Rates of change and derivatives

by

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In Memory of Alexander Wundheiler

1. Introduction. Ever since Lagrange initiated a new epoch in pure analysis by defining the derivative of a function, the logical clarity of applied mathematics has suffered from a confusion of those derivatives with the rate of change of one variable quantity with respect to another. Yet a mere count of the ideas involved in the two concepts clearly demonstrates that the situations studied in pure and in applied mathematics are basically unlike. The derivative associates a function with one function; for instance, the cosine function with the sine function. The rate of change associates a variable quantity with two variable quantities; for instance, the velocity with the distance travelled and the time.

This paper is devoted to the clarification of that difference and to the formulation of articulate rules coordinating the two situations. For the past 200 years, their synthesis has been immensely successful. In fact, the application of the derivative to rates of change in the physical universe has been a contribution of paramount importance to the development of science. But that application has been based on intuitive manipulations rather than on a conceptual foundation.

2. Fluents. One of the principal sources of shortcomings in the traditional literature is the lack of an adequate treatment of variable quantities. Many who use those words fail to explain them by any (either explicit or implicit) definition. Others give a definition — quantities capable of assuming various values — without, however, (either explicitly or implicitly) defining quantity. Still others, finally, confuse variable quantities with number variables such as the letters x and c in the general statement:

$$\mathrm{x}-\mathrm{c}^2=(\sqrt{\mathrm{x}}+\mathrm{c})(\sqrt{\mathrm{x}}-\mathrm{c}) \text{ for any number } \mathrm{x}\geqslant 0 \text{ and any number } \mathrm{c}\,.$$

(„Number” here and in the sequel means: real number.) In order to forestall the latter (particularly obnoxious) confusion, in the present paper I will altogether avoid the words *variable quantity* and replace them by the brief term that Newton used in reference to distance travelled, time, velocity, and the like, namely, *fluents*.

For any class A, a *fluent* with the domain A is a class of ordered pairs in each of which the first member is an element of A, and the second member is a number — a class that is free of what I will call inconsistent pairs, that is, pairs whose first members are equal while their second members are unequal. (Cf. [2]-[5], and [6], especially Chapter VII.) (The reason for calling these classes fluents with the domain A rather than real functions on A will become apparent in Section 3.)

If A is the population of Warsaw and, for any inhabitant a of that city, ha denotes a's height in cm, then the class h of all pairs (a, ha) is a fluent with the domain A.

The observed position on a straight line S of a particle moving along S (more precisely, its directed distance in cm from a certain point on S, called origin) is a fluent if s is defined as the class of all pairs $(\sigma, s\sigma)$ for any act σ of reading the scale mark opposite the particle, where $s\sigma$ denotes the number read on the scale as the result of the act σ.

For a particle at rest relative to the origin, the range (i. e., the class of all values) of s consists of only one number. Such a fluent is said to be *constant* (1).

3. Functions. The cosine function is a fluent if it is defined in the customary way as the class of all pairs $(\mathrm{x}, \cos\mathrm{x})$ for any number x. Any fluent whose domain is a class of real numbers will be called a *function*.

The restriction of the word function to fluents of a special type (namely, to the fluents studied in pure mathematics, whose domains consist of numbers (2)), agrees with the use of the term by Leibniz, who introduced it in order to describe certain connections between Newton's fluents. One may ask what function in Leibniz' sense connects s with the time t; whether s is the sine of t; whether s is the logarithm of some other given fluent; and so on. But one cannot intelligently ask what fluent other than a function in Leibniz' sense connects s with t; whether s is the pressure of t; and whether the position is the temperature of some other given fluent.

(1) The existence of constant fluents shows that the definition of fluents as quantities capable of assuming various values in absence of a definition of quantity is not only incomplete but in some ways too restrictive. For, a constant fluent must by no means be confused with the number that is its only value.

(2) More generally, functions include fluents whose domains consist, for some positive integer n, of ordered n-tuples of numbers; of infinite sequences of numbers; or of mutually similar well-ordered sets of numbers. Fluents whose domains are functions have been called *functionals*. Fluents, in turn, are special cases of *mappings* — classes of mutually consistent pairs of any kind.

4. The substitution of fluents into functions. Let u be any fluent, that is, for some class A (called the *domain* of u and denoted by $\mathrm{Dom}\,u$),

the class of all pairs (a, ua) for any $a \in A$;

and let f be any function whose domain is the class of all numbers. Under these assumptions,

the class of all pairs $(a, f(ua))$ for any $a \in A$

is a fluent with the domain A that may be called the *result of substituting* u into f, and that may be denoted by $f(u)$. If each product of two fluents is designated by placing a dot between the symbols for the factor fluents (as in $f\cdot g$), then one may dispense with the parentheses in the symbols $f(u)$ and $f(ua)$ for results of substitutions and evaluations; and one may write (3)

fu is the class of all pairs (a, fua) for any $a \in A$.

It will be noted that in the preceding definition of the fluent fu, the letter f serves as a function variable while the letter u is what may be called a *fluent variable* (4); that is to say, u may be replaced with designations of specific fluents (e. g., the time t) just as f may be replaced with designations of specific functions (e. g., the cosine function), each such replacement yielding the definition of the result of substituting a specific fluent into a specific function (for instance, $\cos t$).

If f is a function whose domain does not include all numbers, then fu may be defined as

the class of all pairs (a, fua) such that $a \in \mathrm{Dom}\,u$ and $ua \in \mathrm{Dom}\,f$.

(3) Mere juxtaposition of the symbols for a function and a fluent is also the *traditional* designation of the result of substituting the fluent into the function where the latter has a *multi*-letter symbol, as in $\cos t$ and $\log p$ for the cosine of the time and the logarithm of the pressure. No one writes $\cos(t)$ or $\log(p)$. Only when a fluent is substituted into a function with a *single*-letter symbol, such as a Bessel function or a Legendre polynomial, parentheses are in traditional use; and so they are in $f(u)$, where a single letter serves as a function variable. In view of the utter mathematical irrelevance of the number of letters in a symbol for a function, any rational standardization should unify the two cases.

(4) It will be noted that *u is not a number variable*. No meaningful definition results if, in the definition of fu, the letter f is replaced with a function symbol, and u with a numeral. This is a first indication of the dangers in confusing fluents or fluent variables with number variables.

It is easy to describe the domain and the range of the fluent fu if, for any fluent v, any class D, and any class C of numbers, one considers the following two classes:

v (on D), the class of all pairs in v whose 1st member belongs to D;

v (into C), the class of all pairs in v whose 2nd member belongs to C.

In terms of these concepts (which are also useful in other connections) one readily proves:

$$\mathrm{Dom}(fu) = \mathrm{Dom}\,u \text{ (into } \mathrm{Dom} f), \quad \mathrm{Ran}(fu) = \mathrm{Ran} f \text{ (on } \mathrm{Ran}\,u) \tag{1}$$

for any function f and any fluent u.

The following two remarks will clarify the roles of the variables in formulae (1) and in the definition of fu:

1. f is a function variable but not a general fluent variable. Indeed, f may be replaced with the designation of the cosine function but not (if non-vacuous results are to be expected) with the designations of human height or observed position (as defined in Section 2). While the cosine of any fluent u is defined and satisfies formulae (1), the position of a fluent u on a line S remains undefined, and Dom u (into Dom s) is a vacuous symbol for each fluent u.

2. u is a fluent variable but not a number variable. True, upon replacement of u with, say, 3, the symbol fu happens to remain meaningful (namely, to yield the value that f assumes for 3). But formulae (1) and the very definition of fu become nonsensical if u is replaced with the symbol 3, since the number 3 has no domain (5) nor does it assume a value for a.

5. Relative equality of fluents. Fluents being classes, it is clear when two fluents are equal, when u is an extension of v, and when u is a restriction of v. For instance, the fluents v (on D) and v (into C) are proper restrictions of v if D (or, more generally, the intersection of D and Dom u) is a proper subclass of Dom v, and if C (or the intersection of C and Ran v) is a proper subclass of Ran v.

(5) Of course, the number 3 must not be confused with a constant fluent of value 3 (cf. (1)). The constant function *3* of value 3 is the class of all pairs of numbers (x, 3) for any number x. The reader will note that this paper adheres to the typographical convention introduced in the authors book *Calculus. A Modern Approach* [6]. Symbols referring to numbers, to *functions*, and to **operators** are printed in roman, in *italic*, and in **bold face** type, respectively. This standardization greatly simplifies the reading of formulae, and saves some parentheses and a great deal of verbiage.

In pure mathematics, these concepts are satisfactory with regard to the fluents studied, that is, with regard to functions. Two functions, f and g, are equal if and only if any pair belonging to either function also belongs to the other, which is the case if and only if $\mathrm{Dom} f = \mathrm{Dom}\, g$ and $f\mathrm{x} = g\mathrm{x}$ for any element x of that common domain.

In science, however, it is often necessary to relate fluents with disjoint domains — fluents which, mathematically speaking, are unequal. Yet a workable and useful definition of scientific equality of v and u is given, namely, relative to a certain subclass Π of the Cartesian product $\mathrm{Dom}\,u \times \mathrm{Dom}\,v$. I will write

$$v = u \text{ (rel. } \Pi) \quad \text{or} \quad v \underset{\Pi}{=} u$$

if and only if $(\alpha, \beta) \in \Pi$ implies $v\beta = u\alpha$.

For instance, let s' be the observed position of a particle moving along a straight line S' (more precisely, the particle's directed distance from an origin on S'), defined as the class of all pairs $(\sigma', s'\sigma')$ for any act σ' of reading the scale mark opposite the particle. Physicists frequently compare the fluent s' with s (as defined in Section 2) relative to the class Γ_1 of all pairs of simultaneous acts (σ, σ'). The fluents s and s' are equal relative to this class Γ_1 if and only if

$$(\sigma, \sigma') \in \Gamma_1 \quad \text{implies} \quad s'\sigma' = s\sigma .$$

Again, physicists compare s and s' relative to the class Γ_2 of pairs (σ, σ') of acts that are respectively simultaneous with equal readings on two timers (set in motion at different instants and possibly calibrated in different units).

Since Galileo and Boyle, most comparisons of fluents in classical physics have been based, in some way or other, on *simultaneity of acts of observation or of physical states*. The comparison of functions, expressed in the mathematical concept of equality (as defined at the beginning of Section 5) is implicitly based on *equality of numbers*; more precisely, $f = g$ means equality relative to the class of pairs of equal numbers belonging to the intersection of $\mathrm{Dom} f$ and $\mathrm{Dom}\, g$. In contrast, statisticians use a great variety (6) of subclasses of the Cartesian products of the domains (or, as many statisticians say, various *pairings* of the *populations* of the *variates*).

6. Properties of relative equality. For any class Π of pairs, let Π^* denote the class of all pairs (β, α) such that (α, β) belongs to Π. For any two classes Π and P, let $P\Pi$ denote the class of all pairs (α, γ)

(6) In comparing the height h and the weight w in the population of Warsaw, one may study the class Π of all pairs (a, a) for any inhabitant a; or the class Π_1 of all pairs (a, β), where β is the father of a; or the class Π_2 of all pairs of twins, and so on.

for which there exists an element β such that $(\alpha, \beta) \in \Pi$ and $(\beta, \gamma) \in P$. Finally, let $I(C)$ for any class C denote the class of all pairs (γ, γ) such that $\gamma \in C$. It then is clear that relative equality has the following properties:

Symmetry. If $v = u$ (rel. Π), then $u = v$ (rel. Π^*).

Transitivity. If $v = u$ (rel. Π) and $w = v$ (rel. P), then $w = u$ (rel. $P\Pi$). Here, $\Pi \subseteq \mathrm{Dom}\, u \times \mathrm{Dom}\, v$ and $P \subseteq \mathrm{Dom}\, v \times \mathrm{Dom}\, w$, wherefore $P\Pi \subseteq \mathrm{Dom}\, u \times \mathrm{Dom}\, w$.

Reflexivity. $u = u$ (rel. $I(\mathrm{Dom}\, u)$).

Relative equality of a fluent with another fluent that is the result of a substitution is of paramount importance in applications of analysis to science:

$$w = fu \text{ (rel. } \Pi) \quad \text{or} \quad w \underset{\Pi}{=} fu$$

if and only if $(\alpha, \beta) \in \Pi$ implies $w\beta = fu\alpha$.

From the transitivity of relative equality it follows that

if $v = fu$ (rel. Π) and $w = gv$ (rel. P), then $w = gfu$ (rel. $P\Pi$).

7. Rate of change. From this point on, it will be assumed that the domain of any fluent studied is a limit class. There is no difficulty about defining when a fluent is *continuous*. The idea of rate of change is more complex. The velocity of a moving particle, the slope of a curve, and similar examples suggest that the rate of change of a fluent w with respect to a fluent u is itself a fluent. Consequently, its domain must be clearly defined. Moreover, it will appear that an articulate definition of that rate of change must be relative to a subclass Π of $\mathrm{Dom}\, u \times \mathrm{Dom}\, w$; that is to say, that one must define a fluent

$$\frac{\boldsymbol{dw}}{\boldsymbol{du}} \text{ (rel. } \Pi) \quad \text{or} \quad \boldsymbol{dw} \underset{\Pi}{/} \boldsymbol{du}$$

a relativization comparable to that presupposed by a scientifically workable definition of equality.

The salient point of the theory to be expounded is the following. The domain of $\boldsymbol{dw} \underset{\Pi}{/} \boldsymbol{du}$ is a subclass of that class Π; in a formula,

$$\mathrm{Dom}(\boldsymbol{dw} \underset{\Pi}{/} \boldsymbol{du}) \subseteq \Pi.$$

Clearly, since $\mathrm{Dom}\, u$ and $\mathrm{Dom}\, w$ are limit classes, so are (in a natural way) $\mathrm{Dom}\, u \times \mathrm{Dom}\, w$ and $\mathrm{Dom}(\boldsymbol{dw} \underset{\Pi}{/} \boldsymbol{du})$.

To facilitate matters, it will be assumed that $\mathrm{Dom}\, u$ and $\mathrm{Dom}\, w$ are endowed with a continuous semi-metric, by which I mean that a number $d(\alpha', \alpha)$ be associated with any two elements α' and α of $\mathrm{Dom}\, u$ in such

a way that $\lim d(\alpha_n, \alpha) = 0$ if and only if $\lim \alpha_n = \alpha$; and that a similar definition applies to w. These auxiliary metrics as such will be of no significance and may be replaced with any two other metrics that preserve the limits.

For any number $\mathrm{d} > 0$, the pair (ξ, η) of $\mathrm{Dom}\, u \times \mathrm{Dom}\, w$ is said to be a d-*neighbor* of the pair (α, β) if $d(\xi, \alpha) < \mathrm{d}$ and $d(\eta, \beta) < \mathrm{d}$. Two pairs (α, β) and (ξ, η) are said to be u-*discriminating* if $u\alpha \neq u\xi$. A pair $(\alpha, \beta) \in \Pi$ will be called Π-*normal* if, for each number $\mathrm{d} > 0$, the class Π contains a u-discriminating d-neighbor of (α, β).

If (α, β) is Π-normal and c is a (finite) number, then I will call c the value of $\boldsymbol{dw} \underset{\Pi}{/} \boldsymbol{du}$ for (α, β), and I will write

$$\boldsymbol{dw} \underset{\Pi}{/} \boldsymbol{du}(\alpha, \beta) = \mathrm{c}$$

if and only if the following is satisfied.

Condition c. For each positive integer n, there exists a number $\mathrm{d_n} > 0$ such that

$$\left| \frac{w\eta - w\beta}{u\xi - u\alpha} - \mathrm{c} \right| < \frac{1}{\mathrm{n}}$$

for each u-discriminating $\mathrm{d_n}$-neighbor $(\xi, \eta) \in \Gamma$ of (α, β).

Since (α, β) is Π-normal, clearly no two unequal numbers c and c′ can be values of $\boldsymbol{dw} \underset{\Pi}{/} \boldsymbol{du}$ for (α, β). Accordingly, the said rate of change may be defined as the class of all pairs

$$\left((\alpha, \beta),\ \lim_{\substack{\xi \to \alpha \\ \eta \to \beta \\ (\xi, \eta) \in \Pi}} \frac{w\eta - w\beta}{u\xi - u\alpha} \right)$$

for any Π-normal pair (α, β) for which the limit exists; that is to say, such that Condition c is satisfied for some number c.

Clearly, the value of the rate of change for (α, β) depends only upon the values of u and w for the members of neighbor pairs $\in \Pi$. The limit may exist for each pair $(\alpha, \beta) \in \Pi$, or for some but not all of these pairs, or for none. Accordingly, $\mathrm{Dom}(\boldsymbol{dw} \underset{\Pi}{/} \boldsymbol{du})$ may be Π, or a proper subclass of Π, or the vacuous class.

The following remark bears out what has been said in Section 1: The rate of change is a binary operator, and the words "the rate of change of a fluent" remain undefined. The reader should beware of mistaking the quotients

$$\frac{u\xi - u\alpha}{d(\xi, \alpha)}$$

for a possible basis of a unitary rate of change (or fluxion) of the fluent u. For, as far as u is concerned, the auxiliary distance might well be replaced by its double, thereby reducing each of the said quotients to its half.

8. General remarks (7). For any class C and any number p, the constant fluent of value p with the domain C will be denoted by p (on C). For any fluent u, the element α will be called u-*changing* if $\alpha \in \mathrm{Dom}\, u$ and for each positive integer n there exists an element $\xi_n \in \mathrm{Dom}\, u$ such that $d(\xi_n, \alpha) < 1/n$ and $u\xi_n \neq u\alpha$. The class of all u-changing elements will be denoted by $\mathrm{ChDom}\, u$ (Ch for characteristic). In this notation,

$$\frac{du}{du}\big(\text{rel. } I(\mathrm{ChDom}\, u)\big) = 1\,(\text{on } \mathrm{ChDom}\, u), \text{ for any fluent } u\,.$$

Here, $I(\mathrm{ChDom}\, u)$ (see Section 5) is the class of all pairs (α, α) for any $\alpha \in \mathrm{ChDom}\, u$; and the equality is relative to the class of all pairs $\big(\alpha, (\alpha, \alpha)\big)$ for any $\alpha \in \mathrm{ChDom}\, u$. With the same definition of equality, also the following more general formula is self-explanatory.

$$\frac{d(\mathrm{p}u+\mathrm{q})}{du}\big(\text{rel. } I(\mathrm{ChDom}\, u)\big) = p \;\;(\text{on } \mathrm{ChDom}\, u),$$

for any fluent u and any two numbers p and q.

It will be noted that the preceding results do not presuppose that u have a limit anywhere, let alone that u be continuous. If $\mathrm{Domlim}\, u$ denotes the class of all $\alpha \in \mathrm{Dom}\, u$ such that $\lim_{\alpha} u$ exists, then, as one readily proves,

$$\frac{du^2}{du}\big(\text{rel. } I(\mathrm{ChDom}\, u)\big) = u + \lim u, \text{ for any fluent } u\,.$$

The domain of the fluent on the right side is the class of all elements belonging to both $\mathrm{ChDom}\, u$ and $\mathrm{Domlim}\, u$. The fluent assumes the value $u\alpha + \lim_{\alpha} u$ for each α belonging to that domain. The equality is relative to the class of pairs $\big(\alpha, (\alpha, \alpha)\big)$ for any α in that domain. Any polynomial of u can be treated similarly.

If u is continuous at α, and (α, β) belongs to $\mathrm{Dom}(dw \underset{\Pi}{/} du)$, then w is continuous at β in the following restricted sense: For any positive integer n there exists a number $\mathrm{d}'_n > 0$ such that $|w\eta - w\beta| < 1/n$ for each u-discriminating d'_n-neighbor (ξ, η) of (α, β) in Π.

(7) Most of the following remarks were suggested to the author by his colleague Prof. Abe Sklar.

9. The derivative of a function. Df is defined as the class of all pairs

$$\Big(\mathrm{a},\; \lim_{\substack{\mathrm{x}\to\mathrm{a}\\ \mathrm{x}\in\mathrm{Dom} f}} \frac{f\mathrm{x} - f\mathrm{a}}{\mathrm{x} - \mathrm{a}}\Big) \text{ for any } \mathrm{a} \in \mathrm{Dom} f \text{ such that the limit exists.}$$

This definition can be rephrased in such a way as to subsume under the definition of a rate of change. For this purpose it is necessary to have at one's disposal a symbol for the class of all pairs (x, x) for any number x. I will use (8) the letter j, and refer to this traditionally symbolless class as the *identity function*.

The class j plays a twofold role in rephrasing the definition of Df. Firstly, j is the fluent with respect to which the rate of change of the fluent f will be considered. Secondly, j (on $\mathrm{Dom} f$) (that is, the class (9) of all pairs (x, x) for any $\mathrm{x} \in \mathrm{Dom} f$) is the class relative to which I will consider the rate of change of f with respect to j.

In view of the fact that a pair of numbers (x, y) belongs to j (on $\mathrm{Dom} f$) if and only if $\mathrm{x} \in \mathrm{Dom} f$ and $\mathrm{y} = \mathrm{x}$, it is clear that

$$\frac{df}{dj}\big(\text{rel. } j \text{ (on } \mathrm{Dom} f)\big) \quad \text{or} \quad \underset{j\,(\text{on } \mathrm{Dom} f)}{df/dj}$$

is the class of all pairs

$$\Big((\mathrm{a}, \mathrm{a}),\; \lim_{\substack{\mathrm{x}\to\mathrm{a}\\ \mathrm{x}\in\mathrm{Dom} f}} \frac{f\mathrm{x} - f\mathrm{a}}{j\mathrm{x} - j\mathrm{a}}\Big) \quad \text{for any} \quad \mathrm{a} \in \mathrm{Dom}\, Df,$$

that is, for any $\mathrm{a} \in \mathrm{Dom} f$ such that the limit exists.

The domain of this rate of change and $\mathrm{Dom}\, Df$ are disjoint classes, since the latter consists of numbers and the former of pairs of equal numbers (a, a) such that $\mathrm{a} \in \mathrm{Dom}\, Df$. But except for this fact, the rate of change of f with respect to j is equal to Df. The difference between them is comparable to that between the rational number $\frac{3}{1}$ and the integer 3. Hence

THEOREM I. *The derivative of a function f is essentially the rate of change of f with respect to j and relative to j (on* $\mathrm{Dom} f$*). In a formula,*

$$Df \underset{\Pi}{=} \frac{df}{dj}\big(\text{rel. } j \;\; (\text{on } \mathrm{Dom} f)\big)$$

where the equality is relative to the class Π of all pairs $\big((\mathrm{x}, \mathrm{x}), \mathrm{x}\big)$ for any $\mathrm{x} \in \mathrm{Dom}\, Df$.

(8) Cf. [1]. Occasionally, I have used the letter I for the identity function. But it seems preferable to reserve italic capitals for functions of several places and to designate one-place functions by non-capital letters. Also the symbol *id* has been suggested for the identity function.

(9) Of course, j (on $\mathrm{Dom} f$) $= I$ ($\mathrm{Dom} f$) if $I(C)$ is defined as in Section 6.

10. On functionally connected fluents. Suppose that $w \underset{\Pi}{=} fu$ and that (α, β) is Π-normal. Assume further that

$$\underset{\Pi}{\boldsymbol{dw/du}}(\alpha, \beta) = \mathrm{c} \quad \text{and} \quad \boldsymbol{D}fu\alpha = \mathrm{c}' .$$

Since (α, β) is Π-normal, there exist arbitrarily close u-discriminating neighbor pairs (ξ, η) of (α, β). For each sufficiently close pair of this kind,

$$\frac{w\eta - w\beta}{u\xi - u\alpha}$$

is arbitrarily close to c. In view of $w \underset{\Pi}{=} fu$, this means that

$$\frac{fu\xi - fu\alpha}{u\xi - u\alpha}$$

is arbitrarily close to c. If u is continuous at α, then the said difference quotient is also close to c′; wherefore $\mathrm{c} = \mathrm{c}'$. The preceding reasoning shows that, if u is continuous at α, the existence of $\boldsymbol{D}fu\alpha$ implies the existence of $\underset{\Pi}{\boldsymbol{dw/du}}(\alpha, \beta)$.

In order to infer, conversely, from the existence of the latter number that of $\boldsymbol{D}fu\alpha$, one must assume that, for any number $\mathrm{x} \in \mathrm{Dom} f$ that is sufficiently close to $u\alpha$, there is an element $\xi' \neq \alpha$ and an element η' such that (ξ', η') is a close neighbor pair of (α, β) belonging to Π and that $u\xi' = \mathrm{x}$. I will say that u is *Darboux-continuous* at α if u satisfies the following condition: For each positive integer m, each ξ such that $d(\xi, \alpha) < 1/\mathrm{m}$ and $u\xi \neq u\alpha$, and each number x between $u\xi$ and $u\alpha$, there exists an element ξ' of $\mathrm{Dom}\, u$ such that $d(\xi', \alpha) < 1/\mathrm{m}$ and $u\xi' = \mathrm{x}$. I will call Π *quasi-continuous* at (α, β) if for each positive integer m there exists a positive integer n such that $d(\xi, \alpha) < 1/\mathrm{n}$ implies the existence of an element η such that $(\xi, \eta) \in \Pi$ and $d(\eta, \beta) < 1/\mathrm{m}$. In this terminology, the preceding results can be summarized as follows:

THEOREM II. *If $w \underset{\Pi}{=} fu$ and u is continuous at α and $\boldsymbol{D}fu\alpha$ exists, then so does $\underset{\Pi}{\boldsymbol{dw/du}}(\alpha, \beta)$. If u is Darboux-continuous at α, if Π is quasi-continuous at (α, β), and if $\underset{\Pi}{\boldsymbol{dw/du}}(\alpha, \beta)$ exists, then so does $\boldsymbol{D}fu\alpha$. In either case,*

$$\underset{\Pi}{\boldsymbol{dw/du}}(\alpha, \beta) = \boldsymbol{D}fu\alpha .$$

In the preceding theorem, f is only a function variable and, therefore, must not be replaced with designations of fluents other than functions. In contrast, u and w are general fluent variables that may, in particular, be replaced with the designations of functions — for instance, continuous nowhere differentiable functions.

If f is replaced with a designation of the sine function, then, in view of $\boldsymbol{D}\mathit{sin} = \mathit{cos}$, Theorem II yields the following

COROLLARY. *If $w \underset{\Pi}{=} \mathit{sin}\, u$, then $\underset{\Pi}{\boldsymbol{dw/du}} = \mathit{cos}\, u$.*

Here, the fluent variables u and w may be replaced with designations of specific fluents, e. g., the time t and the position s of a certain oscillator, and Π with the designation of a specific class of pairs, e. q., Galileo's class Γ of pairs of simultaneous acts of clock readings and mark readings. Or u and w may be replaced with the abscissa x and the ordinate y along a sine curve in the Cartesian plane, and Π with the class I of pairs of equal points on that curve. In this way one obtains:

If $s \underset{\Gamma}{=} \mathit{sin}\, t$, then $\underset{\Gamma}{\boldsymbol{ds/dt}} = \mathit{cos}\, t$.

If $y \underset{I}{=} \mathit{sin}\, x$, then $\underset{I}{\boldsymbol{dy/dx}} = \mathit{cos}\, x$.

But while u and w are fluent variables, it is perfectly clear that they are not number variables. Replacing in the corollary u and w with designations of numbers one obtains false implications. If u and w are replaced with π and 0, respectively, in the resulting implication

$$\text{if } 0 = \mathit{sin}\, \pi, \text{ then } \boldsymbol{d}0/\boldsymbol{d}\pi = \mathit{cos}\, \pi$$

the antecedent is valid while the consequent is nonsensical.

11. Reciprocal rates of change. The element (α, β) of $\Pi \subseteq \mathrm{Dom}\, u \times \mathrm{Dom}\, w$ will be called *Π-binormal* if for each n the class Π contains an 1/n-neighbor (ξ, η) of (α, β) such that $u\xi \neq u\alpha$ and $w\eta \neq w\beta$. Clearly, if (α, β) is Π-binormal, then (β, α) is Π^*-binormal.

THEOREM III. *If (α, β) is Π-binormal and the numbers $\underset{\Pi}{\boldsymbol{dw/du}}(\alpha, \beta)$ and $\underset{\Pi^*}{\boldsymbol{du/dw}}(\beta, \alpha)$ exist and are $\neq 0$, then*

$$\underset{\Pi}{\boldsymbol{dw/du}}(\alpha, \beta) \cdot \underset{\Pi^*}{\boldsymbol{du/dw}}(\beta, \alpha) = 1 .$$

For each $(\xi, \eta) \in \Pi$ such that $u\xi \neq u\alpha$ and $w\eta \neq w\beta$, one has

$$\frac{w\eta - w\beta}{u\xi - u\alpha} \cdot \frac{u\xi - u\alpha}{w\eta - w\beta} = 1 .$$

Since (α, β) is Π-binormal, Π contains pairs (ξ, η) arbitrarily close to (α, β) for which both difference quotients mentioned in the preceding formula are numbers $\neq 0$. Since by assumption the two rates of change exist and are $\neq 0$, their product equals 1.

12. The chain rule. Consider three fluents u, v, w and two classes $\Pi \subseteq \mathrm{Dom}\, u \times \mathrm{Dom}\, v$ and $P \subseteq \mathrm{Dom}\, v \times \mathrm{Dom}\, w$. Assume that (α, β) be Π-binormal and that P is quasi-continuous at (β, γ). Clearly, $(\alpha, \gamma) \in$

$\epsilon P\Pi \subseteq \mathrm{Dom}\,u \times \mathrm{Dom}\,w$. There exist arbitrarily close neighbors $(\xi, \eta) \epsilon \Pi$ of (α, β) such that $u\xi \neq u\alpha$ and $v\eta \neq v\beta$; and arbitrarily close to γ there exists an element ζ of $\mathrm{Dom}\,w$ such that $(\eta, \zeta) \epsilon P$. Clearly,

$$\frac{v\eta - v\beta}{u\xi - u\alpha} \cdot \frac{w\zeta - w\gamma}{v\eta - v\beta} = \frac{w\zeta - w\gamma}{u\xi - u\alpha}.$$

If the corresponding rates of change exist, they satisfy the chain rule expressed in the following

THEOREM IV. *If (α, β) is Π-binormal and P is quasi-continuous at (β, γ), then*

$$\underset{\Pi}{\boldsymbol{d}v/\boldsymbol{d}u}(\alpha, \beta) \cdot \underset{P}{\boldsymbol{d}w/\boldsymbol{d}v}(\beta, \gamma) = \underset{P\Pi}{\boldsymbol{d}w/\boldsymbol{d}u}(\alpha, \gamma)$$

if the three rates of change exist.

13. The rate of change of totally unconnected fluents. Theorems III and IV hold even for fluents that are not functionally connected with one another. Consider in a Cartesian plane, e. g., the ellipse E given by

$$\{x^2 + 4y^2 = 1 \text{ (on } E)\},$$

that is, the class of all points P in the plane such that $x^2P + 4y^2P = 1$. Clearly, neither of the fluents x (on E) and y (on E) is a function of the other relative to $I(E)$, that is, the class of all pairs (P, P) for any P on E. For instance, y assumes unequal values for two points on E whose abscissae are equal. Notwithstanding the lack of any functional connection between x (on E) and y (on E), Theorem III implies

$$\underset{\Pi}{\boldsymbol{d}y/\boldsymbol{d}x}(P, P) \cdot \underset{\Pi^*}{\boldsymbol{d}x/\boldsymbol{d}y}(P, P) = 1, \quad \text{where} \quad \Pi = \Pi^* = I(E),$$

for any point P on E where both rates of change exist and are $\neq 0$.

Of course, in a neighborhood of each such point, x (on E) and y (on E) are functionally connected. For instance, along the open quarter of E in the first quadrant:

$$y = \tfrac{1}{2}\sqrt{1-x^2} = \tfrac{1}{2}\sqrt{1-j^2}\,x \quad \text{and} \quad x = \sqrt{1-4y^2} = \sqrt{1-4j^2}\,y,$$

where $\sqrt{\ }$ indicates the positive square root, and the juxtaposition of the function and fluent symbols indicates substitution of the fluents into the functions. According to Theorem II,

$$\underset{\Pi}{\boldsymbol{d}y/\boldsymbol{d}x}(P, P) = -\frac{1}{2}\frac{j}{\sqrt{1-j}}\,xP = -\frac{1}{2}\frac{xP}{\sqrt{1-x^2P}}.$$

Similarly, $\underset{\Pi^*}{\boldsymbol{d}x/\boldsymbol{d}y}$ can be connected with y. For each P, the product of the two rates of change is 1, in agreement with Theorem III.

But it is quite different examples that illustrate the true width of the idea of rate of change and the full generality of Theorems III and IV. This is achieved by examples of fluents u and w such that $\underset{\Pi}{\boldsymbol{d}w/\boldsymbol{d}u}$ exists without w being a function of u relative to Π *in any open subclass of* $\mathrm{Dom}\,u$; in other words, in cases where every open subclass of $\mathrm{Dom}\,u$ includes two elements α and α' such that

$$(\alpha, \beta) \epsilon \Pi, \quad (\alpha', \beta') \epsilon \Pi, \quad u\alpha = u\alpha' \quad \text{and} \quad w\beta \neq w\beta'$$

and, nonetheless, $\underset{\Pi}{\boldsymbol{d}w/\boldsymbol{d}u}$ exists. In fact, such examples of u and w exist in the realm of functions, where Π is a subclass of j (that is, a class of pairs of equal numbers).

Let g denote Cantor's nowhere decreasing function such that $g0 = 0$, $g1 = 1$, and which is constant on every interval in the complement of Cantor's discontinuum D. Set

$$g_D = g \text{ (on } D) \quad \text{and} \quad j_D = j \text{ (on } D).$$

Then it is easy to show that

$$\boldsymbol{d}j_D/\boldsymbol{d}g_D = 0 \text{ (on } D);$$

that is to say,

$$\boldsymbol{d}j_D/\boldsymbol{d}g_D(\mathrm{x}, \mathrm{x}) = 0 \quad \text{for each} \quad \mathrm{x} \epsilon D.$$

Yet each open subclass of D includes pairs of unequal numbers x, x' such that $g_D\mathrm{x} = g_D\mathrm{x}'$ while, of course, $j_D\mathrm{x} \neq j_D\mathrm{x}'$.

From a remark in Section 8 it follows that, if f is a differentiable function, then

$$\boldsymbol{d}(fg_D)/\boldsymbol{d}g_D = \boldsymbol{D}fg_D,$$

where $\boldsymbol{D}fg_D$ is the result of substituting g_D into $\boldsymbol{D}f$. Clearly, $g\mathrm{x} = g\mathrm{x}'$ implies $fg\mathrm{x} = fg\mathrm{x}'$. Consequently, just like j_D, the function $j_D + fg_D$ is not a function of g_D in any open subclass of D. Yet, in view of the linearity of the rate of change operator, the rate of change of $j_D + fg_D$ with respect to g_D exists, and, more specifically,

$$\boldsymbol{d}(j_D + fg_D)/\boldsymbol{d}g_D = \boldsymbol{D}fg_D.$$

Any continuous function is the derivative of a differentiable function f. Hence

THEOREM V. *On D, any continuous function connects with g the rate of change with respect to g of some function h even though h is not functionally connected with g in any open subclass of D.*

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Kennzeichnung von Bogen

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Als *Bogen* bezeichnet man bekanntlich jeden topologischen Raum der zu dem abgeschlossenen reellen Intervall $\langle 0, 1\rangle$ homöomorph ist. Eine Kennzeichnung der Bogen liefert der folgende Satz von J. Lennes:

Es sei E ein Hausdorffscher Raum mit abzählbarer Basis, und B sei eine Teilmenge von E, die die Punkte a und b enthält. B ist genau dann ein Bogen, wenn B ein Kontinuum ist und wenn es in B eine Relation $\leqslant$ mit folgenden Eigenschaften gibt: (1) *Die Relation $\leqslant$ ist eine lineare Ordnung von B.* (2) *Für jeden Punkt $p \in B$ gilt $a \leqslant p \leqslant b$.* (3) *Gilt $p < q$, so ist die Menge $\{x\colon p \leqslant x \leqslant q\}$ abgeschlossen.*

Diese Charakterisierung ist deswegen wenig befriedigend, weil in den Voraussetzungen die Existenz einer Ordnungsrelation gefordert wird. Nachstehend soll daher ein anderes Kriterium angegeben werden, das eine Kennzeichnung der Bogen durch innere, rein topologische Eigenschaften gestattet. Interessant ist dabei, daß die wesentlichen Voraussetzungen aus reinen Zusammenhangseigenschaften bestehen.

Satz 1. *Ein topologischer Raum E ist dann und nur dann homöomorph zu $\langle 0, 1\rangle$, $\langle 0, 1)$ oder $(0, 1)$, wenn er folgende Eigenschaften besitzt:*

(1) *E ist ein separabler T_1-Raum mit mindestens zwei Punkten.*

(2) *E ist zusammenhängend und lokal zusammenhängend.*

(3) *Unter je drei nicht-leeren, zusammenhängenden, echten Teilmengen von E gibt es stets zwei, die E nicht überdecken.*

Beweis. Die Bedingungen (1) und (2) sind offenbar notwendig. Im Falle des Intervalls $\langle 0, 1\rangle$ kann man die Notwendigkeit der Eigenschaft (3) folgendermaßen erkennen: Eine zusammenhängende echte Teilmenge von $\langle 0, 1\rangle$ kann die Punkte 0 und 1 nicht gleichzeitig enthalten. Unter je drei nicht-leeren, zusammenhängenden, echten Teilmengen von $\langle 0, 1\rangle$ gibt es daher zwei, die den einen Endpunkt nicht enthalten. In den anderen beiden Fällen schließt man analog, wobei an Stelle des Enthaltenseins eine Häufungseigenschaft zu treten hat. Da die Bedingungen (1)-(3) topologisch invariant sind, ist damit ihre Notwendigkeit allgemein nachgewiesen.