overline{\boldsymbol{E}}$. Then $E$ absorbs $\boldsymbol{R}$, $A$ absorbs $\boldsymbol{P}$, and $E$ is remote from $A$. So $\boldsymbol{R}$ is remote from $\boldsymbol{P}$. Since $E$ belongs to no cluster in $\overline{\boldsymbol{X}}-\boldsymbol{R}$, $X-E$ absorbs $\overline{\boldsymbol{X}}-\boldsymbol{R}$. Also, $B$ absorbs $\boldsymbol{Q}$ and $X-E$ is remote from $B$. So $\overline{\boldsymbol{X}}-\boldsymbol{R}$ is remote from $\boldsymbol{Q}$. Thus (4) holds in $\overline{\boldsymbol{X}}$.

$\boldsymbol{P}$ is close to $\boldsymbol{c}$ if and only if $A$ is close to $B$ whenever $A$ absorbs $\boldsymbol{P}$ and $B$ absorbs $\boldsymbol{c}$. Now the latter condition means just that $B$ belongs to $\boldsymbol{c}$. So by (b), $\boldsymbol{P}$ is close to $\boldsymbol{c}$ if and only if $A$ belongs to $\boldsymbol{c}$ whenever $A$ absorbs $\boldsymbol{P}$. In particular, $\boldsymbol{b}$ is close to $\boldsymbol{c}$ if and only if $A$ belongs to $\boldsymbol{c}$ whenever $A$ belongs to $\boldsymbol{b}$, that is, if and only if $\boldsymbol{b}=\boldsymbol{c}$. So (5) holds in $\overline{\boldsymbol{X}}$.

Given subsets $A$ and $B$ of $X$, $A$ absorbs $\boldsymbol{B}$ if and only if $\bar{A}$ contains $\boldsymbol{B}$ in $X$. So $\boldsymbol{B}$ is close to $\boldsymbol{c}$ if and only if $B$ belongs to $\boldsymbol{c}$. That is, $\overline{\boldsymbol{B}}$ is just the closure of $\boldsymbol{B}$ in $\overline{\boldsymbol{X}}$. So $\boldsymbol{X}$ is dense in $\overline{\boldsymbol{X}}$. Now $\boldsymbol{A}$ is close to $\boldsymbol{B}$ if and only if $C$ is close to $D$ whenever $\bar{C}$ contains $A$ and $\bar{D}$ contains $B$. That is, $\boldsymbol{A}$ is close to $\boldsymbol{B}$ if and only if $A$ is close to $B$. So $\boldsymbol{X}$ is isomorphic to $X$ as a proximity space.

By Theorem 3, since $\boldsymbol{X}$ is dense in $\overline{\boldsymbol{X}}$, every cluster $\boldsymbol{c}'$ from $\overline{\boldsymbol{X}}$ is determined by a cluster $\boldsymbol{c}$ from $X$. Since $B$ belongs to $\boldsymbol{c}$ if and only if $\boldsymbol{c}$ is close to $\boldsymbol{B}$, Theorem 3 implies $\boldsymbol{c}$ belongs to $\boldsymbol{c}'$. So every cluster from $\overline{\boldsymbol{X}}$ has a point. By Theorem 2, $\overline{\boldsymbol{X}}$ is compact.

If $X$ is dense in any compact space $Y$, Theorems 2 and 3 imply that the clusters from $X$ are just those which are determined by points in $Y$. So $Y$ is isomorphic to $\overline{\boldsymbol{X}}$. This makes $\overline{\boldsymbol{X}}$ unique.

We have thus proved the following compactification theorem.

THEOREM 4. *Every proximity space is a dense subspace of a unique compact Hausdorff space in which two sets are close if and only if their closures intersect.*



**5. Extension of proximity mappings.** Let $f$ be a single-valued function from a proximity space $X$ into a proximity space $Y$. We call $f$ a *proximity mapping* if $f$ preserves proximity: $fA$ is close to $fB$ in $Y$ whenever $A$ is close to $B$ in $X$. (See [1].) Taking $A$ to be a point, we clearly see that a proximity mapping is continuous. If $X$ is compact, the converse follows from the corollary in section 3. We shall show that every proximity mapping on $X$ is induced by a unique continuous mapping on $\overline{\boldsymbol{X}}$.

THEOREM 5. *Every proximity mapping $f$ with domain $X$ has a unique extension to a continuous mapping $\bar{f}$ which maps the compactification of $X$ onto the compactification of $fX$.*

Proof. Let $Y=fX$. Given a cluster $\boldsymbol{c}$ from $X$ define a class $\boldsymbol{d}$ of subsets of $Y$ as follows: A subset $P$ of $Y$ belongs to $\boldsymbol{d}$ if and only if $P$ is close to $fC$ for every $C$ belonging to $\boldsymbol{c}$. Clearly, $fC$ belongs to $\boldsymbol{d}$ for every $C$ in $\boldsymbol{c}$, since $f$ is proximity-preserving and $\boldsymbol{c}$ satisfies (a). We contend that $\boldsymbol{d}$ is a cluster.

Let $P$ be remote from $Q$ in $Y$. Then there exists $S$ remote from $Q$ with $Y-S$ remote from $P$. Let $E=f^{-1}S$. For $C$ in $\boldsymbol{c}$ we have either $C\cap E$ or $C-E$ in $\boldsymbol{c}$, by (c). Since $f(C\cap E)$ is a subset of $S$, $Q$ is remote from $f(C\cap E)$, by (i). Similarly, $P$ is remote from $f(C-E)$. So either $P$ or $Q$ fails to belong to $\boldsymbol{d}$. Contrapositively, if $P$ and $Q$ belong to $\boldsymbol{p}$, $P$ is close to $Q$. So $\boldsymbol{d}$ satisfies (a).

If $P$ is close to every $Q$ in $\boldsymbol{d}$, then $P$ is close to $fC$ for every $C$ in $\boldsymbol{c}$, since $fC$ belongs to $\boldsymbol{d}$. So $P$ belongs to $\boldsymbol{d}$. Hence, $\boldsymbol{d}$ satisfies (b).

Let $P\cup Q$ belong to $\boldsymbol{d}$ with $P$ not in $\boldsymbol{d}$. Then $P$ is remote from $fA$ for some $A$ in $\boldsymbol{c}$. Hence, there exists $S$ remote from $P$ with $Y-S$ remote from $fA$, by (4). Let $E=f^{-1}S$. For every $C$ in $\boldsymbol{c}$, $f(C-E)$, being contained in $Y-S$, is remote from $fA$ by (i). So $C-E$ is remote from $A$ since $f$ is proximity-preserving. Thus by (a), $C-E$ cannot belong to $\boldsymbol{c}$. So by (c), $C\cap E$ belongs to $\boldsymbol{c}$. Hence, $P\cup Q$ is close to $f(C\cap E)$. Since $f(C\cap E)$ is a subset of $S$ which is remote from $P$, $f(C\cap E)$ is remote from $P$. Hence, $f(C\cap E)$ is close to $Q$. So $fC$ is close to $Q$. Thus $Q$ belongs to $\boldsymbol{d}$, which proves that $\boldsymbol{d}$ satisfies (c). So $\boldsymbol{d}$ is a cluster from $Y$.

Define a mapping $\bar{f}$ from $\overline{\boldsymbol{X}}$ into $\overline{\boldsymbol{Y}}$ by letting $\bar{f}\boldsymbol{c}=\boldsymbol{d}$. For $\boldsymbol{x}$ the cluster of sets close to $x$, $\bar{f}\boldsymbol{x}$ is clearly the cluster of sets close to $fx$. So $\bar{f}$ on $\boldsymbol{X}$ agrees with $f$ on $X$ under the identification of $\boldsymbol{x}$ with $x$.

To show that $\bar{f}$ is a proximity mapping, hence continuous, let $\boldsymbol{A}$ be close to $\boldsymbol{B}$ in $\overline{\boldsymbol{X}}$. We must show that $\bar{f}\boldsymbol{A}$ is close to $\bar{f}\boldsymbol{B}$ in $\boldsymbol{Y}$. Let $P$ absorb $\bar{f}\boldsymbol{A}$ and $Q$ absorb $\bar{f}\boldsymbol{B}$. We have only to prove $P$ is close to $Q$ in $Y$.

Suppose that $P$ is remote from $Q$. Then by (4), there exist remote sets $R$ and $S$ with $P$ remote from $Y-R$ and $Q$ remote from $Y-S$.

Since $Q$ belongs to every cluster in $\bar{f}\boldsymbol{B}$, $Y-S$ belongs to no cluster in $\bar{f}\boldsymbol{B}$. Thus $f^{-1}(Y-S)$, which equals $X-f^{-1}S$, belongs to no cluster in $\boldsymbol{B}$. So by (c), $f^{-1}S$ belongs to every cluster in $\boldsymbol{B}$. Similarly, $f^{-1}R$ belongs to every cluster in $\boldsymbol{A}$. Since $\boldsymbol{A}$ is close to $\boldsymbol{B}$, $f^{-1}R$ is close to $f^{-1}S$. Since $f$ preserves proximity, $R$ is close to $S$, a contradiction. So $P$ must be close to $Q$. Hence $\bar{f}$ preserves proximity.

Now $Y\subset\bar{f}\bar{X}\subset\bar{Y}$ with $\bar{f}\bar{X}$ compact. Thus, since $Y$ is dense in $\bar{Y}$, $\bar{Y}=\bar{f}\bar{X}$. So $\bar{f}$ maps $\bar{X}$ *onto* $\bar{Y}$.

Let $\bar{f}$ and $\bar{g}$ be continuous mappings of $\bar{X}$ onto $\bar{Y}$ with $\bar{f}\boldsymbol{c}$ distinct from $\bar{g}\boldsymbol{c}$ for some $\boldsymbol{c}$ in $\bar{X}$. Since $\bar{Y}$ is Hausdorff, there exists a neighborhood $\boldsymbol{E}$ of $\boldsymbol{c}$ with $\bar{f}\boldsymbol{E}$ and $\bar{g}\boldsymbol{E}$ disjoint. Since $X$ is dense in $\bar{X}$, there exists some $\boldsymbol{x}$ in $\boldsymbol{E}\cap X$. For such $\boldsymbol{x}$, $\bar{f}\boldsymbol{x}$ is distinct from $\bar{g}\boldsymbol{x}$. This proves that the extension $\bar{f}$ of $f$ is unique, completing the proof of the theorem.

**6. Applications of the compactification theory.** The Urysohn lemma and Theorems 4 and 5 immediately give the following result, which is proved directly in [1].

THEOREM 6. *$A$ is remote from $B$ in a proximity space $X$ if and only if there exists a proximity mapping of $X$ into the interval* $[0,1]$ *which maps $A$ onto* 0 *and $B$ onto* 1.

Similarly, the Tietze extension theorem assumes the following form.

THEOREM 7. *Let $A$ be any subspace of a proximity space $X$ and $f$ be a proximity mapping of $A$ into the interval* $[-1,1]$. *Then $f$ can be extended to a proximity mapping of $X$ into* $[-1,1]$.

Let $\mathfrak{A}$ be the class of all bounded, real-valued, proximity mappings on a proximity space $X$. By Theorem 5, each $f$ in $\mathfrak{A}$ is uniquely defined by a continuous mapping $\bar{f}$ of the compact Hausdorff space $\bar{X}$ into the real numbers. Thus $\mathfrak{A}$ is a Banach algebra under the uniform norm.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be distinct clusters from $X$, that is, distinct points in $\bar{X}$. Then there exist $A$ in $\boldsymbol{a}$ and $B$ in $\boldsymbol{b}$ with $A$ remote from $B$. If $fA$ is remote from $fB$, then $\bar{f}\boldsymbol{a}$ is distinct from $\bar{f}\boldsymbol{b}$.

We thus obtain the following extension of the Stone-Weierstrass theorem [15].

THEOREM 8. *Let $\mathfrak{A}$ be the algebra of all bounded, real-valued, proximity mappings on a proximity space $X$. Let $\mathfrak{B}$ be a subalgebra of $\mathfrak{A}$ such that:*

(I) *$\mathfrak{B}$ contains a mapping whose range is bounded away from* 0,

(II) *Given $A$ remote from $B$ in $X$, there exists some $f$ in $\mathfrak{B}$ with $fA$ remote from $fB$.*

*Then $\mathfrak{B}$ is uniformly dense in $\mathfrak{A}$.*



By our discussion above, condition (II) means just that for $\boldsymbol{a}$ and $\boldsymbol{b}$ distinct points in $\bar{X}$ there exists $f$ in $\mathfrak{B}$ with $\bar{f}\boldsymbol{a}$ and $\bar{f}\boldsymbol{b}$ distinct. Condition (I) means that for every point $\boldsymbol{c}$ in $\bar{X}$ there exists $f$ in $\mathfrak{B}$ with $\bar{f}\boldsymbol{c}$ distinct from 0.

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