Solution of a problem of Tarski

by

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1. Introduction. In his classical paper "A decision method for elementary algebra and geometry" Note 21 ([6], p. 57) A. Tarski raises the question of providing a decision procedure for elementary sentences concerning the field of real numbers which, in addition to equality, order, addition, and multiplication, contain also the relation (atomic predicate) $A(x)$, to be satisfied exclusively by the *algebraic* real numbers. In the present paper, we solve this problem by specifying a complete set of axioms for the above-mentioned relations, including $A(x)$, such that the real numbers constitute a model of that set (sections 3, 4). The corresponding problem for the field of complex numbers is of a somewhat simpler nature (section 5).

We shall be concerned with algebraic fields in which certain additional relations have been defined, more particularly the relation of order, $Q(x,y)$ (i. e. $x \leqslant y$), the relation $A(x)$, and a set of relations $D_{nk}(x_1, \ldots, x_n)$, $n, k = 1, 2, \ldots$, which will be detailed presently. Accordingly, when we say that a field M' is an extension of a field M *including* $Q(x,y)$, and (or) $A(x)$ and (or) $D_{nk}(x_1, \ldots, x_n)$, $n, k = 1, 2, \ldots$ we shall mean by this that, for the elements of M, the relations $Q(x,y)$, $A(x)$, or $D_{nk}(x_1, \ldots, x_n)$ hold in M' if and only if they hold in M. Similarly, when we say that two fields, M and M', are isomorphic, *including* $Q(x,y)$, $A(x)$, and (or) $D_{nk}(x_1, \ldots, x_n)$, we shall mean that there exists an isomorphic correspondence between the two fields such that the relations in question hold, or do not hold, simultaneously for corresponding elements. In particular, an isomorphism which includes $Q(x,y)$ is simply an order-preserving isomorphism.

We shall also require the notion of relativisation with respect to the relation $A(x)$ (compare, e. g., [7], p. 24). This is defined inductively as follows (1.1-1.3)

1.1. The relativised form of an atomic formula X (e. g. $E(x,y)$, $S(x,a,b)$) is the formula itself. $R(X) = X$.

1.2. $R(\sim X) = \sim R(X)$, $R(X \wedge Y) = R(X) \wedge R(Y)$, $R(X \vee Y) = R(X) \vee \vee R(Y)$ if $\sim, \wedge, \vee$ are the basic connectives of the language. If other connectives are regarded as basic, the list 1.2 has to be extended accordingly.

1.3. $R((\exists z)\, Y(z)) = (\exists x)\, [A(z) \wedge R(Y(z))]$,

$R((z)\, Y(z)) = (z)[A(z) \supset R(Y(z))]$.

2. Auxiliary notions and results from algebra. We shal suppose that the reader is familiar with the concept of a real-closed field. A real-closed ordered field can be characterized as an ordered field in which every positive number has a square root and every polynomial of odd degree has a root. It can also be characterized as an ordered field in which every polynomial of positive degree can be decomposed into linear and quadratic factors, the latter being of the form $(x-a)^2+b$, with $b>0$.

Let M be a real-closed ordered field which is an order-preservig extension of an ordered field M_0. Let M_1 be the field of elements of M which are algebraic with respect to M_0. Then it is not difficult to show that M_1 is real-closed.

Let M be an ordered field which contains real-closed subfields M_1 and M_2. Let $M_0 = M_1 \cap M_2$. Then M_0 is a real-closed ordered field.

Let M be a field containing subfields M_1 and M_2 and let M_0 be a field contained in both M_1 and M_2. Suppose that every set of elements of M_1 which is algebraically independent with respect to M_0 is algebraically independent also with respect to M_2. Then every set of elements of M_2 which is algebraically independent with respect to M_0 is algebraically independent also with respect to M_1 (see [8], p. 3) and M_1 and M_2 are said to be algebraically independent over M_0.

If, in the last two sentences, we replace "algebraically independent" everywhere by "linearly disjoint" then we obtain a parallel result, and a parallel definition related to the notion of linear disjointness in place of algebraic independence (see [8], B. 4).

2.1. THEOREM. *Let M be a field which contains algebraically closed subfields M_1 and M_2, and let $M_0 = M_1 \cap M_2$. Suppose that M_1 and M_2 are algebraically independent over M_0. Then M_1 and M_2 are linearly disjoint over M_0.*

Proof. Suppose, contrary to the conclusion of the theorem, that there exists a set of elements of M_1, $\{a_1, \ldots, a_n\}$ say, $n \geqslant 1$, which is linearly dependent over M_2 although it is linearly independent over M_0. Consider a set of this kind for which n is a minimum. By assumption, there exist elements $b_1, \ldots, b_n$ in M_2, not all zero, such that

2.2. $b_1 a_1 + b_2 a_2 + \ldots + b_n a_n = 0$.

Moreover, by the minimum property of $\{a_1, \ldots, a_n\}$ $b_n \neq 0$, and so we may suppose that more particularly $b_n = 1$. Indeed, if this is not the case from the outset then it can be achieved by dividing the equation by the last coefficient.

Now if all of $b_1, \ldots, b_{n-1}$ belong to M_0 then we have finished since in that case the set $\{a_1, \ldots, a_n\}$ would be linearly dependent over M_0. Accordingly we may suppose that b_1 does not belong to M_0, $b_1 \in M_2 - M_0$. We may supplement b_1 by a (finite of infinite) number of elements of $M_2 - M_0$ so as to obtain a transcendence base S of M_2 over M_0. Then for some finite subset S' of S, $S' = \{\sigma_1, \ldots, \sigma_k\}$, say, the coefficients $b_1, \ldots, b_{n-1}$ all depend algebraically on $M_0(\sigma_1, \ldots, \sigma_k)$, and we may suppose $\sigma_1 = b_1$. Let $\overline{M}$ be the algebraic closure of $M_1(\sigma_1, \ldots, \sigma_k)$, then the quantities which appear on the left hand side of 2.2 all belong to $\overline{M}$.

The elements $\sigma_1, \ldots, \sigma_k$ of $M_2 - M_0$ are algebraically independent over M_0 and hence, by the assumption of the theorem, are algebraically independent over M_1. It follows that the one-one correspondence under which the elements of M_1 correspond to themselves while

$$\sigma_1 \leftrightarrow \sigma_1 + 1, \quad \sigma_2 \leftrightarrow \sigma_2, \quad \ldots, \quad \sigma_k \leftrightarrow \sigma_k$$

can be extended to an automorphism of $\overline{M}$ in which

$$a_1 \leftrightarrow a_1, \quad a_2 \leftrightarrow a_2, \quad \ldots, \quad a_n \leftrightarrow a_n$$

and $b_1 \leftrightarrow b_1 + 1$ (since $\sigma_1 = b_1$) while $b_2 \leftrightarrow b_2', \ldots, b_n \leftrightarrow b_n'$, say. Hence, from 2.2,

2.3. $(b_1+1)\, a_1 + b_2' a_2 + \ldots + b_n' a_n = 0$.

Subtracting 2.2 from 2.3, we obtain

$$a_1 + (b_2' - b_2)\, a_2 + \ldots + (b_n' - b_n)\, a_n = 0 .$$

Now the b_j, $j = 2, \ldots, n-1$, satisfy non-vanishing polynomials $q_j(x)$ whose coefficients are polynomials of $\sigma_1, \ldots, \sigma_k$ with coefficients in M_0. The above-mentioned automorphism transforms $q_j(x)$ into $q_j'(x)$, also non-vanishing and with coefficients in $M_0[\sigma_1, \ldots, \sigma_k]$, $j = 2, \ldots, n$. It follows that the b_j', $j = 2, \ldots, n-1$, are algebraic with respect to $M(\sigma_1, \ldots, \sigma_k)$, and hence belong to M_2. Thus, the quantities

$$b_j'' = b_j' - b_j, \qquad j = 2, \ldots, n-1 ,$$

all belong to M_2 while $b_n = 1$ by our simplifying assumption, and so $b_n' = 1$, $b_n' - b_n = 0$. Hence

$$a_1 + b_2'' a_2 + \ldots + b_{n-1}'' a_{n-1} = 0 ,$$

showing that the set $\{a_1, \ldots, a_{n-1}\}$ is linearly dependent over M_2. The minimum property of n now implies that $\{a_1, \ldots, a_{n-1}\}$, and hence $\{a_1, \ldots, a_n\}$, is linearly dependent over M_0. This involves a contradiction and proves 2.1.

Let M_0 be the intersection of two fields, M_1 and M_2, which are contained in a joint extension M. For M_1 and M_2 to be algebraically independent over M_0, it is necessary that if $\{s_1, \ldots, s_k\}$ and $\{t_1, \ldots, t_n\}$ are any two algebraically independent sets over M_0 which belong to M_1 and M_2 respectively, then $\{s_1, \ldots, s_k, t_1, \ldots, t_n\}$ also is algebraically independent over M_0. It is not difficult to see that this condition is also sufficient, for if a set $\{s_1, \ldots, s_k\}$ of elements of M_1 satisfies a non-vanishing polynomial with coefficients in M_2, then there must exist a non-vanishing polynomial $p(x_1, \ldots, x_k)$, whose coefficients are polynomials (with coefficients in M_0) of a set of elements of M_2, $\{t_1, \ldots, t_n\}$, $n \geqslant 0$, which is algebraically independent over M_0, such that $p(s_1, \ldots, s_k) = 0$. And the existence of such an equation signifies that the set $\{s_1, \ldots, s_k, t_1, \ldots, t_n\}$ is algebraically dependent over M_0.

2.4 THEOREM. *Let S and T be two fixed transcendence bases over M_0 (maximal sets of algebraically independent elements), of the fields M_1 and M_2 respectively, where $M_0 = M_1 \cap M_2$. In order that M_1 and M_2 be algebraically independent over M_0 it is necessary and sufficient that the set $S \cup T$ be independent over M_0.*

Proof. Necessity is obvious. To prove sufficiency, let $\{s_1, \ldots, s_k\}$, $\{t_1, \ldots, t_n\}$ be algebraically independent sets over M_0, and belonging to M_1 and M_2 respectively. Then the elements $s_1, \ldots, s_k$ depend on a finite subset $\{\sigma_1, \ldots, \sigma_j\}$ of S over M_0, and the elements $t_1, \ldots, t_n$ depend on a finite subset $\{\tau_1, \ldots, \tau_m\}$ of T over M_0, $j \geqslant k$, $m \geqslant n$. By the exchange theorem of Steinitz, the set $\{\sigma_1, \ldots, \sigma_j\}$ is, for a certain numbering of the σ, equivalent to $\{s_1, \ldots, s_k, \sigma_{k+1}, \ldots, \sigma_j\}$ and the set $\{\tau_1, \ldots, \tau_m\}$ is, in a similar way, equivalent to $\{t_1, \ldots, t_n, \tau_{n+1}, \ldots, \tau_m\}$. It follows that the elements $\sigma_1, \ldots, \sigma_j, \tau_1, \ldots, \tau_m$ depend algebraically on $\{s_1, \ldots, s_k, \sigma_{k+1}, \ldots, \sigma_j, t_1, \ldots, t_n, \tau_{n+1}, \ldots, \tau_m\}$ over M_0. But the $j+m$ elements $\sigma_1, \ldots, \tau_m$ are algebraically independent over M_0 and so the set $\{s_1, \ldots, \sigma_{k+1}, \ldots, \sigma_j, t_1, \ldots, \tau_m\}$ also is algebraically independent over M_0, and in particular, $\{s_1, \ldots, s_k, t_1, \ldots, t_n\}$ is algebraically independent over M_0.

2.5 THEOREM. *Let M be an ordered field which contains real closed subfields M_1 and M_2, and let $M_0 = M_1 \cap M_2$. Suppose that M_1 and M_2 are algebraically independent over M_0. Then M_1 and M_2 are linearly disjoint over M_0.*

Proof. Let M_0^*, M_1^*, M_2^* be the algebraic closures of M_0, M_1, M_2, respectively, $M_0^* = M_0(i)$, $M_1^* = M_1(i)$, $M_2^* = M_2(i)$, $M_0^* = M_1^* \cap M_2^*$. Let

S and T be transcendence bases of M_1 and M_2, respectively over M_0. By the assumption of the theorem, $S \cap T$ is algebraically independent over M_0, and hence over M_0^*. But S and T are also transcendence bases of M_1^* and M_2^* over M_0^*, and so, by Theorem 2.4, M_1^* and M_2^* are algebraically independent over M_0^*. It then follows from 2.1 that M_1^* and M_2^* are also linearly disjoint over M_0^*.

Now let $\{s_1, \ldots, s_k\}$ be a set of elements of M_1, which is linearly independent over M_0. Then the set is linearly independent over $M_0^* = M_0(i)$ also. It follows that $\{s_1, \ldots, s_k\}$ is linearly independent over M_2, and hence that it is linearly independent over M_2^*, as asserted by the theorem.

To illustrate a subsequent remark, we shall now display two real-closed Archimedean ordered fields which are not algebraically independent (and not linearly disjoint) over their intersection.

Let M_0 be the field of real algebraic numbers, and let s_1, s_2, t_1 be three real transcendental numbers, such that $\{s_1, s_2, t_1\}$ is algebraically independent over M (i. e. absolutely). Put

$$t_2 = -s_1 t_1 - s_2 .$$

Then it will be seen that $\{t_1, t_2\}$ is algebraically independent over M_0 and the same applies to $\{s_2, t_2\}$.

Let M_1 be the real closure of $M_0(s_1, s_2)$ within the field of real numbers, let M_2 be the real closure of $M_0(t_1, t_2)$, and similarly, let M be the real closure of $Q_0 = M_0(s_1, s_2, t_1, t_2) = M_0(s_1, s_2, t_1)$. Then M is the real closure of the compositum of M_1 and M_2. All these fields are ordered by their natural order in the field of real numbers. We wish to show that the intersection of M_1 and M_2 is M_0. If not, there exists a real transcendental number ξ in M which satisfies polynomials $p(x)$, $q(x)$ with coefficients in $M_0[s_1, s_2]$ and $M_0[t_1, t_2]$ respectively, and irreducible in these rings. Moreover, we may suppose that $p(x)$ and $q(x)$ are primitive in these rings, respectively.

Put $R_0 = M_0[s_1, s_2, t_1, t_2] = M_0[s_1, s_2, t_1]$, then we have also that $R_0 = M_0[t_1, t_2, s_1]$ since $s_2 = -s_1 t_1 - t_2$. Let a_0 be the field of quotients of R_0. Since $p(x)$ and $q(x)$ have a common root in an extension of Q_0 they must have a common factor in $Q_0[x]$. The coefficients of $p(x)$ belong to R_0, and so if $p(x)$ is reducible in $Q_0[x]$ it must be reducible also in $R_0[x]$. But R_0 is obtained from $M_0[s_1, s_2]$ by the adjunction of t_1, which is transcendental with respect to $M_0[s_1, s_2]$ and $p(x)$ is irreducible in $M_0[s_1, s_2][x]$. It follows that $p(x)$ is irreducible in $R_0[x]$. Similarly, taking into account that $R_0 = M_0[t_1, t_2, s_1]$, we find that $q(x)$ is irreducible in $R_0[x]$. But $p(x)$ and $q(x)$ have a common factor, and so

$$p(x) = dq(x)$$

where d belongs to Q_0. Writing d in the form

$$d = d_1(s_1, s_2, t_1)/d_2(s_1, s_2, t_1)$$

where d_1 and d_2 are polynomials with real algebraic coefficients and without a (non-trivial) common factor, we then obtain

$$d_2(s_1, s_2, t_1)p(x) = d_1(s_1, s_2, t_1)q(x).$$

It follows that $d_1(s_1, s_2, t_1)$ divides the coefficients of $p(x)$. But $p(x)$ is primitive in $M_0[s_1, s_2][x]$ and hence also in $R_0[x]$. It follows that d_1 belongs to M_0. Also, writing $-s_1t_1-t_2$ for s_2 in d_2 we obtain a polynomial of t_1, t_2, s_1 with coefficients in M_0 and we conclude in the same way that d_2 also reduces to an element of M_0. Hence $p(x) = dq(x)$ where $d \in M_0$.

Let $n \geqslant 0$ and compare the coefficients of x^n in $p(x)$ and $q(x)$. These are polynomials of s_1, s_2 and of t_1, t_2 respectively, with coefficients in M_0, and will be denoted by $p_n(s_1, s_2)$ and $q_n(t_1, t_2)$. Then

2.6. $p_n(s_1, s_2) = dq_n(t_1, t_2)$

and so

2.7. $p_n(s_1, s_2) = dq_n(t_1, -s_1t_1-s_2)$.

Now s_1, s_2, and t_1 are algebraically independent over M_0, so we obtain a valid relation by writing 0 for s_1 in 2.7. Then

2.8. $p_n(0, s_2) = dq_n(t_1, -s_2)$.

But the left hand side of 2.8 is independent of t_1 and so the same applies to the right hand side. It follows that $q_n(t_1, t_2)$ is independent of t_1 and similarly, that $p_n(s_1, s_2)$ is independent of s_1. If so, then 2.6 yields an algebraic relation with coefficients in M_0 between s_2 and t_2. But s_2 and t_2 are algebraically independent over M_0 and so the coefficients p_n, q_n all reduce to elements of M_0. It follows that ξ is an algebraic number, contrary to assumption. We conclude that $M_1 \cap M_2 = M_0$. M_1 and M_2 are not algebraically independent over M_0 since $\{s_1, s_2\}$ is algebraically independent over M_0 while satisfying the equation

$$s_1t_1+s_2+t_2 = 0$$

whose coefficients belong to M_2.

3. Distinction of the algebraic numbers in the field of real numbers. Let K be a set of axioms for the concept of a real-closed ordered field, formulated in the lower predicate calculus in terms of the relations of equality, $E(x, y)$, addition, $S(x, y, z)$, multiplication, $P(x, y, z)$, and order, $Q(x, y)$, and without individual constants (compare [2], p. 43). Let K_A be the set of sentences obtained by relativising the sentences of K with respect to the relation $A(x)$, and define sentences X_1, X_2, X_3, and X_4 by

3.1. $X_1 = (\exists x)A(x)$, $X_2 = (\exists x)\sim A(x)$,

3.2. $X_3 = (x)(y)[E(x, y) \supset [A(x) \supset A(y)]]$,

3.3. $X_4 = (x)(y)(\exists z)[Q(x, y) \wedge \sim E(x, y) \supset Q(x, z) \wedge Q(z, y) \vee A(z)]$,

respectively. (We observe that X_1 may or may not be a consequence of K_A depending on the detailed formulation of the axioms of K. If the system of [2] is used, X_1 follows from K_A in view of the last axiom on p. 38 of that reference.)

Let M be a model of K which satisfies X_1 and let M_A be the set of elements of M which satisfy $A(x)$. Then M_A is a model of K. X_4 requires that M_A be dense in M.

Let n and k be two integers, positive but otherwise arbitrary. It is not difficult to formulate in the lower predicate calculus, in terms of the relations E, S, and P, a predicate $Q_{nk}(x_1, \ldots, x_n)$ which states that "$x_1, \ldots, x_n$ satisfy a non-vanishing polynomial of degree not exceeding k with coefficients in M_A (i. e. satisfying $A(x)$).". More precisely, we may formulate $Q_{nk}(x_1, \ldots, x_n)$ as an *existential* predicate, i. e. in prenex normal form with existential quantitiers only.

We now define the *sentence* X_{nk} by

3.4. $X_{nk} = (x_1)\ldots(x_n)[D_{nk}(x_1, \ldots, x_n) \equiv Q_{nk}(x_1, \ldots, x_n)]$

where $D_{nk}(x_1, \ldots, x_n)$ is a new relation. Let K_D be the set of sentences $\{X_{nk}\}$, $n, k = 1, 2, 3, \ldots$, and let

3.5. $K^* = K \cup K_A \cup \{X_1, X_2, X_3, X_4\} \cup K_D$.

Then

3.6 THEOREM. *The set K^* is model-complete.*

In due course, it will be shown by means of an example that the introduction of the D_{nk} is essential for the model-completeness of K^*.

For the proof of 3.6 we shall make use of the model-completeness test of [2], p. 16. To apply this test, consider any model M^* of K^*. M^* is a real-closed ordered field containing a proper subfield which is also real-closed and which consists of all elements of M^* that satisfy $A(x)$. Let

3.7. $X = (\exists y_1)\ldots(\exists y_i)Z(y_1, \ldots, y_i)$

— Z free of quantifiers — be a primitive sentence formulated in terms of (some of) the relations E, S, P, Q, A, and D_{nk}, and in terms of (some of) the individual constants of M^* such that X holds in some extension M^{**} of M^* which is a model of K^*. Then we have to establish that X holds already in M^*. (A primitive sentence is a sentence in prenex normal form whose quantifiers — if any — are existential, and whose matrix is a *conjunction* of atomic formulae and (or) of the negations of such formulae.)

Suppose that X is satisfied by $y_1 = a_1, \ldots, y_i = a_i$ in M^{**}, i. e. that $Z(a_1, \ldots, a_i)$ holds in M^{**} but that X does not hold in M^*. The elements of M^{**} which satisfy $A(x)$ constitute a real-closed proper subfield of M^{**}, M_{AA} say, such that $M^* \cap M_{AA} = M_A$.

We dispose first of the case $M_{AA} = M_A$. In this case, we may suppose M^{**} to be of finite degree of transcendence over M^*. Indeed, if this is not the case from the outset, consider the algebraic closure $\bar{M}$ of $M^*(a_1, \ldots, a_i)$ within M^{**} (the set of elements of M^{**} which are algebraic with respect to $M^*(a_1, \ldots, a_i)$). $\bar{M}$ is real closed. Define that $A(x)$ holds in $\bar{M}$ for the same elements as in M^{**}, i. e. for the elements of M_A. Then $\bar{M}$ is an extension of M^* including $A(x)$ (and, of course, including $Q(x, y)$). It follows that $\bar{M}$ satisfies X, and X_2, and also X_3. Moreover, $M_A = M_{AA}$ is dense in M^{**} and so it is certainly dense in $\bar{M}$. Thus, $\bar{M}$ satisfies X_4 as well.

Finally, we define that for any $b_1, \ldots, b_n$ in $\bar{M}$, $n \geqslant 1$ and for any integer $k \geqslant 1$, $D_{nk}(b_1, \ldots, b_n)$ holds in $\bar{M}$ if it holds in M^{**}. And since $M_{AA} = M_A$ consists of the elements of $\bar{M}$ which satisfy $A(x)$ it follows that $D_{nk}(b_1, \ldots, b_n)$ holds in $\bar{M}$ if and only if $Q_{nk}(b_1, \ldots, b_n)$ holds in that structure. Thus, $\bar{M}$ satisfies K_D, and hence all sentences of K^*.

Moreover, with these definitions, M^{**} is an extension of $\bar{M}$ including Q, A, and the D_{nk}. Hence $Z(a_1, \ldots, a_i)$, and with it X, holds in $\bar{M}$. The degree of transcendence of $\bar{M}$ over M^* does not exceed i and this completes our argument. Accordingly, we shall suppose from now on that M^{**} is of finite degree of transcendence, $m \geqslant 1$, over M^*, while retaining the previous assumption that $M_{AA} = M_A$.

We may then interpolate an ascending sequence of real closed ordered fields between M^* and M^{**}

$$M^{**} = M^{(0)} \subset M^{(1)} \subset \ldots \subset M^{(m)} = M^{**}$$

such that the degree of transcendence of any member of the chain over its predecessor (if any) is 1. We turn each $M^{(j)}$ into a model of K^* by restricting the relations A, D_{nk} of M^{**} to $M^{(j)}$, i. e. by stipulating that these relations hold for given elements of $M^{(j)}$ whenever they hold for the same elements within M^{**}. It will be seen that the resulting structure (which will still be denoted by $M^{(j)}$) satisfies K, K_A, and X_1, X_2, X_3, X_4. To check that $M^{(j)}$ satisfies also K_D consider any X_{nk} as given by 3.4. We have to verify that for any $b_1, \ldots, b_j$ in $M^{(j)}$, $D_{nk}(b_1, \ldots, b_n)$ is indeed equivalent to $Q_{nk}(b_1, \ldots, b_n)$. But this is true since the equivalence in question holds in M^{**}, and since $Q_{nk}(x_1, \ldots, x_n)$ now refers to polynomials with coefficients in the same field M_A for both M^{**} and $M^{(j)}$.

Now let $M^{(l)}$ be the first $M^{(j)}$ in which X holds, $0 < l \leqslant m$. Then $M^{(l-1)}$ does not satisfy X. The set of elements which satisfy $A(x)$ is the same in both fields and constitutes the field M_{AA}. We propose to show that this situation cannot arise.

Let t be an arbitrary but fixed element (individual constant) of $M^{(l)}$ which does not belong to $M^{(l-1)}$. Define the set of sentences, H, as the union of the following sets:

3.8. The set K^* (see 3.5 above);

3.9. the diagram N of $M^{(l-1)}$ including the relations Q, A, and D_{nk} (see [2], p. 6 for the definition of a diagram);

3.10. the set of all sentences $\sim E(a, t)$ where a varies over the elements of $M^{(l-1)}$, together with all the sentences $Q(a, t)$ and $Q(t, a)$ which hold in M^{**} for elements a of $M^{(l-1)}$; and

3.11. for any finite set of elements $\{t_1, \ldots, t_r\}$ of $M^{(l-1)}$ which is algebraically independent over M_A, $r \geqslant 0$, include all sentences

$$\sim D_{r+1,k}(t, t_1, \ldots, t_r), \quad k = 1, 2, \ldots$$

We note that if $\{t_1, \ldots, t_r\}$ is algebraically independent over M_A then $\{t, t_1, \ldots, t_r\}$ also is algebraically independent over M_A, in view of the fact that t is transcendental over $M^{(l-1)}$. Thus, the sentences of 3.11 are all satisfied by $M^{(l)}$. The same applies to the sentences of 3.8-3.10 and so $M^{(l)}$ is a model of H, H is consistent.

Let M_H be an arbitrary model of H. We propose to show that M_H contains a partial structure which is isomorphic to $M^{(l)}$ including Q, A, D_{nk}. Indeed M_H contains $M^{(l-1)}$ since it is a model of 3.9 (or, M_H contains a partial structure which is isomorphic to $M^{(l)}$ depending on the precise definition of the concept of a model). Also, by 3.10, M_H contains the individual constant t, which is different, within M_H, from all elements of $M^{(l-1)}$. Hence $M^{(l-1)}(t)$ is a proper extension of $M^{(l-1)}$ within M_H.

Let $\bar{M}$ be the algebraic closure of $M^{(l-1)}(t)$ within M_H. The ordering of $M^{(l-1)}(t)$ by $Q(x, y)$ within M_H is the same as the ordering of $M^{(l-1)}(t)$ within $M^{(l)}$ since both orderings are determined completely by 3.10 (compare [3], p. 44, 45). It follows that there exists an order preserving isomorphism between $\bar{M}$ and the real closure of $M^{(l-1)}(t)$ within $M^{(l)}$, which is $M^{(l)}$ itself. In this isomorphism, the elements of $M^{(l-1)}$ correspond to themselves. Moreover, since M_H is an extension of $M^{(l-1)}$ including $A(x)$, all elements of M_A satisfy $A(x)$ also within M_H. We maintain that these are the only elements of $\bar{M}$ which satisfy $A(x)$ within M_H.

Suppose on the contrary that there exists an element c of $\bar{M}$ which does not belong to M_A such that $A(c)$ holds in M_H. c depends on t algebraically over $M^{(l-1)}$. Hence, if $T=\{t_\nu\}$ is a transcendence base of $M^{(l-1)}$ over M_A then c depends algebraically on $T\cup\{t\}$ over M_A, and hence on t together with a finite subset T' of T, $T'=\{t_1,\dots,t_r\}$, $r\geqslant 0$. In other words, c satisfies a polynomial equation

3.12. $p(t,t_1,\dots,t_r,c)=0$

with coefficients in M_A, not all zero. On the other hand, by 3.11 there is in M_H no polynomial with coefficients satisfying $A(x)$ and not all zero which is satisfied by $t,t_1,\dots,t_r$. And since $A(c)$ holds in M_H by assumption, we conclude that the polynomials of c, which are the coefficients of the left hand side of 3.12, regarded as a polynomial of its first $r+1$ variables are all equal to zero. But these are polynomials in c with coefficients in M_A, and since c is transcendental with respect to M_A, they can be equal to zero only if all their coefficients vanish. We conclude that the coefficients of the polynomial of 3.12, regarded as a function of $r+2$ variables, all vanish, and thereby arrive at a contradiction. Thus the relation $A(x)$ holds for corresponding elements of $\bar{M}$ and $M^{(l)}$, more precisely, it holds both in $\bar{M}$ and in $M^{(l)}$ only for the elements of M_A.

Next we wish to show that any D_{nk} holds, or does not hold, simultaneously for corresponding elements of $\bar{M}$ and $M^{(l)}$ (when the former are taken as elements of M_H). Suppose first that some relation $D_{nk}(b_1,\dots,b_n)$ holds in $M^{(l)}$. This is equivalent to the existence of a non-vanishing polynomial of degree not exceeding k with coefficients in M_A, $p(x_1,\dots,x_n)$ say, such that $p(b_1,\dots,b_n)=0$ holds in $M^{(l)}$. Let $b_1',\dots,b_n'$ be the corresponding elements in $\bar{M}$, then $p(b_1',\dots,b_n')=0$ holds in M_H (since the correspondence is an isomorphism) and so $D_{nk}(b_1,\dots,b_n)$ holds in M_H.

Conversely, suppose that for some $b_1',\dots,b_n'$ in $\bar{M}$, $n\geqslant 1$, a certain D_{nk} holds in M_H. That is to say, there exists a non-vanishing polynomial of degree not exceeding k with coefficients in M_{HA}, $p(x_1,\dots,x_n)$ say, such that $p(b_1',\dots,b_n')=0$ holds in M_H. In this statement, M_{HA} is the real closed field which consists of the elements of M_H that satisfy $A(x)$. Let $b_1,\dots,b_n$ be the corresponding elements of $M^{(l)}$, then we have to show that there exists a non-vanishing polynomial $q(x_1,\dots,x_n)$ of degree $\leqslant k$ with coefficients in M_A such that $q(b_1,\dots,b_n)=0$ holds in $M^{(l)}$. But this will be the case precisely if $q(b_1',\dots,b_n')=0$ in M_H and so it is sufficient to establish the existence of a non-vanishing polynomial q of degree $\leqslant k$ with coefficients in M_A which satisfies the latter condition. Now the existence of polynomials p and q as mentioned is equivalent to the linear dependence of the product of powers of the b_ν',

$$b_1'^{m_1}b_2'^{m_2}\dots b_n'^{m_n},\qquad m_\nu\geqslant 0,\qquad \sum m_\nu\leqslant k$$

over M_{HA} and over M_A respectively. Accordingly, we only have to show that any set of elements of $\bar{M}$ which is linearly independent over M_A is linearly independent also over M_{HA}. We note that $M_A=\bar{M}\cap M_{HA}$. Thus we have to prove that $\bar{M}$ and M_{HA} are linearly independent over their intersection or, by 2.5, that $\bar{M}$ and M_{HA} are algebraically independent over their intersection, M_A.

Let T be a transcendence base of $M^{(l-1)}$ over M_A; then $T\cup\{t\}$ is a transcendence base of $\bar{M}$ over M_A. Let $\{c_1,\dots,c_\lambda\}$ be a set of elements of M_{HA} which is algebraically dependent over $\bar{M}$. Then $\{c_1,\dots,c_\lambda\}$ is dependent algebraically already over some field $M_A(t_1,\dots,t_s,t)$ where $\{t_1,\dots,t_s\}$ is a finite subset of T. Thus, the $c_1,\dots,c_\lambda$ satisfy a non-vanishing polynomial $p(x_1,\dots,x_\lambda)$ with coefficients in $M_A(t_1,\dots,t_s,t)$ or, more particularly, in $M_A[t_1,\dots,t_s,t]$. In other words, there exists a non-vanishing polynomial $q(y,y_1,\dots,y_s,x_1,\dots,x_\lambda)$ with coefficients in M_A such that

$$q(t,t_1,\dots,t_s,c_1,\dots,c_\lambda)=0\,.$$

Consider now the polynomial $Q(y,y_1,\dots,y_s)=q(y,y_1,\dots,s_s,c_1,\dots,c_\lambda)$ which is a non-vanishing polynomial with coefficients in M_{HA}. Then

$$Q(t,t_1,\dots,t_s)=0\,.$$

But the set $\{t_1,\dots,t_s\}$ is algebraically independent over M_A, and so, by 3.11, $\{t,t_1,\dots,t_s\}$ is algebraically independent over M_{HA}. It follows that the coefficients of $Q(y,y_1,\dots,y_s)$ all vanish, i. e. that the coefficients of $y,y_1,\dots,y_s$ in $q(y,y_1,\dots,y_s,x_1,\dots,x_\lambda)$ are polynomials of $x_1,\dots,x_\lambda$ with coefficients in M_A which vanish for $x_1=c_1,\dots,x_\lambda=c_\lambda$. But these polynomials do not all vanish identically and so the set $\{c_1,\dots,c_\lambda\}$ is algebraically dependent over M_A. Thus, we have shown that $\bar{M}$ and M_{HA} are algebraically independent over their intersection M_A. As explained above, this entails, in view of 2.5, that any D_{nk} holds, or does not hold, simultaneously for corresponding elements of $\bar{M}$ and $M^{(l)}$.

We have therefore shown that $\bar{M}$ regarded as a partial structure of M_H including Q, A, and the D_{nk} is isomorphic to $M^{(l)}$ including Q, A, and the D_{nk}, by an isomorphism under which the elements of $M^{(l-1)}$ correspond to themselves. It follows that X holds in $\bar{M}$ and (being an existential sentence) also in M_H. Since M_H is an arbitrary model of H we conclude that *X is deducible from H*.

More particularly, X is deducible from $K^*\cup N$ together with finite subsets of 3.10 and 3.11. Thus, there exist predicates $Y_1(z)$ and $Y_2(z)$ of the form

3.13. $Y_1(z)=\sim E(a_1,z)\wedge\dots\wedge\sim E(a_k,z)\wedge Q(a_{k+1},z)\wedge\dots\wedge Q(a_k,z)\wedge$
$\wedge Q(z,a_{j+1})\wedge\dots\wedge Q(z,a_m)$

and

3.14. $Y_2(z) = \sim D_{r_1+1,k_1}(z, t_1^1, \dots, t_{r_1}^1) \wedge \dots \wedge \sim D_{r_j+1,k_j}(z, t_1^j, \dots, t_{r_j}^j)$,

— where the a_μ, t_ν^μ belong to $M^{(l-1)}$, and where, for given μ the t_ν^μ are algebraically independent over M_A — such that

$$Y_1(t) \wedge Y_2(t) \supset X$$

is deducible from $K^* \cup N$. But t does not belong to $M^{(l-1)}$ and does not appear in X and so the sentence

$$[(\exists z)[Y_1(z) \wedge Y_2(z)]] \supset X$$

also must be deducible from $K^* \cup N$. K^* and N are satisfied by $M^{(l-1)}$. Accordingly, in order to establish that X holds in $M^{(l-1)}$ we only have to verify that

$$(\exists z)[Y_1(z) \wedge Y_2(z)]$$

also holds in $M^{(l-1)}$, i. e. that $Y_1(a) \wedge Y_2(a)$ holds for some a in $M^{(l-1)}$.

Let $M_0 = M_A(t_1^1, \dots, t_{r_1}^1, \dots, t_1^j, \dots, t_{r_j}^1)$ and let $k_0 = \max(k_1, \dots, k_j)$. Then we may replace $Y_2(z)$ by the stronger condition that z does not satisfy any equation of degree not exceeding k_0 with coefficients in M_0. Similarly (compare the argument in [2], p. 46), we may replace $Y_1(z)$ by the condition that z belongs to the open interval (a', a'') where a' and a'' are specified elements of $M^{(l-1)}$, $a' < a''$. And since M_A is dense in $M^{(l-1)}$, we may specialise this condition by supposing that a' and a'' belong to M_A.

If $M_0 = M_A$, choose an arbitrary element b of $M^{(l-1)}$. If $M_0 \neq M_A$, so that set $\{t_\nu^\mu\}$ is not empty, choose a transcendence base of M_0, S say, from among the elements of $\{t_\nu^\mu\}$. Then M_0 is finite algebraic over $M_A(S)$, of degree h_0 say. Let $s \in S$ and define b in $M^{(l-1)}$ as the positive root od the polynomial $z^k - |s|$ where $k = k_0 h_0 + 1$. This polynomial is irreducible in $M_A(s)$ and hence in $M_A(S)$, and so b is of degree k over $M_A(S)$, and of degree $> k_0$ over M_0. Thus, for both $M_0 = M_A$ and $M_0 \neq M_A$ our choice of b ensures that it does not satisfy any polynomial of degree $\leqslant k_0$ with coefficients in M_0. The same applies to any element

$$a = \frac{c_1 + c_2 b}{c_3 + c_4 b}, \qquad c_1 c_4 - c_2 c_3 \neq 0$$

where c_1, c_2, c_3, c_4 belong to M_A, and so it only remains to choose c_1, c_2, c_3, c_4 in such a way that $z = a$ belongs to the open interval (a', a''). A suitable choice is

$$a = \frac{a' + a'' b}{1 + b}$$

which is a weighted mean of a' and a''.

We have now shown that X holds in $M^{(l-1)}$ contrary to our original assumption. This leaves us with the case $M_{AA} \neq M_A$ where we recall that M_{AA} and M_A are the fields determined by $A(x)$ in M^{**} and M^* respectively. By assumption, X holds in M^{**} but not in M^*.

Let M^{***} be the algebraic closure of the compositum of M^* and M_{AA} in M^{**}. M^{***} is real closed and is the intersection of all real-closed subfields of M^{**} which include both M^* and M_{AA}. Defining the relations A and D_{nk} for elements of M^{***} as in M^{**} we see without difficulty that K, K_A, and $\{X_1, X_2, X_3\}$ are satisfied. Also, since M_{AA} is dense in M^{**} it is certainly dense in M^{***}, showing that X_4 is satisfied as well. Moreover, for any given $n, k = 1, 2, \dots$, the equivalence between Q_{nk} and D_{nk} holds in M^{***} since the set of elements which satisfies $A(x)$ is the same in M^{***} as in M^{**} (i. e. M_{AA}). Thus, the sentences X_{nk} all hold in M^{***}, M^{***} is a model of K^*.

Now suppose that X does not hold in M^{***} although it clearly is defined in that structure. Replacing the M^* of the general problem by M^{***} we then have the case in which $A(x)$ determines the same field, M_{AA}, in the two given models of K^*. But we have just shown that in that case it is impossible that X holds in one model but not in the other. We conclude that X holds in M^{***}. But M^{***} is an extension of M^* including Q, A, and the D_{nk}. Replacing M^{**} by M^{***} (if necessary) we may therefore suppose from the outset that M^{**} is the algebraic closure of the compositum of M^* and M_{AA}.

Let T be a transcendence base of M_{AA} over M_A. Since the elements of M^{**} are algebraically dependent on $M_{AA} \cup M^*$, T is a transcendence base of M^{**} over M^* as well. Suppose that X is satisfied by $y_1 = a_1, \dots, y_i = a_i$ in M^{**} (see 3.7). Then there exists a finite subset of T, $S = \{t_1, \dots, t_m\}$ say, such that the elements $a_1, \dots, a_i$ are algebraically dependent on S over M^*. Thus, if $\bar{M}$ is the algebraic closure of the ordered field $M^*(S)$ in M^{**}, then $\bar{M}$ includes $a_1, \dots, a_i$. $\bar{M}$ is real-closed and by defining A and D_{nk} in $\bar{M}$ as in M^{**} we turn $\bar{M}$ into an extension of M^* including Q, A, and the D_{nk}. The elements of $\bar{M}$ which satisfy $A(x)$ constitute a subfield $\bar{M}_A$ of M. $\bar{M}_A$ includes M_A and S, and hence also $M_A(S)$. Moreover, $\bar{M}_A$ is the algebraic closure of $M_A(S)$ within M^{**}, and hence within $\bar{M}$. To see this, we prove first that the fields M_{AA} and M^* are algebraically independent over M_A. Indeed, suppose that a set of elements of M^*, $b_1, \dots, b_n$ is algebraically independent over M_A. Then $\sim D_{nk}(b_1, \dots, b_n)$ holds in M^* for $k = 1, 2, 3, \dots$ But if so then all these sentences hold also in M^{**} where the relations D_{nk} refer to polynomials with coefficients in M_{AA}. Thus, the elements $b_1, \dots, b_n$ are algebraically independent also over M_{AA}, M_{AA} and M^* are algebraically independent over M_A.

Now suppose that $\bar{M}_A$ contains an element t which is not algebraic over $M_A(S)$ although, by construction, it is algebraic over $M^*(S)$. Then the set $\{t\}\cup S$ is algebraically dependent over M^*. It follows, by what has just been proved concerning the algebraic independence of M_{AA} and M^* over M_A, that $\{t\}\cup S$ is algebraically dependent over M_A. But S is algebraically independent over M_A, and so t depends on $M_A(S)$ algebraically, $\bar{M}_A$ is the algebraic closure of $M_A(S)$ within M^{**} and within $\bar{M}$. Moreover

$$\bar{M}_A = M_{AA} \cap \bar{M}\,, \qquad M_A = M_{AA} \cap M^* = \bar{M}_A \cap M^*\,,$$

and $\bar{M}$ is the algebraic closure of the compositum of $\bar{M}_A$ and M^* in M^{**}. Now $\bar{M}_A$ is a subfield of M_{AA} and so $\bar{M}_A$ and M^* also are algebraically independent over M_A. Hence, by 2.5, $\bar{M}_A$ and M^* are linearly independent over M_A, and this in turn implies that if the elements $b_1, \ldots, b_n$ of M^* do not satisfy any non-vanishing polynomial of degree $\leqslant k$ with coefficients in M_A, for *specified* k then they do not satisfy any such polynomial with coefficients in $\bar{M}_A$ either. In other words, Q_{nk} holds in $\bar{M}$ if and only if it holds in M^*. And since $D_{nk}(b_1, \ldots, b_n)$ holds in $\bar{M}$ also if and only if it holds in M^*, it follows that X_{nk} holds in $\bar{M}$. K and K_A evidently hold in $\bar{M}$ and so do X_1, X_2, X_3. However, we do not claim that X_4 holds in $\bar{M}$, i. e. that $\bar{M}_A$ is dense in $\bar{M}$.

Let $p = p(x_1, \ldots, x_m)$ be a polynomial with coefficients in M^*. Then it is not difficult to formulate, in the lower predicate calculus, and by means of the relations E, S, P, Q, and of some of the individual constants of M^*, a predicate $Q_p(x_1, \ldots, x_m)$ which states that $p(x_1, \ldots, x_m) > 0$.

Disregarding an earlier notation, we now denote by H the union of the following sets, where we recall that $S = \{t_1, \ldots, t_m\}$:

3.15. The set K^*;

3.16. the diagram N of M^*;

3.17. the set of all sentences $Q_p(t_1, \ldots, t_m)$ for which $p(t_1, \ldots, t_m) > 0$ holds in $\bar{M}$;

3.18. the set $\{A(t_1), A(t_2), \ldots, A(t_m)\}$.

H is consistent, for M^{**} is a model of H. Let M_H be any other model of H. We propose to show that M_H contains a partial model which is isomorphic to $\bar{M}$ including Q, A, D_{nk}, under an isomorphism under which the elements of M^* correspond to themselves.

M_H is an extension of M^*, by 3.16, and contains $S = \{t_1, \ldots, t_m\}$, by 3.18. Moreover, S is algebraically independent over M^* in H for by 3.17 we have, for any non-vanishing polynomial p with coefficients in M^* that either $Q_p(t_1, \ldots, t_m)$ or $Q_{-p}(t_1, \ldots, t_m)$ holds in M_H, and hence that $p(t_1, \ldots, t_m) \neq 0$ in M_H.

Let $M_1 = M^*[t_1, \ldots, t_m]$ in M_H; let $M_2 = M^*(t_1, \ldots, t_m)$ in M_H so that M_2 is the field of quotients of M_1; and let $\bar{M}_H$ be the algebraic closure of M_2 within M_H. Then M_1 is isomorphic to the ring $M^*[t_1, \ldots, t_m]$ in M^{**} by the correspondence by which the elements of M^* as well as $t_1, \ldots, t_m$ correspond to themselves, since both rings are purely transcendental extensions of M^* by the algebraically independent elements $t_1, \ldots, t_m$. Moreover, by 3.17, the order defined in both rings is the same, and so the isomorphism includes Q. Passing to the fields of quotients, we find that M_2 is, similarly, isomorphic to the field $M^*(t_1, \ldots, t_m)$ in M^{**}. Also, M_H is a real-closed field, and so the algebraic closure of M_2 within M_H, $\bar{M}_H$, also is real closed. But any ordered field determines uniquely (including the ordering) its real closed algebraic extension. It follows that $\bar{M}_H$ is isomorphic, including Q, to $\bar{M}$, by an extension of the isomorphism just considered. We propose to show that if in $\bar{M}_H$ we define A and the D_{nk} as in M_H, then the isomorphism includes these relations as well.

Let $\bar{M}_{HA}$ be the set of elements of $\bar{M}_H$ which satisfy $A(x)$ under this definition. $\bar{M}_{HA}$ is a real closed field which includes M_A, by 3.16, and the elements $t_1, \ldots, t_m$, by 3.18. Hence $\bar{M}_{HA}$ includes the algebraic closure M_3 of $M_A(t_1, \ldots, t_m)$ within $\bar{M}_H$, and $\bar{M}_H$ itself is the algebraic closure in M_H of the compositum of $M_A(t_1, \ldots, t_m)$ and of M^*. But M_3 is the set of elements of $\bar{M}_H$ that correspond to elements of $\bar{M}_A$, and so we only have to show that $\bar{M}_{HA} = M_3$ in order to establish that the isomorphism includes A.

Suppose on the contrary that $A(x)$ is satisfied in $\bar{M}_H$ by an element t which does not belong to M_3. Let R be a transcendence base of M^* over M_A, then all elements of $\bar{M}_H$ and in particular t depend algebraically on $S\cup R$ over M_A. It follows that t depends on some finite subset R' of R over $M_A(t_1, \ldots, t_m)$, $R = \{r_1, \ldots, r_j\}$ say. Thus, there exists a non-vanishing polynomial $p(x)$ with coefficients in $M_A\{t_1, \ldots, t_m, r_1, \ldots, r_j\}$ such that $p(t) = 0$. In other words, there exists a non-vanishing polynomial $q(y_1, \ldots, y_m, z_1, \ldots, z_j, x)$ with coefficients in M_A such that

3.19. $q(t_1, \ldots, t_m, r_1, \ldots, r_j, t) = 0\,.$

Let M_{HA} be the set of elements of M_H which satisfy $A(x)$, so that $\bar{M}_{HA} = M_{HA} \cap \bar{M}_H$ and $M_A = M_{HA} \cap M^*$. As before, M^* and M_{HA} are algebraically independent over M_A. The same applies to M^* and $\bar{M}_{HA}$ since the latter is a subfield of M_{HA}. But $\{r_1, \ldots, r_j\}$ is algebraically independent over M_A and so it must be algebraically independent over $\bar{M}_{HA}$ also. Now the elements $t_1, \ldots, t_m, t$ all belong to $\bar{M}_{HA}$. It follows that if we denote the coefficients of the products of powers of $z_1, \ldots, z_j$ in $q(y_1, \ldots, y_m, z_1, \ldots, z_j, x)$ by $q_\lambda(y_1, \ldots, y_m, x)$, then in view of 3.19,

3.20. $q_\lambda(t_1, \ldots, t_m, t) = 0$ for all λ.

If we can show that the coefficients of at least one $q_\lambda(t_1, \dots, t_m, x)$ do not all vanish, then we have finished for in that case 3.20 shows that t belongs to M_3 after all. Now $\{t_1, \dots, t_m\}$ is algebraically independent over M_A, and so if the coefficients of all $q_\lambda(t_1, \dots, t_m, x)$ vanish, then so do the coefficients of all $q_\lambda(y_1, \dots, y_m, x)$ and hence of $q(y_1, \dots, y_m, z_1, \dots, z_j, x)$. This is contrary to assumption. We conclude that the isomorphism between $\bar{M}$ and $\bar{M}_H$ includes $A(x)$. Moreover, the axioms X_{nk} hold both in M^{**} and in M_H. Accordingly, in order to prove for any given D_{nk} that it holds for corresponding elements in $\bar{M}$ and in $\bar{M}_H$, we only have to show that Q_{nk} holds for corresponding elements. But if a set of elements $b_1, \dots, b_n$ in $\bar{M}$ satisfies a polynomial $p(x_1, \dots, x_n)$ with coefficients in $\bar{M}_A$, then the corresponding elements in $\bar{M}_H$ satisfy the corresponding polynomial in $\bar{M}_H$, and the coefficients of that polynomial belong to $\bar{M}_{HA}$. Thus Q_{nk} holds for corresponding elements, D_{nk} holds for corresponding elements, the isomorphism includes D_{nk}.

Since X holds in $\bar{M}$, the existence of the isomorphism in question shows that X holds also in $\bar{M}_H$. But X is an existential statement, and so X holds also in M_H, which is an extension of $\bar{M}_H$ including Q, A, D_{nk}. It follows that X is deducible from H. More particularly, there exist polynomials $p_1(x_1, \dots, x_n), \dots, p_j(x_1, \dots, x_n)$ with coefficients in M^*, $j \geqslant 0$, such that the sentence

$$A(t_1) \wedge A(t_2) \wedge \dots \wedge A(t_m) \wedge Q_{p_1}(t_1, \dots, t_m) \wedge \dots \wedge Q_{p_j}(t_1, \dots, t_m) \supset X$$

is deducible from $K^* \cup N$. But $t_1, \dots, t_m$ are included neither in X nor in $K^* \cup N$ and so the sentence

3.21. $$[(\exists y_1) \dots (\exists y_m)[A(y_1) \wedge \dots \wedge A(y_m) \wedge Q_{p_j}(y_1, \dots, y_m) \wedge \dots \wedge Q_{p_j}(y_1, \dots, y_m)]] \supset X$$

also is deducible from $K^* \cup N$. Now K^* and N hold in M^*. Hence, in order to prove that X holds in M^* we only have to show that M^* satisfies the implicans of 3.21. In other words we have to show that the system of inequalities

3.22. $$p_1(y_1, \dots, y_m) > 0, \quad \dots \dots \quad p_j(y_1, \dots, y_m) > 0$$

has a solution in M^* with the auxiliary condition that the elements of the solution all belong to M_A.

3.22 has a solution in $\bar{M}$, i. e.

3.23. $$Q_{p_1}(y_1, \dots, y_m) \wedge \dots \wedge Q_{p_j}(y_1, \dots, y_n)$$

is satisfied in $\bar{M}$, by $y_1 = t_1, \dots, y_m = t_m$. $\bar{M}$ is a real-closed field which is an extension of M^* and the elementary theory of real closed fields is model-complete ([2], p. 44). It follows that 3.23 possesses a solution already in M^*, e. g. $y_1 = c_1, \dots, y_m = c_m$. Now it is not difficult to show that if 3.22 is satisfied by $c_1, \dots, c_m$ then we can find in M^* a positive quantity ε such that 3.22 is satisfied also by any $d_1, \dots, d_m$ such that $c_i \leqslant d_i \leqslant c_i + \varepsilon$. But M_A is dense in M^* and so we may choose the d_i as elements of M_A. Thus the implicans of 3.21 holds in M^* and the same therefore applies to the implicate, X. This is contrary to our original assumption, and completes the proof of 3.6.

4. Completeness and decidability.

4.1. Theorem. *The set K^** (see 3.5) *is complete.*

Proof. Let M_A be the ordered field of real algebraic numbers and let t be an arbitrary but fixed transcendental number. Let M^* be the algebraic closure of $M_A(t)$ within the field of real numbers. Then M^* is real-closed. Define that $A(x)$ holds in M^* precisely for the elements of M_A. Define that $D_{nk}(x_1, \dots, x_n)$ holds for elements $a_1, \dots, a_n$ of M^* if $Q_{nk}(a_1, \dots, a_n)$ holds in M^*. With these definitions, M^* becomes a model of K^*.

Now let X be a sentence which is defined in K^*. Thus X may contain the relations E, S, P, Q, A, and D_{nk}. X does not contain any individual constants. (However, all real algebraic numbers can be characterised without the use of individual constants.) We have to show that either X or $\sim X$ is deducible from K.

Let N be the diagram of M^*. Then $K^* \cup N$ is complete, by 3.6. It follows that either X or $\sim X$ is deducible from $K^* \cup N$. We may suppose that the former is the case. If so, we shall show that X is already deducible from K^* alone. Clearly this is sufficient to prove 4.1.

We define the set of sentences H as the union of the following sets:

4.2. The set K^*;

4.3. the diagram N_A of M_A, with respect to the relations E, S, P, Q only;

4.4. the set containing the single sentence $\sim A(t)$;

4.5. the set of sentences $Q(t, c)$ or $Q(c, t)$ according as $Q(t, c)$ or $Q(c, t)$ holds in M^*, where c varies over the elements of M_A.

H is consistent for it is satisfied by M^*. Let M_H be any model of H. By 4.3, M_H is an extension of M_A; it contains the element t which is different from (not equal to) all the elements of M_A since the latter satisfy $A(x)$ by 4.3, while t satisfies $\sim A(x)$ by 4.4. Let $\bar{M}$ be the algebraic closure of $M_A(t)$ within M_H, and define the relations A and D_{nk} in $\bar{M}$ as in M_H. We propose to show that by this definition $\bar{M}$ is isomorphic to M^* including Q, A, D_{nk}, such that the elements of M_A correspond to themselves under the isomorphism.

$M_A(t)$ is a simple transcendental extension of M_A in M_H as it is in M^*, and its order is the same in both cases, by 4.5. Since the real-closed algebraic extension of an ordered field is uniquely determined also as to order, it follows that $\bar{M}$ and M are isomorphic, including Q, by an extension of the isomorphism between $M_A(t)$ in M_A and $M_A(t)$ in M_H (under which all elements correspond to themselves). The elements of M_A satisfy $A(x)$ both in $\bar{M}$ and in M^*. Since no other elements of M^* satisfy $A(x)$, we have to show that no other element of $\bar{M}$ satisfies $A(x)$ either.

Suppose that $A(s)$ holds in M_H, and hence in $\bar{M}$ for an element s of $\bar{M}$ which does not belong to M_A. Let M' be the algebraic closure of $M_A(s)$ in $\bar{M}$. Since M_H satisfies K_A, it follows that all elements of M' satisfy $A(x)$. But $\bar{M}$ is of degree of transcendence 1 over M_A and so M' which is a subfield of $\bar{M}$ must actually coincide with it. It follows in particular that $A(t)$ holds in $\bar{M}$ which contradicts 4.4. Thus, the only elements of $\bar{M}$ which satisfy $A(x)$ are the elements of M_A, the isomorphism between M^* and $\bar{M}$ includes $A(x)$. Moreover, the axioms X_{nk} hold both in M^* and in $\bar{M}$. Thus, we again have to show only that Q_{nk} holds for corresponding elements, and this follows immediately from the fact that the isomorphism includes $A(x)$. Accordingly, the isomorphism includes $D_{nk}(x_1, \dots, x_k)$ as well, $k = 1, 2, \dots$ It follows that X holds also in $\bar{M}$. But K^* is model-complete, and M_H is an extension of $\bar{M}$ including Q, A, D_{nk}. Accordingly, X holds also in M_H, implying that X is deducible from H. Thus, there exist elements $a_1, \dots, a_k$ in M_A such that the sentences

4.6. $[\sim A(t) \wedge Q(a_1, t) \wedge \dots \wedge Q(a_l, t) \wedge Q(t, a_{l+1}) \wedge \dots \wedge Q(t, a_k)] \supset X$

and hence

4.7. $$[(\exists x)[\sim A(x) \wedge Q(a_1, x) \wedge \dots \wedge Q(a_l, x) \wedge Q(x, a_{l+1}) \wedge \dots \wedge Q(x, a_k)]] \supset X$$

are deducible from $K^* \cup N_A$. By an argument used previously (see the sequel to 3.13 and 3.14 above) and explained in detail elsewhere ([2], p. 46) we may replace 4.7 by the condition

4.8. $[(\exists x)[\sim A(x) \wedge Q(a', x) \wedge Q(x, a'')]] \supset X$

where a' and a'' are two particular elements of M_A, $a' < a''$.

Now let M be an arbitrary model of K^*. We have to show that X holds in M.

M contains the real algebraic numbers, and so is a model of $K^* \cup N_A$. Hence in order to show that X holds in M, we only have to verify that the implicans of 4.8 holds in that structure. In other words, we have to show that to any pair of real algebraic numbers, a', a'' such that $a' < a''$ we can find an element a which belongs to the open interval (a', a'') and which does not satisfy $A(x)$. (Briefly, the elements of M which do not satisfy $A(x)$ are dense in the set of real algebraic numbers.) Now, by X_2, M contains at least one element which does not satisfy $A(x)$. Thus there exist a positive element of this kind, b say. It follows that the element $a = (a' + ba'')/(1+b)$ does not satisfy $A(x)$ either. But a is a weighted mean of a' and a'' and belongs to the open interval (a', a''), as required. Accordingly X is deducible from K^*. Similarly, if $\sim X$ holds in M^* then $\sim X$ is deducible from K^*. This proves 4.1.

Let $K^{**} = K \cup K_A \cup \{X_1, X_2, X_3, X_4\}$, so that K^{**} is obtained from K^* by the exclusion of the sentences of K_D, X_{nk}, $n, k = 1, 2, \dots$ Let X be any sentence which is defined in K^{**}, i. e. which is formulated in terms of the relations E, S, P, Q, A and without individual constants. Suppose that X holds in a model M_1 of K^{**} while $\sim X$ holds in a model M_2 of K^{**}. Now both in M_1 and in M_2 we may introduce the D_{nk} in such a way that the sentences X_{nk} are satisfied by simply defining that $D_{nk}(x_1, \dots, x_n)$ holds for elements $a_1, \dots, a_n$ of M_1, or of M_2, if $Q_{nk}(x_1, \dots, x_n)$ holds for these elements. In this way we turn M_1 and M_2 into models of K^*. But K^* is complete and so it is impossible that X holds in one model of K^* and $\sim X$ in another. Thus either X holds in all models of K^{**} or $\sim X$ holds in all models of K^{**}. We have proved

4.9. Theorem. *The set K^{**} is complete.*

Now both K^* and K^{**} may be supposed to be recursively enumerable (and even recursive, e. g. if based on [2]). Hence ([7], p. 14, and [1])

4.10. Theorem. *The theories of K^* and K^{**} are decidable.*

A particular model of K^{**} is the ordered field of real numbers within which the real algebraic numbers constitute the set which satisfies $A(x)$. If we supplement this by a suitable definition of the relations D_{nk}, as above, we obtain a model of K^*. Using either K^* or K^{**} we have therefore solved Tarski's problem.

In [6], p. 45 and 57, Tarski also raises the question of finding a decision procedure for the theory of (i. e. the set of all sentences which hold for the) real numbers, in which the relations E, S, P, Q have been supplemented by a relation for the exponential function to the base 2 (e. g. $F(x, y)$, to denote the relation $2^x = y$). This problem is still unsolved. Tarski points out that the corresponding problem for the complex numbers possesses a negative answer since the existence of a decision procedure for that case would imply the existence of a decision procedure for the elementary theory of positive integers and that theory is known to be absolutely undecidable. It is of interest to note that if to E, S, P, Q and F we add the relation $A(x)$ for the real algebraic numbers, then the resulting theory also is undecidable. Indeed, in that case we can define the rational numbers x by the condition

$$A(x) \wedge (y)[F(x, y) \supset A(y)]$$

by virtue of the theorem of Gelfond and Schneider (compare [5], p. 75) and the theory of rational numbers is absolutely undecidable by a result of J. Robinson [4]. A familiar argument [7] now implies that the theory under consideration also is absolutely undecidable.

Our decision procedure for the relations E, S, P, Q, A (and D_{nk}) is not based on an elimination method such as was provided by Tarski for the relations E, S, P, Q. However, it has been shown [3] that in certain circumstances model-completeness ensures the existence of an elimination method. We are now going to discuss this point informally. Let $Q(x_1, \ldots, x_n)$ be a predicate which is formulated in terms of the relations of K^* and without individual constants. Then it is known ([2], p. 21) that there exists an existential predicate,

$$Q'(x_1, \ldots, x_n) = (\exists y_1) \ldots (\exists y_m) Z(x_1, \ldots, x_n, y_1, \ldots, y_m)$$

where Z is free of quantifiers such that Q' is equivalent to Q with respect to K^* i. e. such that

4.11. $(x_1) \ldots (x_n)[Q(x_1, \ldots, x_n) \equiv Q'(x_1, \ldots, x_n)]$

is deducible from K^*.

We now pass from K to a set $\bar{K}$ which contains sentences with universal quantifiers only, the existential quantifiers having been replaced by "Skolem-Herbrand functors". This can be done in a mechanical fashion, by replacing, for example, a sentence of the form $(\exists y)(z)(\exists w)Y(y, z, w)$, Y free of quantifiers, by $(z)\, Y\big(\varphi, z, \psi(z)\big)$. However, we have some freedom in the choice of the sentences of $\bar{K}$ and we may try to specify them in such a way that the functors introduced are as simple as possible. For the axioms of an ordered field, we require only the individual constants (functors of order 0) 0, 1 as well as functors for the sum, the product, the inverse with respect to addition, and the inverse with respect to multiplication. For the latter, we may define $\varphi(0) = 0^{-1} = 0$, with the axiom

$$(x)\big[E(x, 0) \vee P\big(x, \varphi(x), 1\big)\big]$$

The axioms of order do not require the introduction of additional functors, but in order to state that the field is real-closed we require a functor for the square root — where we may take the positive square root for positive numbers and 0 for all non-positive numbers — and functors for the roots of equations of odd degree, $n = 3, 5, \ldots$, regarded as functions of the coefficients. These can be made one-valued by choosing always the smallest root of the equation in question, but this point is inessential for the sequel.

Next, in place of K_A, we introduce sentences which state that the application of the functors defined so far to arguments which satisfy $A(x)$ yields values which satisfy $A(x)$. In particular we have $A(0)$, $A(1)$. Let the resulting set be called $\bar{K}_A$.

In the group X_1—X_4, X_1 is now redundant and X_3 is already in a suitable form, with universal quantifiers only. In place of X_2, we introduce the sentence $\sim A(t)$ which includes the new constant t. In order to replace X_4 by a sentence without existential quantifiers we require a functor $\psi(x, y)$ which yields an element of M_A for $x < y$ and which may be taken as 0 for $x \geqslant y$. If x, y are real numbers whose decimal expansion is known, and M_A is the field of rational numbers, then it is not difficult to make a suitable choice for $\psi(x, y)$. Call the resulting sentence $\bar{X}_4$.

Finally consider the sentences X_{nk}. Each of these may be replaced by a pair of sentences,

4.12. $(x_1) \ldots (x_n)[D_{nk}(x_1, \ldots, x_n) \supset Q_{nk}(x_1, \ldots, x_n)]$

and

4.13. $(x_1) \ldots (x_n)[Q_{nk}(x_1, \ldots, x_n) \supset D_{nk}(x_1, \ldots, x_n)]$.

Now, as stated, the predicates Q_{nk} may be written in prenex normal form with existential quantifiers only. It follows that if in 4.13 we replace $\supset$ by $\sim \ldots \vee$ in the familiar way, and transform to prenex normal form, we obtain a sentence with universal quantifiers only, so that no additional functors are required. On the other hand, in order to replace 4.12 by sentences in prenex normal form with universal quantifiers only, we require functors $\varphi_i^{(k)}(x_1, \ldots, x_n)$ which constitute the coefficients of a non-vanishing polynomial $p(x_1, \ldots, x_n)$ of degree $\leqslant k$ which is satisfied by $x_1, \ldots, x_n$ if D_{nk} holds, and which may be chosen arbitrarily if D_{nk} does not hold for the arguments in question. Denote the set of sentences obtained in this way from 4.12 and 4.13 by $\bar{K}_D$ and define

$$\bar{K}^* = \bar{K} \cup \bar{K}_A \cup \{\sim A(t), X_3, \bar{X}_4\} \cup \bar{K}_D .$$

Then K^* is deducible from $\bar{K}^*$ and so 4.11 is deducible from $\bar{K}^*$ as well. Now the sentences of $\bar{K}^*$ are in prenex normal form with universal quantifiers only. We may therefore conclude by means of the extended first ε-theorem (compare [3]) that there exists a predicate $Q''(x_1, \ldots, x_n)$ formulated in terms of relations and functors of $\bar{K}^*$ and free of quantifiers such that

4.14. $(x_1) \ldots (x_n)[Q'(x_1, \ldots, x_n) \equiv Q''(x_1, \ldots, x_n)]$.

Combining 4.11 with 4.14 we obtain that

4.15. $(x_1)\dots(x_n)[Q(x_1,\dots,x_n)\equiv Q''(x_1,\dots,x_n)]$

is deducible from $\bar{K}^*$. Thus, $Q''(x_1,\dots,x_n)$ represents the result of "eliminating the quantifiers" from $Q(x_1,\dots,x_n)$. This has to be achieved at the cost of introducing in addition to some "natural" functors like the sum and product also some very unnatural ones like $\psi(x,y)$ and the $\varphi_i^{(k)}(x_1,\dots,x_n)$. The form of $Q''(x_1,\dots,x_n)$ is independent of the particular choice of the functors (e. g. whether we choose the largest or the smallest root of an equation of odd degree). But the use of Q'' to decide whether or not Q holds for a given set of arguments presupposes that we are actually able to compute these functors, or rather to decide whether their iterates satisfy the atomic relations contained in Q''. Moreover, not all the steps which are involved in the derivation of Q'' from Q are constructive and so our arguments do not yield an effective procedure of elimination in their present form.

Finally we wish to give an example which shows that the introduction of the relations $D_{nk}(x_1,\dots,x_n)$ was essential in order to establish model-completeness. More precisely, we shall establish that the set of axioms K^{**}, though complete by 4.9, is not model-complete. For this purpose we refer to the fields M_0, M_1, M_2, M described at the end of section 2 above. If in M_1, regarded as an ordered field, we define that $A(x)$ holds precisely for the elements of M_0, then the resulting structure is a model of K, K_A, and X_1, X_2, X_3 (see 3.1, 3.2). It also satisfies 3.3 since M_0, which is the field of real algebraic numbers is dense in any other Archimedean field. Thus M_1, with the specified definition of $A(x)$, constitutes a model of K^{**}. Similarly, if in M we define that $A(x)$ holds precisely for the elements of M_2, then we thereby turn M into a model of K^{**}. Moreover $M_1\cap M_2=M_0$ and so M is an extension of M_2 including $A(x)$. Now M_1 satisfies the following sentence which can obviously be formulated in the first order predicate calculus — "For all x_1 and x_2, $s_1x_1+s_2+x_2\neq 0$". But this sentence does not hold in M since $s_1t_1+s_2+t_2=0$. Thus K^{**} is not model-complete.

5. Distinction of the algebraic numbers in the field of complex numbers. We now come to the corresponding problem in the theory of algebraically closed fields. Disregarding some of our earlier notation, let K be a set of axioms for the concept of an algebraically closed field, formulated in terms of E, S, and D, and let K_A be obtained by the relativisation of the sentences of K with respect to $A(x)$. Let the sentences X_1, X_2, X_3, and X_{nk}, $n, k=1,2,3,\dots$, be defined by 3.1, 3.2, and 3.4, as previously, and put $K^*=K\cup K_A\cup\{X_1,X_2,X_3\}\cup K_D$ where K_D is the set of all X_{nk}. Then

5.1. Theorem. *The set K^* is model-complete.*

This theorem can be proved by a method similar to that used in the proof of 3.6. Although the details are somewhat less complicated, they still require considerable space. For that reason we shall omit this proof here and shall instead give an independent proof of one of the corresponding theorems for ordinary completeness. However, in the first instance we shall accept 5.1 and shall derive theorems for ordinary completeness on that basis.

K^* is not complete since it does not determine the characteristic of its models. We define extensions K_p^* of K^*, $p=0$ or p positive and prime, as follows.

For positive p, we add to K^* a sentence X_p which states that the repeated addition of any element to itself, p times, yields zero, while for $p=0$ we obtain K_p^* by adding to K^* the sequence of sentences $\sim X_2, \sim X_3, \sim X_5, \dots$ Then the models of K_p^*, $p\geqslant 0$, are characterised by the property that they are fields of characteristic p which are models of K^*. Any K_p^* possesses a prime model in the sense of [2], p. 72, which is obtained as follows. Let M_p^* be an algebraically closed field of transcendence degree 1 over the field M_p of absolutely algebraic numbers of the characteristic in question. Within M_p^* we ascribe $A(x)$ precisely to the elements of M_p, and we define that $D_{nk}(x_1,\dots,x_n)$ holds precisely when $Q_{nk}(x_1,\dots,x_n)$ holds, for any set of n elements of M_p^*. By this definition, M_p^* is a model of K_p^*. Any other model M of K_p^* contains a partial structure M' which is isomorphic to M_p^* including A and D_{nk}. Such a structure is obtained by choosing an element t in M which does not satisfy $A(x)$. Such a t exists, by X_2, and is by necessity transcendental. We define M' as the algebraic closure of $M_p(t)$ in M, and we maintain that M' is isomorphic to M_p^* including $A(x)$ and the $D_{nk}(x_1,\dots,x_n)$. Indeed, let s be any element of M_p^* which does not satisfy $A(x)$. Then the natural isomorphism between $M_p(t)$ in M' and $M_p(s)$ in M_p^* can be extended to an isomorphism between M' and M_p^* which satisfies the required conditions. Hence, in view of the prime model test of [2], p. 74,

5.2. Theorem. *The sets K_p^* are complete, $p=0,2,3,5,\dots$*

Now let $K_p^{**}=K\cup K_A\cup\{X_1,X_2,X_3\}$ so that K_p is obtained from K_p^* by removing the sentences of K_D. Using the method by which we passed from Theorem 4.1 to Theorem 4.9 above, we arrive at

5.3. Theorem. *The sets K_p^{**} are complete.*

An independent proof of 5.3 will now be given.

Let $S=\{s_1,s_2,s_3,\dots\}$ and $T=\{t_1,t_2,t_3,\dots\}$ be two infinite sequences of individual constants such that S and T have no element in common. We define a set of sentences H as the union of the following sets:

5.4. The set K_p^{**}.

5.5. The set of sentences $\{A(s_1), A(s_2), A(s_3), \ldots\}$.

5.6. The set of sentences $\{Y_q\}$ where $q = q(x_1, \ldots, x_n)$ varies over all non-vanishing polynomials with integer coefficients (all elements of the prime field for finite characteristic), and where Y_q states that $q(s_1, \ldots, s_n) \neq 0$. Thus, $\{Y_q\}$ ensures that the set S is algebraically independent over the prime field (i. e. absolutely).

5.7. The set of sentences $\{Z_{nk}\}$, $n, k = 1, 2, \ldots$, which states that there is no non-vanishing polynomial $r(x_1, \ldots, x_n)$ of degree $\leqslant k$ with coefficients all satisfying $A(x)$ such that $r(t_1, \ldots, t_n) = 0$.

It is easy to show by means of a suitable example that H is consistent. Suppose that the sentence X is defined in K_p^{**} (i. e. formulated in terms of E, S, P and without individual constants), and that both X and $\sim X$ are consistent with H. Let M_1 be a model of $H \cup \{X\}$ and M_2 a model of $H \cup \{\sim X\}$. By the theorem of Löwenheim-Skolem we may suppose that both M_1 and M_2 are countable. Let M_{1A} and M_{2A} be the subfields of elements satisfying $A(x)$ in M_1 and M_2 respectively. Then both M_{1A} and M_{2A} are of absolute degree of transcendence $\aleph_0$, by 5.5 and 5.6. Accordingly M_{1A} is isomorphic to M_{2A}. Also, by 5.7, the set T is algebraically independent over M_{1A} and M_{2A}, in M_1 and M_2 respectively, and accordingly, M_1 is of degree of transcendence $\aleph_0$ over M_{1A} and M_2 is of degree of transcendence $\aleph_0$ over M_{2A}. It follows that any given isomorphism between M_{1A} and M_{2A} can be extended to an isomorphism between M_1 and M_2. Thus, there exists an isomorphism between M_1 and M_2 which includes $A(x)$. But in this case it is impossible that X holds in M_1 and $\sim X$ in M_2. We conclude that either X is deducible from H or $\sim X$ is deducible from H. It will be sufficient to consider only the first of these two possibilites further.

Since X is deducible from H, there exist finite subsets of 5.5-5.7 from which, together with K_p^{**}, we can deduce X. And a little reflection shows that, in consequence, there exists a sentence of the form

5.8. $A(s_1) \wedge \ldots \wedge A(s_m) \wedge Y_{q_1} \wedge \ldots \wedge Y_{q_j} \wedge Z_{nk} \supset X$

which is deducible from K_p^{**}. In this sentence, $q_1, \ldots, q_j$ are non-vanishing polynomials with integer coefficients, $q_1 = q_1(x_1, \ldots, x_{m_1}), \ldots, q_j = q_j(x_1, \ldots, x_{m_j})$ (see 5.6 above), and we may simplify 5.8 by replacing $Y_{q_1} \wedge \ldots \wedge Y_{q_j}$ by Y_q where $q = q_1 \ldots q_j$. Also, by trivial modifications we may achieve that the number of variables in q is the same as the number of terms $A(s_i)$ which appear in the implicans. Next, we observe that Y_q is of the form $Q(s_1, \ldots, s_m)$ where $Q(x_1, \ldots, x_m)$ is a predicate which signifies that $x_1, \ldots, x_m$ do not satisfy the *particular* polynomial $q(x_1, \ldots, x_m)$, while Z_{nk} is of the form $R(t_1, \ldots, t_n)$ where $R(x_1, \ldots, x_n)$ is a predicate which signifies that $x_1, \ldots, x_n$ do not satisfy *any* non-vanishing polynomial of degree $\leqslant k$ coefficients satisfying $A(x)$. Then 5.8 becomes

5.9. $A(s_1) \wedge \ldots \wedge A(s_m) \wedge Q(s_1, \ldots, s_m) \wedge R(t_1, \ldots, t_n) \supset X$.

Since 5.9. is deducible from K_p^{**} we may conclude in a familiar fashion that

5.10. $$[(\exists x_1)\ldots(\exists x_m)(\exists y_1)\ldots(\exists y_n)[A(x_1) \wedge \ldots \wedge A(x_m) \wedge \wedge Q(x_1, \ldots, x_m) \wedge R(y_1, \ldots, y_n)]] \supset X$$

also is deducible from K_p^{**}. Hence, in order to prove that X is deducible from K_p^{**} alone we only have to show that the implicans of 5.10 is deducible from K_p^{**}, i. e. that it holds in every model of K_p^{**}.

Let then M be a model of K_p^{**} and let M_A be the subfield of M which consists of the elements of M that satisfy $A(x)$. Then M_A is infinite and so we can find elements in M_A that satisfy $Q(x_1, \ldots, x_m)$, i. e. we can find elements in M that satisfy $A(x_1) \wedge \ldots \wedge A(x_m) \wedge Q(x_1, \ldots, x_m)$. I remains to be shown that in M we can find elements $a_1, \ldots, a_k$ which do not satisfy any non-vanishing polynomial of degree $\leqslant k$ with coefficients in M_A.

Choose an element a of M, which does not belong to M_A, and put

5.10. $a_i = a^{(k+1)^{i-1}}$, $i = 1, 2, \ldots, n$.

(This is Kronecker's substitution "in reverse".) Suppose that there exists a non-vanishing polynomial $r(x_1, \ldots, x_n)$ of degree $\leqslant k$ with coefficients in M_A such that

$$r(a_1, \ldots, a_n) = 0 .$$

Now a is transcendental with respect to M_A (which is algebraically closed), while by a familiar argument the substitution 5.10 transforms different products of powers of the a_i into different powers of a. Hence the substitution of a for the a_i in $r(a_1, \ldots, a_n)$ — or more precisely, the substitution of y for the x_i in $r(x_1, \ldots, x_n)$ by means of

$$x_i = y^{(k+1)^{i-1}}$$

yields a polynomial whose coefficients are all zero, and the same must therefore be true also of $r(x_1, \ldots, x_n)$. This is contrary to assumption and shown that the a_i given by 5.10 satisfy the required condition. Thus M satisfies

$$(\exists y_1)\ldots(\exists y_n) R(y_1, \ldots, y_n)$$

and hence, satisfies the implicans of 5.9, and hence satisfies X. This completes the proof of 5.3

In conclusion, the author wishes to express his indebtedness to Dr. A. Lévy for reading this paper in manuscript, and for suggesting a number of improvements.

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Reçu par la Rédaction le 1. 10. 1958

On clusters in proximity spaces *

by

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1. Introduction. The topology in a metric space is determined by stating which points are close to each given set, a point x being close to a set B if the distance between x and B is zero. A continuous mapping is just a function which preserves proximity between points and sets: fx is close to fB whenever x is close to B. In 1922 K. Kuratowski [3] had abstracted the proximity relation "x is close to B" by axiomatically characterizing the set $\bar{B}$ of all points close to B.

Now the uniform topology in a metric space is determined by stating which sets are close to each given set, a set A being close to a set B if the distance between A and B is zero. A uniformly continuous mapping is just a function which preserves proximity between sets: fA is close to fB whenever A is close to B. (See [17].) This immediately suggests abstracting uniform topology by axiomatizing the proximity relation "A is close to B" as a binary relation on subsets of a set X.

Strangely enough, this remained undone until 1952 when V. A. Efremovich [1] introduced a set of axioms characterizing proximity relations and thus launched the theory of proximity spaces. This theory is an elegant generalization of uniform topology in metric spaces, yet is more specific than the theory of uniform structures. (See [6].)

The compactification of proximity spaces was first treated by Yu. M. Smirnov [8]. Smirnov's treatment involves constructions using transfinite induction. In this paper we introduce an alternative approach to the compactification of a proximity space based on the simple concept of a "cluster", which is extrinsically just the class of all sets close to some fixed point. We avoid transfinite induction by using the axiom of choice in the following form: Given a class of elements, every subclass having a property of finite character is contained in some maximal subclass having that property [16].

2. Proximity spaces. Let X be an abstract set. A *point* is a subset of X having no proper subsets. A *proximity relation* in X is a binary

* Research sponsored by the Research Council of Rutgers, The State University.