Functionals on uniformly closed rings of continuous functions

by

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In this paper we are concerned with the following problem: Suppose X is a completely regular space and let R be a linear ring of continuous real-valued functions defined on X which satisfies the following conditions:

1° *All constant functions belong to* R.

2° R *is closed with respect to the uniform convergence (i. e. if* $\{f_n\}$ *uniformly converges to* f *and* $f_n \in R$ $(n = 1, 2, \ldots)$, *then* $f \in R$*).*

Under what conditions imposed on X and R each non-trivial linear multiplicative functional φ (1) defined on R is of the form

$$(*) \qquad \varphi(f) = f(p_0)$$

where p_0 is a fixed point of X?

We note some results related to this problem:

If R *is the ring of all bounded continuous functions on* X, *then the answer to our problem is positive if and only if* X *is a compact space* (Stone [4]).

If R *is the ring of all continuous functions on* X *then the answer to the problem is positive if and only if* X *is a* Q*-space* (Hewitt [1], [2]).

The main role in our considerations is played by the evaluation mapping of X into the Tihonov cube build up by means of all members f of R which satisfy the inequality $0 \leqslant f(p) \leqslant 1$ (i. e. denote by R^* the set of all members f of R which satisfy the above inequality and agree that the coordinates of points of the Tihonov cube $I^{\mathfrak{m}}$ $(\mathfrak{m} = \overline{\overline{R}})$ are enumerated by means of members of R^*. Then the evaluation mapping can be described as a mapping which carries a point $p \in X$ into the point $a \in I^{\mathfrak{m}}$ whose fth coordinate is equal to $f(p)$). We denote this evaluation mapping by F_R.

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(1) A functional φ is said to be *non-trivial* provided that φ does not vanish identically.

If each member of a ring R is bounded, then the answer to our problem is rather uninteresting; it is quite similar to that of the above-mentioned Stone result; namely, it is positive if and only if $F_R(X)$ is compact. The more interesting case is the case where R contains possibly many unbounded functions, i. e. where R contains the inverse of each member of R whose each value is different from 0. The answer to our problem in this case is given in Theorem 2.

I. Some properties of the mapping F_R. In this section X denotes a fixed completely regular space; R denotes a fixed linear ring of real-valued continuous functions defined on X which satisfies the conditions 1° and 2°; $\overline{F_R(X)}$ denotes the closure of $F_R(X)$ with respect to the Tihonov cube I^m.

(i) *If f is a bounded function in R, then there is a continuous real-valued function h defined on $\overline{F_R(X)}$, such that $f(p) = h(F_R(p))$ for each p in X.*

Let $f^*(p) = \alpha f(p) + \beta$, where $\alpha \neq 0$ and β are real numbers, chosen in such a way that $0 \leqslant f^*(p) \leqslant 1$ for each p in X. Then $f^* \in R^*$. Let

$$h(a) = \frac{1}{\alpha}(p_{f^*}(a) - \beta),$$

for a in $\overline{F_R(X)}$, where $p_{f^*}(a)$ denotes the f^*th coordinate of a. Then h is the required function.

(ii) *If h is a continuous real-valued function defined on $\overline{F_R(X)}$, then the function f defined on X by the equality $f(p) = h(F_R(p))$ belongs to R.*

Let $C = \{p_f\}_{f \in R^*}$ be the family of all coordinate functions of points in $\overline{F_R(X)}$. Since $\overline{F_R(X)}$ is compact and C distinguishes points of $\overline{F_R(X)}$, by the Stone-Weierstrass approximation theorem for each positive ε there exists a polynomial $W(t_1, \ldots, t_k)$ of real variables $t_1, \ldots, t_k$ and members $f_1, \ldots, f_k$ of R^* such that

$$\left|h(a) - W\big(p_{f_1}(a), \ldots, p_{f_k}(a)\big)\right| < \varepsilon$$

for each a in $\overline{F_R(X)}$.

By the definition of F_R and f, we obtain

$$\left|f(p) - W\big(f_1(p), \ldots, f_k(p)\big)\right| < \varepsilon$$

for each p in X. Since $W(f_1, \ldots, f_k) \in R$ and R is closed with respect to the uniform convergence, $f \in R$.

II. Theorem 1. *If R is a linear ring of bounded real-valued continuous functions on a completely regular space X satisfying the conditions 1° and 2°, then each non-trivial linear multiplicative functional φ defined on R is of the form $(*)$ if and only if $F_R(X)$ is compact.*

Proof. Suppose that $F_R(X)$ is compact and let φ be any non-trivial linear multiplicative functional defined on R. Denote by R_1 the ring of all continuous real-valued functions defined on $F_R(X)$. Let h be any member of R_1 and let us set $\varphi_1(h) = \varphi(f)$, where f is a function in R satisfying the equality $f(p) = h(F_R(p))$ (by (ii), $f \in R$). Then φ_1 is a non-trivial linear multiplicative functional on R_1, whence, by the Stone theorem, there is a point $a_0 \in F_R(X)$ such that $\varphi_1(h) = h(a_0)$ for each h in R_1. Let p_0 be any point of X with $F_R(p_0) = a_0$. If f is any member of R, then there is an $h \in R_1$ such that $f(p) = h(F_R(p))$ for each p in X. We have

$$\varphi(f) = \varphi_1(h) = h(a_0) = h(F_R(p_0)) = f(p_0),$$

whence φ is of the form $(*)$.

Conversely, suppose that $F_R(X)$ is not compact. Then $\overline{F_R(X)} \neq F_R(X)$; let a_0 be any point of $\overline{F_R(X)} \setminus F_R(X)$. Let us set $\varphi(f) = h(a_0)$ for any f in R, where h is a continuous function on $\overline{F_R(X)}$ such that $f(p) = h(F_R(p))$ for each p in X. Then φ is a non-trivial linear multiplicative functional on R. If p_1 is any point of X, then there is a continuous function h on $\overline{F_R(X)}$ with $h(a_0) = 0$ and $h(a_1) = 1$, where $a_1 = F_R(p_1)$. By (ii), there is a member f in R such that $f(p) = h(F_R(p))$ for each p in X. We have $\varphi(f) = 0$ and $f(p_1) = 1$ and it follows that φ is not of the form $(*)$.

III. In the sequel the following definition is needed: a subset P of a topological space S is said to be *Q-closed* (in S) provided that for each point $p \in S \setminus P$ there is a G_δ-set which contains p and is disjoint from P [2].

If S is a completely regular space, then we have the following:

(iii) *A set $P \subset S$ is Q-closed in S if and only if for each p in $S \setminus P$ there is a continuous real-valued function f on S such that $f(p) = 0$ and $f(q) \neq 0$ for each q in P.*

The simple proof of (iii) can be left to the reader.

We are interested in the case where a space is Q-closed in a certain compactification of itself.

(iv) *If $b'S$ and $b''S$ are compactifications of a completely regular space S, $b'S \leqslant b''S$ (i. e. there is a continuous mapping F of $b''S$ onto $b'S$ such that $F(p) = p$ for each p in S) and S is Q-closed in $b'S$, then S is Q-closed in $b''S$.*

Indeed, if $p \in b''S \setminus S$, then $q = F(p) \in b'S \setminus S$ [3], whence there is a G_δ-set $G \subset b'S$ which contains q and is disjoint from S. Then $F^{-1}(G)$ is a G_δ-set in $b''S$ which contains p and is disjoint from S.

[2] This definition was introduced in [3].

[3] This can be proved in the following way: suppose $q = F(p) \in S$. Then p and q are distinct points of $b''S$, whence there exists a neighbourhood U of p with $q \notin \overline{S \cap U}$ (the bar indicates the closure with respect to S). Since S is dense in $b''S$, $p \in \overline{S \cap U}$ (the bar indicates the closure with respect to $b''S$). On the other hand, $F(S \cap U) = S \cap U$, and since F is continuous, $q = F(p) \in \overline{F(S \cap U)} = \overline{S \cap U}$ (the bar indicates the closure with respect to S) and this leads to a constradiction.

The extreme case is explained by the following:

(v) *A space S is Q-closed in βS if and only if S is a Q-space.*

This statement is given in [3].

(vi) *A space S is Q-closed in each of its compactifications if and only if S is a Lindelöf space (i. e. each open covering of S contains a countable subcovering).*

Suppose that S is a Lindelöf space. Let bS be a compactification of S and let $p_0 \in bS \setminus S$. For each p in S there is a neighbourhood U_p of p such that $p_0 \notin \overline{U}_p$. Since $\{U_p\}_{p \in S}$ is an open covering of S, there is a countable covering $U_{p_1}, U_{p_2}, \ldots$ of S. Then $G = \bigcap_n \{bS \setminus \overline{U}_{p_n}\}$ is a G_δ-set in bS which contains p_0 and is disjoint from S.

Conversely, suppose that S is not a Lindelöf space. Then there is a family $U = \{U_\alpha\}_{\alpha \in A}$ of open subsets of βS which covers S and such that no countable subfamily of U covers S. Let $H = \beta S \setminus \bigcup \{U_\alpha\colon \alpha \in A\}$ and let bS be the compactification of S which is obtained from βS by the identification of all points of H to a single point; denote this point by p_0. Suppose that there is a G_δ-set $G \subset bS$ which contains p_0 and is disjoint from S. Let $F = bS \setminus G$. Then $S \subset F \subset \bigcup \{U_\alpha\colon \alpha \in A\}$. But F, being an F_σ-set in a compact space, can be covered by a countable infinity of sets U_α. This leads to a contradiction, whence S is not Q-closed in bS.

An immediate consequence of (vi) is the following:

(vii) *A locally compact space is Q-closed in its minimal one-point compactification if and only if it is a Lindelöf space.*

IV. In this section we shall consider the case of rings containing unbounded functions. We assume the following condition:

3° *If $f \in R$ and $f(p) \neq 0$ for each p in X, then $1/f \in R$.*

We shall prove some elementary properties of such rings (in (viii)-(xii) R is a fixed linear ring of continuous functions on a fixed space X satisfying the conditions 1°-3°).

(viii) *If $f \in R$, then $|f| \in R$.*

At first, suppose that f is a bounded function. Then one can assume without lost of generality that $|f(p)| < \frac{1}{2}$ for each p in X. We have $|f| = \sqrt{1-(1-f^2)}$, whence $|f|$ can be written as the sum of a uniformly convergent series of polynomials with respect to f. Thus, by 2°, $|f| \in R$.

Now, suppose that f is an arbitrary function in R. Let

$$f_1 = \frac{f}{1+f^2}.$$

Then f_1 is a bounded function, and, by 3°, $f_1 \in R$. Consequently $|f_1| \in R$ and $|f| = |f_1| \cdot (1+f^2) \in R$.

(ix) *If $f, g \in R$, then $\max\{f, g\} \in R$ and $\min\{f, g\} \in R$.*

This follows from (viii) and the formulas:

$$\max\{f, g\} = \frac{f+g+|f-g|}{2}; \quad \min\{f, g\} = \frac{f+g-|f-g|}{2}.$$

(x) *Each member f of R can be written as the difference of two non-negative members of R.*

In fact, $f = f^+ - f^-$, where $f^+ = \max\{f, 0\}$, $f^- = -\min\{f, 0\}$.

(xi) *Each member f of R can be written in the form $f = 1/f_1 - 1/f_2$, where f_1 and f_2 are bounded positive functions in R.*

Indeed, let

$$f_1 = \frac{1}{f^+ + 1}; \quad f_2 = \frac{1}{f^- + 1},$$

where f^+ and f^- have the same meaning as in the proof of (x).

(xii) *For each f in R there is a continuous function h defined on $F_R(X)$ such that $f(p) = h(F_R(p))$ for each p in X.*

In virtue of (x), one can assume that f is a non-negative function. Then, by 3°, $g = 1/(f+1) \in R^*$ (see the definition of the mapping F_R given at the beginning of this paper). Let

$$h(a) = \frac{1-p_g(a)}{p_g(a)},$$

where $p_g(a)$ denotes the gth coordinate of a point $a \in F_R(X)$. Then h is the required function (this function is well-defined on $F_R(X)$, since the gth coordinate of the point $a \in F_R(X)$ lies in the interval $0 < t \leqslant 1$).

THEOREM 2. *If R is a linear ring of continuous real-valued functions defined on a topological space X which satisfies the conditions 1°-3°, then each non-trivial linear multiplicative functional φ defined on R is of the form (*) if and only if $F_R(X)$ is Q-closed in $\overline{F_R(X)}$.*

Proof. Suppose that $F_R(X)$ is Q-closed in $\overline{F_R(X)}$ and let φ be a non-trivial linear multiplicative functional defined on X. Denote by R_1 the ring of all continuous functions defined on $\overline{F_R(X)}$. In virtue of (i) and (ii) a one-to-one correspondence can be established between bounded members of R and *all* members of R_1; corresponding functions f, h ($f \in R$, $h \in R_1$) satisfy the equality $f(p) = h(F_R(p))$ for each p in X. Let us set $\varphi_1(h) = \varphi(f)$. Then φ_1 is a non-trivial linear multiplicative functional defined on R_1. Since $\overline{F_R(X)}$ is compact, there is a point $a_0 \in \overline{F_R(X)}$ such that $\varphi_1(h) = h(a_0)$ for each h in R_1. We shall show that $a_0 \in F_R(X)$.

Indeed, if $a_0 \in \overline{F_R(X)} \setminus F_R(X)$, then, by (iii), there is a continuous function h defined on $\overline{F_R(X)}$ such that $h(a_0) = 0$ and $h(a) \neq 0$ for each a in $F_R(X)$. Let f be the function in R which corresponds to h. Then $f(p) \neq 0$ for each p in X, whence, by 3°, $1/f \in R$, and it follows that $\varphi(f) \neq 0$. On the other hand, $\varphi(f) = \varphi_1(h) = h(a_0) = 0$ and this leads to a contradiction.

Now, let p_0 be any point in X with $F_R(p_0) = a_0$. Using (i) and the definition of the functional φ_1, one can easily show that $\varphi(f) = f(p_0)$ for each bounded function f in R. Using (xi) we infer that the above equality holds true for each function f in R. Thus the first part of our theorem is proved.

Conversely, suppose that $F_R(X)$ is not Q-closed in $\overline{F_R(X)}$. Then, by (iii), there is a point $a_0 \in \overline{F_R(X)} \setminus F_R(X)$ such that for each continuous function h defined on $\overline{F_R(X)}$ which is strictly positive on $F_R(X)$ we have $h(a_0) > 0$. Let f be any member of R. By (xii), there is a continuous function h defined on $F_R(X)$ such that $f(p) = h(F_R(p))$ for each p in X. We shall show that h admits a continuous extension over $F_R(X) \cup \{a_0\}$. In fact, let

$$f = \frac{1}{f_1} - \frac{1}{f_2},$$

where f_1, f_2 are bounded positive functions in R. By (i), there are continuous functions h_1, h_2 defined on whole $\overline{F_R(X)}$ such that $f_i(p) = h_i(F_R(p))$ for each p in X $(i = 1, 2)$. Since h_i $(i = 1, 2)$ are strictly positive on $F_R(X)$, $h_i(a_0) > 0$ $(i = 1, 2)$ and it follows that the function $\frac{1}{h_1} - \frac{1}{h_2}$ is continuous on $F_R(X) \cup \{a_0\}$ and clearly it is an extension of h.

Let us set $\varphi(f) = h^*(a_0)$, where h^* is the continuous extension of h over $F_R(X) \cup \{a_0\}$. Then φ is a non-trivial linear multiplicative functional defined on R. Using (iii), it is easy to show that φ is not of the form $(*)$.

V. Consequences of theorem 2. THEOREM 3. *If X is a Lindelöf space and R is any linear ring of continuous functions defined on X satisfying the conditions* 1°-3°, *then each non-trivial linear multiplicative functional φ defined on R is of the form $(*)$.*

Conversely, if X is not a Lindelöf space, then there is a linear ring R of continuous functions on X satisfying the conditions 1°-3° *and a non-trivial linear multiplicative functional φ defined on R which is not of the form $(*)$.*

Proof. If X is a Lindelöf space, then each continuous image of X is a Lindelöf space and a Lindelöf space is Q-closed in each of its compactifications (clearly $\overline{F_R(X)}$ is a compactification on $F_R(X)$).

Conversely, if X is not a Lindelöf space, then there is a compactification bX of X such that X is not Q-closed in bX. Let R be the least ring satisfying the conditions 1°-3° that contains all functions on X which admit a continuous extension over bX. Then F_R is a homeomorphism and F_R can be extended to a continuous mapping of bX onto $\overline{F_R(X)}$. It follows that $F_R(X)$ is not Q-closed in $\overline{F_R(X)}$.

THEOREM 4. *If R_1, R_2 are linear rings of continuous functions on X satisfying the conditions* 1°-3°, *R_1 distinguishes points and closed sets* [4] *and $R_1 \subset R_2$, then if each non-trivial linear multiplicative functional φ on R_1 is of the form $(*)$, the same holds true for the ring R_2.*

Proof. The mappings F_{R_1} and F_{R_2} are homeomorphisms, whence $\overline{F_{R_1}(X)}$ and $\overline{F_{R_2}(X)}$ can be regarded as compactifications of X. One can easily verify that $\overline{F_{R_1}(X)} \leqslant \overline{F_{R_2}(X)}$, whence the statement of the theorem follows directly from Theorem 2 and (iv).

References

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[4] I. e. for each closed set $A \subset X$ and each point $p \in X \setminus A$ there is a f in R with $f(p) \notin \overline{f(A)}$.

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