Dense families of continuous selections *

by

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1. Introduction. Let X be a metric space, Y a Banach space, $\mathcal{C}(Y)$ the family of non-empty, closed, convex subsets of Y, and let $\varphi\colon X\to\mathcal{C}(Y)$ be lower semi-continuous (i. e. $\{x\in X \mid \varphi(x)\cap U\neq\varphi\}$ is open in X for every open $U\subset Y$). Under these circumstances, it was proved in [4], Theorem 3.2″ (see also Theorem 1 of the expository paper [3]) that there exists a *selection* f for φ, that is, a continuous $f\colon X\to Y$ such that $f(x)\in\varphi(x)$ for every $x\in X$. In the present paper, this result is applied to prove Theorem 1.1 below and some of its consequences. A special case of Theorem 1.1 will be used by V. L. Klee [2].

THEOREM 1.1. *For every infinite cardinal a, there exists a family Φ of selections for φ, with card $\Phi\leqslant a$, such that, whenever $x\in X$ and $\varphi(x)$ has a dense subset of cardinality $\leqslant a$, then $\{f(x)\}_{f\in\Phi}$ is dense in $\varphi(x)$* (1).

Our first corollary generalizes the well-known result that the Banach space of continuous, real-valued functions on a compact metric space is separable.

COROLLARY 1.2. *If X is compact and if, for some infinite cardinal a, $\varphi(x)$ has a dense subset of ordinality $\leqslant a$ for all $x\in X$, then the space of selection for φ has a uniformly dense subset of cardinality $\leqslant a$.*

If $C\subset\mathcal{C}(Y)$, then a *face* of C is a closed, convex subset F of C such that any line segment in C, which has an interior point in F, must be entirely in F; the *inside* of C, denoted by $I(C)$, is the set of points in C which lie in no face of C. It is known that every separable $C\subset\mathcal{C}(Y)$ has a non-empty inside ([4], Lemma 5.1). As another application of Theorem 1.1, we have the following result, which was obtained in [4], Theorem 3.1‴, for separable Y.

COROLLARY 1.3. *There exists a selection f for φ such that $f(x)\in I\big(\varphi(x)\big)$ whenever $\varphi(x)$ is separable.*

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(1) For separable Y, with $a=\aleph_0$, this result was already obtained in [4], Lemma 5.2.

Section 2 contains the proof of Theorem 1.1, and sections 3 and 4 contain the proofs of Corollaries 1.2 and 1.3, respectively. In section 5, finally, we observe that Theorem 1.1 remains true, with unchanged proof, in the more general situation described in Theorem 5.1.

2. Proof of Theorem 1.1. We begin with the following lemma, which follows easily from a result of A. H. Stone [7].

LEMMA 2.1. *Every metric space X has a σ-discrete* (2) *collection $\mathcal{A}$ of closed subsets such that, if $x\in U\subset X$ with U open, then $x\in A\subset U$ for some $A\in\mathcal{A}$.*

Proof. A. H. Stone [7] proved (and R. H. Bing [1], p. 179, explicitly observed) that X has a σ-discrete base $\mathcal{B}$ for the open sets. But then the collection $\mathcal{A}=\{\bar{B}\,|\,B\in\mathcal{B}\}$ satisfies all our requirements, and the proof of the lemma is complete.

We now turn to the proof of Theorem 1.1. Let $\mathcal{V}$ be a σ-locally finite base for the open sets in Y. For each $V\in\mathcal{V}$, let

$$U_V=\{x\in X\mid \varphi(x)\cap V\neq\varphi\};$$

then each U_V is open because φ is lower semi-continuous. Let $X'\subset X$ consist of all those $x\in X$ such that $\varphi(x)$ has a dense subset of cardinality $\leqslant\alpha$. Note that each $x\in X'$ is in U_V for at most α elements of V, for if D_x is a dense subset of $\varphi(x)$ of cardinality $\leqslant\alpha$, then each $y\in D_x$ is in at most countably many elements of $\mathcal{V}$, and therefore D_x, and thus also $\varphi(x)$, intersects at most $\aleph_0\alpha=\alpha$ elements of $\mathcal{V}$.

Now let $\mathcal{A}$ be as in Lemma 2.1, and for each $A\in\mathcal{A}$, let

$$\mathcal{V}_A=\{V\in\mathcal{V}\mid U_V\supset A\}.$$

Let

$$\mathcal{A}'=\{A\in\mathcal{A}\mid A\cup X'\neq\varphi\}.$$

Then $\operatorname{card}\mathcal{V}_A\leqslant\alpha$ for all $A\in\mathcal{A}'$, and hence, letting Λ be a set of cardinality α, we can write

$$\mathcal{V}_A=\{V_{A,\lambda}\}_{\lambda\in\Lambda}$$

for every $A\in\mathcal{A}'$.

Let $A\in\mathcal{A}'$ and $\lambda\in\Lambda$, and note that $\varphi(x)\cap V_{A,\lambda}\neq\varphi$ for all $x\in A$. We can therefore define φ: $A_{A,\lambda}\to\mathcal{C}(Y)$ by

$$\varphi_{A,\lambda}(x)=\big(\operatorname{conv}(\varphi(x)\cap V_{A,\lambda})\big)^{-}\ (^3)$$

(2) A collection $\mathcal{S}$ of subsets of a topological space is *discrete* if it is locally finite and its elements have disjoint closures; S is *σ-discrete* if $\mathcal{S}=\bigcup_{i=1}^{\infty}\mathcal{S}_i$, with each $\mathcal{S}_i$ discrete.

(3) $\operatorname{conv}(S)$ denotes the convex hull of a set $S\subset Y$.

and this $\varphi_{A,\lambda}$ is lower semi-continuous by [4], Propositions 2.6 and 2.3. The selection theorem quoted in the first paragraph on the introduction therefore provides us with a selection $g_{A,\lambda}$ for $\varphi_{A,\lambda}$.

By assumption, $\mathcal{A}=\bigcup_{i=1}^{\infty}\mathcal{A}_i$, with each $\mathcal{A}_i$ discrete. For each i, let $\mathcal{A}'_i=\mathcal{A}_i\cap\mathcal{A}'$, and let $C_i=\bigcup\mathcal{A}'_i$. Since $\mathcal{A}'_i$ is discrete, we have C_i closed, and the function $g_{i,\lambda}$: $C_i\to Y$, defined by

$$g_{i,\lambda}(x)=g_{A,\lambda}(x),\qquad x\in A\in\mathcal{A}'_i,$$

is continuous. Now define $\varphi_{i,\lambda}$: $X\to\mathcal{C}(Y)$ by

$$\varphi_{i,\lambda}(x)=\begin{cases}\{g_{i,\lambda}(x)\} & \text{if}\quad x\in C_i,\\ \varphi(x) & \text{if}\quad x\notin C_i.\end{cases}$$

Then $\varphi_{i,\lambda}$ is lower semi-continuous ([4], Example 1.3*), and hence has a selection $f_{i,\lambda}$. Let

$$\Phi=\{f_{i,\lambda}\mid \lambda\in\Lambda,\ i=1,2,\ldots\},$$

and let us show that Φ satisfies our requirements.

Clearly each $f\in\Phi$ is a selection for φ, and $\operatorname{card}\Phi\leqslant\aleph_0\alpha=\alpha$. It remains to check that $\{f(x)\mid f\in\Phi\}$ is dense in $\varphi(x)$ for every $x\in X'$. Since $\mathcal{V}$ is a base for Y, it suffices to show that, if $x\in X'$, $V\in\mathcal{V}$, and $\varphi(x)\cap V\neq\varphi$, then there exists a positive integer i and a $\lambda\in\Lambda$ such that

$$(*)\qquad f_{i,\lambda}(x)\in\big(\operatorname{conv}(\varphi(x)\cap V)\big)^{-}.$$

Now since $\varphi(x)\cap V\neq\varphi$, we have $x\in U_V$. Because of the property of $\mathcal{A}$ guaranteed in Lemma 2.1, there exists an i, and an $A\in\mathcal{A}_i$, such that

$$x\in A\subset U_V.$$

Hence $V\in\mathcal{V}_A$. But since $x\in(X'\cap A)$, we have $A\in\mathcal{A}'$, and hence $V=V_{A,\lambda}$ for some $\lambda\in\Lambda$. To show that the i and λ thus obtained work, note that $x\in C_i$, and hence

$$f_{i,\lambda}(x)=g_{i,\lambda}(x)=g_{A,\lambda}(x)\in\varphi_{A,\lambda}(x)=\big(\operatorname{conv}(\varphi(x)\cap V)\big)^{-},$$

which completes the proof.

3. Proof of Corollary 1.2. Let $\mathcal{V}$ be a countable base for X. For each finite, ordered subcovering $\{V_i\}_{i=1}^{n}$ of $\mathcal{V}$, pick a partition of unity

$\{p_i\}_{i=1}^n$ which is subordinated to it (4). Let Φ be as in Theorem 1.1, and let Φ' be the family of all functions $g: X \to Y$ of the form

$$g(x) = \sum_{i=1}^{n} p_i(x) f_i(x),$$

where $\{p_i\}_{i=1}^n$ is one of the above ordered partitions of unity, and $\{f_i\}_{i=1}^n$ is a finite sequence of elements of Φ. Clearly every $f \in \Phi'$ is a selection for φ, and card $\Phi' \leqslant \aleph_0 a = a$. It therefore remains to prove that Φ' is uniformly dense in the space of selections for φ.

Let f be a selection for φ, and let $\varepsilon > 0$. We must find a $g \in \Phi'$ such that $\varrho(f(x), g(x)) < \varepsilon$ for all $x \in X$, where ϱ is the metric in Y. For each $x \in X$, pick an $f_x \in \Phi$ such that $\varrho(f_x(x), f(x)) < \varepsilon$, and pick a neighborhood V_x of x, with $V_x \in \mathcal{V}$ such that $\varrho(f_x(x'), f(x')) < \varepsilon$ for all $x' \in V_x$. Then $\{V_x\}_{x \in X}$ is an open covering of X, and hence has a finite subcovering $\{V_{x_i}\}_{i=1}^n$. Let $\{p_{x_i}\}_{i=1}^n$ be the partition of unity subordinated to $\{V_{x_i}\}_{i=1}^n$ which was picked in the first paragraph of this proof. Now define g: $X \to Y$ by

$$g(x) = \sum_{i=1}^{n} p_{x_i}(x) f_{x_i}(x), \quad x \in X.$$

Then $g \in \Phi'$, and $\varrho(g(x), f(x)) < \varepsilon$ for all $x \in X$ because spheres in Y are convex sets. This completes the proof.

4. Proof of Corollary 1.3. This corollary is an immediate consequence of Theorem 1.1 and [4], Lemma 5.1. In fact, let $X' = \{x \in X \mid \varphi(x)$ is separable$\}$. Then by Theorem 1.1, with $a = \aleph_0$, there exists a sequence $\{g_i\}_{i=1}^\infty$ of selections for φ such that $\{g_i(x)\}_{i=1}^\infty$ is dense in $\varphi(x)$ for every $x \in X'$. Now, for every $x \in X$, let

$$f_i(x) = g_1(x) + \frac{g_i(x) - g_1(x)}{1 + \|g_i(x) - g_1(x)\|} \quad (i = 1, 2, \ldots),$$

$$f(x) = \sum_{i=1}^{\infty} (\tfrac{1}{2})^i f_i(x).$$

Then f is continuous, and $f(x) \in I(\varphi(x))$ for every $x \in X'$ by [4], Lemma 5.1.

(4) This means that each p_i is a continuous function from X to the closed interval $[0, 1]$, that p_i vanishes outside V_i, and that $\sum_{i=1}^{n} p_i(x) = 1$ for every $x \in X$.

5. A generalization of Theorem 1.1. An inspection of the proof of Theorem 1.1 reveals that it is applicable in any situation where there are "enough" selections. To be precise, let X and Y be metric spaces, $\mathcal{S}$ a family of non-empty subsets of Y, and $\varphi: X \to \mathcal{S}$. If $V, U \subset Y$, let us say that $V \prec U$ if, whenever $A \subset X$ is closed and $\varphi(x) \cap V \neq \varnothing$ for every $x \in A$, then there exists a selection for $\varphi \mid A$ with $f(x) \in U$ for every $x \in A$. Suppose now that

(a) For every $y \in Y$ and neighborhood U of y, there exists a neighborhood V of y with $V \prec U$.

(b) If $A \subset X$ is closed, then every selection for $\varphi \mid A$ can be extended to a selection for φ.

We now have the following generalization of Theorem 1.1.

THEOREM 5.1. *Theorem 1.1 remains true if X, Y, $\mathcal{S}$ and φ are as above.*

Proof. Let us first of all show that (a) above implies

(a′) There exists a σ-locally finite base $\mathcal{V}$ for Y, and for each $V \in \mathcal{V}$ a $U(V) \subset Y$ with $V \prec U(V)$, such that whenever $y \in Y$ and W is a neighborhood of y then there exists a $V \in \mathcal{V}$ such that $x \in V \subset U(V) \subset W$.

To prove (a′), let n be a positive integer, and let $\mathcal{U}_n$ be the family of open $(1/n)$-spheres about points in Y. Using (a) and the paracompactness of Y, there exists a locally finite open covering $\mathcal{V}_n$ of Y, and for each $V \in \mathcal{V}_n$ a $U(V, n) \in \mathcal{U}_n$, such that $V \prec U(V, n)$. Let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$, and for each $V \in \mathcal{V}$ let $n_V = \sup\{n \mid V \in \mathcal{V}_n\}$ and let $U(V) = U(V, n_V)$. It is easily checked that the above choices of $\mathcal{V}$ and $U(V)$ satisfy the conditions of (a′).

The proof of Theorem 5.1 now proceeds just like that of Theorem 1.1, with the following minor changes. First, we must take care to pick the base $\mathcal{V}$ for Y to satisfy (a′). Next, we no longer need $\varphi_{A,\lambda}$ to define $g_{A,\lambda}$; instead, we use (a′) to pick a selection $g_{A,\lambda}$ for $\varphi \mid A$ with $g_{A,\lambda}(x) \in U(V_{A,\lambda})$ for all $x \in A$. The functions $g_{i,\lambda}$ are now defined as in the proof of Theorem 1.1. However, we no longer need $\varphi_{i,\lambda}$ to define $f_{i,\lambda}$; instead, we use assumption (b) to pick a selection $f_{i,\lambda}$ for φ which extends $g_{i,\lambda}$. Finally, (a′) implies that requirement (*) can be replaced by

$$(**) \qquad f_{i,\lambda}(x) \in U(V).$$

The verification of (**) runs just like that of (*), taking into account the new definition of $g_{A,\lambda}$. This completes the proof.

Some situations to which Theorem 5.1 is applicable are covered in the following examples, where Example 5.3 is more general than Example 5.2.

EXAMPLE 5.2. Y is a Banach space, $\mathcal{S} = \mathcal{C}(Y)$, and $\varphi\colon X \to \mathcal{S}$ is lower semi-continuous. (This is the situation covered in Theorem 1.1.)

EXAMPLE 5.3. Y is a complete metric space with a convex structure (see [6] for the relevant definitions and theorems), $\mathcal{S}$ consists of closed, convex sets, and $\varphi\colon X \to \mathcal{S}$ is lower semi-continuous.

EXAMPLE 5.4. $\dim X \leqslant n+1$, Y is complete, $\mathcal{S}$ is an equi-LC^n family of closed sets (see [5] for the relevant definitions and theorems), and $\varphi\colon X \to \mathcal{S}$ is lower semi-continuous.

References

[1] R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. 3 (1951), p. 175-186.

[2] V. L. Klee, *Some new results on smoothness and strict convexity in normed linear spaces*, to appear in Math. Ann.

[3] E. Michael, *Selected selection theorems*, Amer. Math. Monthly 63 (1956), p. 233-238.

[4] — *Continuous selections I*, Ann. of Math. 63 (1956), p. 361-382.

[5] — *Continuous selections II*, Ann. of Math. 64 (1956), p. 562-580.

[6] — *Convex structures and continuous selections*, to appear in Canadian Math. J.

[7] A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. 54 (1948), p. 977-982.

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Solution of a problem of Tarski

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1. Introduction. In his classical paper "A decision method for elementary algebra and geometry" Note 21 ([6], p. 57) A. Tarski raises the question of providing a decision procedure for elementary sentences concerning the field of real numbers which, in addition to equality, order, addition, and multiplication, contain also the relation (atomic predicate) $A(x)$, to be satisfied exclusively by the *algebraic* real numbers. In the present paper, we solve this problem by specifying a complete set of axioms for the above-mentioned relations, including $A(x)$, such that the real numbers constitute a model of that set (sections 3, 4). The corresponding problem for the field of complex numbers is of a somewhat simpler nature (section 5).

We shall be concerned with algebraic fields in which certain additional relations have been defined, more particularly the relation of order, $Q(x, y)$ (i. e. $x \leqslant y$), the relation $A(x)$, and a set of relations $D_{nk}(x_1, \ldots, x_n)$, $n, k = 1, 2, \ldots$, which will be detailed presently. Accordingly, when we say that a field M' is an extension of a field M *including* $Q(x, y)$, and (or) $A(x)$ and (or) $D_{nk}(x_1, \ldots, x_n)$, $n, k = 1, 2, \ldots$ we shall mean by this that, for the elements of M, the relations $Q(x, y)$, $A(x)$, or $D_{nk}(x_1, \ldots, x_n)$ hold in M' if and only if they hold in M. Similarly, when we say that two fields, M and M', are isomorphic, *including* $Q(x, y)$, $A(x)$, and (or) $D_{nk}(x_1, \ldots, x_n)$, we shall mean that there exists an isomorphic correspondence between the two fields such that the relations in question hold, or do not hold, simultaneously for corresponding elements. In particular, an isomorphism which includes $Q(x, y)$ is simply an order-preserving isomorphism.

We shall also require the notion of relativisation with respect to the relation $A(x)$ (compare, e. g., [7], p. 24). This is defined inductively as follows (1.1-1.3)

1.1. The relativised form of an atomic formula X (e. g. $E(x, y)$, $S(x, a, b)$) is the formula itself. $R(X) = X$.