$\mathcal{N}$-lattices and constructive logic with strong negation *

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The subject of this paper is the algebraic method of the examination of the system $\mathcal{S}$ of the constructive propositional calculus with strong negation (cf. Nelson [3], Markov [2], Vorob'ev [5], [6]).

The system $\mathcal{S}$ determines a type of abstract algebras which will be called $\mathcal{N}$-lattices. They are some kind of distributive lattices. The purpose of this paper is to give a topological representation of $\mathcal{N}$-lattices. For this aim we shall apply a method similar to that of Stone [4].

Some applications of this theorem will be given in a separate paper.

This paper contains also at the end of § 3 some remarks of A. Białynicki-Birula, which simplify the formulation of the representation's theorem for $\mathcal{N}$-lattices.

§ 1. A constructive propositional calculus with strong negation. The system $\mathcal{S}$ of the constructive propositional calculus with strong negation can be briefly described as follows (see Vorob'ev [5]).

The symbols of the system $\mathcal{S}$ consist of infinitely many propositional variables $p_1, p_2, \ldots$ of the constants and the parentheses. There are the following constants: the disjunction sign $+$, the conjunction sign $\cdot$, the sign of strong negation $\sim$, the implication sign $\rightarrow$, and the sign of intuitionistic negation $\neg$.

The class of formulas is the smallest class of expressions of this system, which contains all the variables and satisfies the following conditions:

(i) if α is a formula, then so are $(\sim\alpha)$ and $(\neg\alpha)$;

(ii) if α and β are arbitrary formulas, then so are $(\alpha+\beta)$, $(\alpha\cdot\beta)$, $(\alpha\rightarrow\beta)$.

We introduce for convenience the following abbreviations:

$$(\alpha\leftrightarrow\beta) \quad \text{for} \quad \big((\alpha\rightarrow\beta)\cdot(\beta\rightarrow\alpha)\big),$$

$$(\alpha\equiv\beta) \quad \text{for} \quad \big((\alpha\leftrightarrow\beta)\cdot\big((\sim\alpha)\leftrightarrow(\sim\beta)\big)\big).$$

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A formula is called an *axiom* if it is of one of the following kinds (where α, β, γ are arbitrary formulas):

A1 $(\alpha\to(\beta\to\alpha))$,

A2 $((\alpha\to(\beta\to\gamma))\to((\alpha\to\beta)\to(\alpha\to\gamma)))$,

A3 $((\alpha\cdot\beta)\to\alpha)$,

A4 $((\alpha\cdot\beta)\to\beta)$,

A5 $((\gamma\to\alpha)\to((\gamma\to\beta)\to(\gamma\to(\alpha\cdot\beta))))$,

A6 $(\alpha\to(\alpha+\beta))$,

A7 $(\beta\to(\alpha+\beta))$,

A8 $((\alpha\to\gamma)\to((\beta\to\gamma)\to((\alpha+\beta)\to\gamma)))$,

A9 $((\alpha\to(\neg\beta))\to(\beta\to(\neg\alpha)))$,

A10 $((\neg\alpha)\to(\alpha\to\beta))$,

A11 $((\sim\alpha)\to(\alpha\to\beta))$,

A12 $((\sim(\alpha\to\beta))\leftrightarrow(\alpha\cdot(\sim\beta)))$,

A13 $((\sim(\alpha\cdot\beta))\leftrightarrow((\sim\alpha)+(\sim\beta)))$,

A14 $((\sim(\alpha+\beta))\leftrightarrow((\sim\alpha)\cdot(\sim\beta)))$,

A15 $((\sim(\neg\alpha))\leftrightarrow\alpha)$,

A16 $((\sim(\sim\alpha))\leftrightarrow\alpha)$.

All the formulas of the form A1-A10, where α, β, γ contain no sign of strong negation, constitute the system of axioms for the intuitionistic propositional calculus.

A formula α is said to be *provable* if α belongs to the smallest set of formulas which includes all the axioms and is closed under the operation of detachment. We shall then write $\vdash \alpha$.

It is known (cf. Vorob'ev [5]) that the provability of a formula $(\alpha\leftrightarrow\beta)$ does not generally imply the provability of the formula $((\sim\alpha)\leftrightarrow(\sim\beta))$.

It is also known or very simple to demonstrate that the following formulas are provable (α, β, γ being arbitrary formulas):

P1 $((\neg\alpha)\equiv(\alpha\to(\sim\alpha)))$,

P2 $((\alpha\equiv\beta)\to(\gamma(\alpha)\equiv\gamma(\beta)))$,

where $\gamma(\alpha)$ is an arbitrary formula containing a formula α as its part and $\gamma(\beta)$ is obtained from $\gamma(\alpha)$ by replacing a part α by β.

P3 $(\alpha\to\alpha)$,

P4 $((\alpha\to\beta)\to((\beta\to\gamma)\to(\alpha\to\gamma)))$,

P5 $((\alpha\to(\beta\to\gamma))\leftrightarrow((\alpha\cdot\beta)\to\gamma))$,

P6 $((\alpha\to(\beta\to\gamma))\to(\beta\to(\alpha\to\gamma)))$,

P7 $((\alpha\cdot(\beta+\gamma))\leftrightarrow((\alpha\cdot\beta)+(\alpha\cdot\gamma)))$,

P8 $((\alpha+(\beta\cdot\gamma))\leftrightarrow((\alpha+\beta)\cdot(\alpha+\gamma)))$,

P9 $((\alpha\to\beta)\to((\neg\beta)\to(\neg\alpha)))$,

P10 $(\neg(\alpha\cdot(\neg\alpha)))$,

P11 $((\alpha\to(\neg\beta))\to((\alpha\to\beta)\to(\neg\alpha)))$,

P12 $(((\neg\alpha)+\beta)\to(\alpha\to\beta))$,

P13 $((\alpha+\beta)\to((\neg\alpha)\to\beta))$,

P14 $((\alpha+\beta)\to((\alpha\to\gamma)\to(\beta+\gamma)))$,

P15 $((\sim(\sim\alpha))\equiv\alpha)$,

P16 $((\sim(\alpha+\beta))\equiv((\sim\alpha)\cdot(\sim\beta)))$,

P17 $((\sim(\alpha\to(\sim(\beta\to\beta))))\to(\sim(\neg\alpha)))$ (A12, A3, A15, P4),

P18 $((\alpha\cdot(\sim\alpha))\to\beta)$,

P19 $((\alpha\to(\sim(\beta\to\beta)))\to(\neg\alpha))$ (A12, P18, P4, P9, P10, A1, P11),

P20 $((\sim(\neg\alpha))\to(\sim(\alpha\to(\sim(\beta\to\beta)))))$ (A12, P15, P2, P5, P3, A15, P4),

P21 $((\alpha+\beta)\to((\sim\beta)\to(\sim(\neg\alpha))))$ (A15, A1, P6, A11, P6, A8),

P22 $((\alpha+\beta)\to((\sim(\beta+\gamma))\to(\sim(\alpha\to\gamma))))$
(A12, P5, A1, P6, P5, P16, P2, A11, P6, A1, P6, P5, P16, P2, A8),

P23 $(((\beta\cdot\alpha)\to(\neg\alpha))\to(\beta\to(\neg\alpha)))$.

§ 2. Algebraical characterization of the system $\mathcal{S}$. In this section we shall use the familiar notion of a matrix. The matrices considered are abstract algebras $\mathfrak{M} = \langle A, e, +, \cdot, \sim, \to, \neg\rangle$, where A is the set of elements of $\mathfrak{M}$, e is a distinguished element of A, $+, \cdot, \to$ are binary operations defined over A, and $\sim$, $\neg$ are unary operations defined over A.

The matrix $\mathfrak{M} = \langle A, e, +, \cdot, \sim, \to, \neg\rangle$ is said to be *regular* if it satisfies the following conditions:

(i) if $e\to x = e$, then $x = e$ for every $x \in A$,

(ii) if $a\to b = e$, $b\to a = e$, $\sim a\to\sim b = e$ and $\sim b\to\sim a = e$, then $a = b$.

Given a formula α of $\mathcal{S}$, we can interpret it as a polynomial $\alpha_{\mathfrak{M}}$ of a matrix $\mathfrak{M} = \langle A, e, +, \cdot, \sim, \to, \neg \rangle$ by conceiving the propositional variables occuring in α as variables running over the set A, and the logical operations as the respective algebraical operations in $\mathfrak{M}$.

A formula α is said to be *satisfied* by a matrix $\mathfrak{M}$ if $\alpha_{\mathfrak{M}} = e$ identically. A regular matrix $\mathfrak{M}$ is called a *matrix of the system* $\mathcal{S}$ if it satisfies all the axioms of this system. A regular matrix $\mathfrak{M}$ is called a *characteristic matrix of* $\mathcal{S}$, if for every formula α of S the provability of α is equivalent to the satisfiability of α in $\mathfrak{M}$.

It is easy to see that

2.1. *If α is provable, then α is satisfied in every regular matrix $\mathfrak{M}$ of the system $\mathcal{S}$.*

We shall say that an abstract algebra $\langle A, +, \cdot, \sim \rangle$ is a *quasi-Boolean algebra* (cf. Białynicki and Rasiowa [1]) when:

(a) $\langle A, +, \cdot \rangle$ is a distributive lattice with the zero element 0 and the unit element e,

(b) $\sim$ is a unary operation which satisfies the following conditions

$$\sim\sim a = a, \quad \sim(a+b) = \sim a \cdot \sim b \quad \text{for any} \quad a, b \in A.$$

The operation $\sim$ is called the *operation of quasi-complement*. In any quasi-Boolean algebra

$$\sim(a \cdot b) = \sim a + \sim b \quad \text{for any } a, b \in A, \quad \sim 0 = e \quad \text{and} \quad \sim e = 0.$$

2.2. *If $\mathfrak{M} = \langle A, e, +, \cdot, \sim, \to, \neg \rangle$ is a regular matrix of the system $\mathcal{S}$, then the following conditions are satisfied:*

(1) *the set A is quasi-ordered by the relation $\prec$ defined as follows*

$$a \prec b \text{ if and only if } a \to b = e \text{ for any } a, b \in A;$$

(2) *the abstract algebra $\langle A, +, \cdot, \sim \rangle$ is a quasi-Boolean algebra with the unit element e; moreover, the relation $\subset$ defined by the equivalence*

$$a \subset b \text{ if and only if } a \prec b \text{ and } \sim b \prec \sim a \text{ for any } a, b \in A,$$

is the partly ordering relation of this lattice;

(3) $a \prec c$ *and* $b \prec c$ *imply* $a + b \prec c$;

(4) $c \prec a$ *and* $c \prec b$ *imply* $c \prec a \cdot b$;

(5) $\sim(a \to b) \prec (a \cdot \sim b)$;

(6) $(a \cdot \sim b) \prec \sim(a \to b)$;

(7) $a \prec \sim\neg a$;

(8) $\sim\neg a \prec a$;

(9) $a \cdot \sim a \prec b$;

(10) $a \prec b \to c$ *if and only if* $a \cdot b \prec c$;

(11) $\neg a = a \to \sim e$.

The conditions (3)-(11) *hold for arbitrary elements a, b, c of A.*

To prove this theorem, let us suppose that $\mathfrak{M} = \langle A, e, +, \cdot, \sim, \to, \neg \rangle$ is a regular matrix of the system $\mathcal{S}$.

On account of P3 (§ 1) and the definition of the regular matrix of the system $\mathcal{S}$, we have $a \to a = e$ for every $a \in A$. Hence $\prec$ is a reflexive relation. Let us suppose that $a \prec b$ and $b \prec c$. Hence $a \to b = e$ and $b \to c = e$. By P4 $(a \to b) \to ((b \to c) \to (a \to c)) = e$. Consequently, by (i) $a \to c = e$. Then the relation $\prec$ is a transitive one. Thus (1) holds.

The conditions (3), (4), (5), (6), (7), (8), (9), (10) follow immediately from A8, A5, A12, A3, A4, A15, P18, P5 and the definition of the regular matrix of the system $\mathcal{S}$.

Obviously the conditions (1), (3), (4) and A6, A7, A3, A4 imply that if $a \prec b$ and $c \prec d$, then $a + c \prec b + d$ and $a \cdot c \prec b \cdot d$ for any $a, b, c, d \in A$.

On account of (1) we easily infer that the relation $\subset$ is reflexive and transitive. By (ii) it is antisymmetric. Hence the set A is partly ordered by the relation $\subset$.

It easily follows from A16 that

(12) $$a = \sim\sim a \quad \textit{for every} \quad a \in A.$$

Moreover,

(13) $$a \subset b \quad \textit{if and only if} \quad \sim b \subset \sim a.$$

In fact, this follows immediately from the definition of the relation $\subset$ and (12).

We shall prove that

(14) $$\sim(a \cdot b) = \sim a + \sim b \quad \textit{for any} \quad a, b \in A.$$

Indeed, by A13 and A3, A4

$$\sim(a \cdot b) \prec \sim a + \sim b \quad \text{and} \quad \sim a + \sim b \prec \sim(a \cdot b).$$

Using A14 and (12) we obtain

$$\sim(\sim a + \sim b) \prec \sim\sim(a \cdot b) \quad \text{and} \quad \sim\sim(a \cdot b) \prec \sim(\sim a + \sim b).$$

In consequence, we obtain (14).

It follows immediately from (12) and (14) that

(15) $$\sim(a + b) = \sim a \cdot \sim b.$$

Now we shall prove that $a \cdot b$ is the meet of a and b, the relation $\subset$ being partly ordering.

By A3 and A4 we obtain $a\cdot b \prec a$ and $a\cdot b\prec b$. On the other hand, using A6, A7, and (14) we infer that $\sim a \prec \sim(a\cdot b)$ and $\sim b\prec\sim(a\cdot b)$. Consequently

$$a\cdot b \subset a \quad\text{and}\quad a\cdot b\subset b\,. \tag{16}$$

It remains to prove that if $c\subset a$ and $c\subset b$, then $c\subset a\cdot b$. Indeed, on account of (4) we have $c\prec a\cdot b$. On the other hand $\sim a\prec\sim c$ and $\sim b\prec\sim c$ implies by (3) that $\sim a+\sim b\prec\sim c$. Hence by (14) $\sim(a\cdot b)\prec\sim c$ and consequently $c\subset a\cdot b$.

Now we shall show that $a+b$ is the join of the elements a and b. By A6 and A7 $a\prec a+b$ and $b\prec a+b$. Moreover, using A3, A4 and (15) we infer that $\sim(a+b)\prec\sim a$ and $\sim(a+b)\prec\sim b$. Consequently

$$a\subset a+b \quad\text{and}\quad b\subset a+b\,. \tag{17}$$

Suppose $a\subset c$ and $b\subset c$. By (3) $a+b\prec c$. It follows from (4) that $\sim c\prec\sim a\cdot\sim b$. Hence by (15) $\sim c\prec\sim(a+b)$. Thus if $a\subset c$ and $b\subset c$, then $a+b\subset c$. In consequence $\langle A, +, \cdot\rangle$ is a lattice, the relation $\subset$ being the partly ordering one of this lattice.

We shall show that e is the unit element of this lattice. On account of A1 we have $e\to(a\to e)=e$ for every $a\in A$. Hence $a\prec e$. By P18 $b\cdot\sim b\prec\sim a$ for any $a, b\in A$. Hence, by (1) and (5) $\sim(b\to b)\prec\sim a$. Since $b\to b=e$, we obtain $\sim e\prec\sim a$. Thus $a\subset e$ for every $a\in A$.

Since for every $a\in A$, $\sim a\subset e$, we infer by (13) and (12) that $\sim e\subset a$ for every $a\in A$. Hence the element $0=\sim e$ is the zero element of the lattice $\langle A, +, \cdot\rangle$.

Now we shall prove that the lattice $\langle A, +, \cdot\rangle$ is a distributive one. For this purpose it suffices to show that

$$a\cdot(b+c)=a\cdot b+a\cdot c \quad\text{for any}\quad a, b, c\in A\,.$$

It follows from P7, P8, A3, A4 that

$$a\cdot(b+c)\prec a\cdot b+a\cdot c\,, \tag{18}$$

$$a\cdot b+a\cdot c\prec a\cdot(b+c)\,, \tag{19}$$

$$a+b\cdot c\prec(a+b)\cdot(a+c)\,, \tag{20}$$

$$(a+b)\cdot(a+c)\prec a+b\cdot c\,. \tag{21}$$

On the other hand, by (15), (14) we obtain

$$\sim(a\cdot b+a\cdot c)\prec(\sim a+\sim b)\cdot(\sim a+\sim c)\,.$$

Hence by (21) and (1)

$$\sim(a\cdot b+a\cdot c)\prec\sim a+\sim b\cdot\sim c\,.$$

Then by (15), (14)

$$\sim(a\cdot b+a\cdot c)\prec\sim\big(a\cdot(b+c)\big)\,.$$

Hence, by (18) we obtain

$$a\cdot(b+c)\subset a\cdot b+a\cdot c\,. \tag{22}$$

It follows from (14) and (15) that

$$\sim\big(a\cdot(b+c)\big)\prec\sim a+(\sim b\cdot\sim c)\,.$$

Using (20) we obtain

$$\sim\big(a\cdot(b+c)\big)\prec(\sim a+\sim b)\cdot(\sim a+\sim c)\,.$$

Hence by (14) and (15)

$$\sim\big(a\cdot(b+c)\big)\prec\sim(a\cdot b+a\cdot c)\,.$$

In consequence by (19) we infer that

$$a\cdot b+a\cdot c\subset a\cdot(b+c)\,. \tag{23}$$

It follows from (22) and (23) that $a\cdot(b+c)=a\cdot b+a\cdot c$. This completes the proof of (2).

It remains to show (11).

On account of A10, P3, (2) and P17 we have

$$\neg a\prec a\to 0 \quad\text{and}\quad \sim(a\to 0)\prec\sim\neg a\,.$$

Hence $\neg a\subset a\to 0$. On the other hand by P19, P20 and (2) we obtain

$$a\to 0\prec\neg a \quad\text{and}\quad \sim\neg a\prec\sim(a\to 0)\,.$$

Hence $a\to 0\subset\neg a$ and consequently $a\to 0=\neg a$.

2.3. *If the conditions* (1)-(11) *of the theorem* 2.2 *hold for a matrix* $\mathfrak{M}=\langle A, e, +, \cdot, \to, \neg\rangle$, *then it is a regular matrix of the system* $\mathcal{S}$.

Let us suppose that the conditions (1)-(11) are satisfied for $\mathfrak{M}$. First of all we shall show that $\mathfrak{M}$ is a regular matrix. To prove (i) suppose that for an element x of A, $e\to x=e$. Since e is the unit element of the lattice $\langle A, +, \cdot\rangle$ we have for every $a\in A$, $a\subset e$. Consequently $a\subset e\to x$. Thus $a\prec e\to x$ and $\sim(e\to x)\prec\sim a$. Hence, by (10) $a\cdot e\prec x$ and by (6) $e\cdot\sim x\prec\sim a$. In consequence $a\subset x$. Thus (i) is fulfilled. The condition (ii) follows immediately from (1) and (2). Hence $\mathfrak{M}$ is a regular matrix.

Now we shall prove that every axiom of $\mathcal{S}$ is satisfied in $\mathfrak{M}$.

In fact the axioms A3 and A4 are satisfied since $a\cdot b\subset a$ and $a\cdot b\subset b$ for any $a, b\in A$, and consequently

$$(*)\qquad a\cdot b\preccurlyeq a \quad\text{and}\quad a\cdot b\preccurlyeq b.$$

On account of (10) we obtain $a\preccurlyeq b\to a$. Hence the axiom A1 is satisfied. It follows from $(*)$ and (4) that

$$a\preccurlyeq c \text{ and } b\preccurlyeq d \text{ imply that } a\cdot b\preccurlyeq c\cdot d.$$

The reflexivity of the relation $\preccurlyeq$ and (10) imply that

$$(**)\qquad a\cdot(a\to b)\preccurlyeq b \quad\text{for any}\quad a, b\in A.$$

It is easy to see that on account of $(*)$, $(**)$ and (4)

$$a\cdot(a\to b)\cdot\big(a\to(b\to c)\big)\preccurlyeq c.$$

Hence using (10) we obtain $\big(a\to(b\to c)\big)\preccurlyeq\big((a\to b)\to(a\to c)\big)$. Thus the axiom A2 is satisfied.

It easily follows from $(*)$, $(**)$ and (4) that $c\cdot(c\to a)\cdot(c\to b)\preccurlyeq a\cdot b$. Using (10) we infer that $(c\to a)\preccurlyeq\big((c\to b)\to(c\to a\cdot b)\big)$. Hence the axiom A5 is satisfied.

Obviously $a\subset a+b$ and $b\subset a+b$ for any $a, b\in A$. Consequently $a\preccurlyeq a+b$ and $b\preccurlyeq a+b$. Thus the axioms A6 and A7 are satisfied.

It is easy to show making use of $(**)$, (10) and (3) that $a+b\preccurlyeq\big((a\to c)\to\big((b\to c)\to c\big)\big)$. Hence by (10) $(a\to c)\preccurlyeq\big((b\to c)\to(a+b\to c)\big)$. Thus the axiom A8 is satisfied.

We easily prove using $(**)$ and $(*)$ that $a\cdot b\cdot\big(a\to(b\to 0)\big)\preccurlyeq 0$. Hence by (10) $b\cdot\big(a\to(b\to 0)\big)\preccurlyeq a\to 0$. By (11) we obtain $b\cdot(a\to\neg b)\preccurlyeq\neg a$. Hence using (10) we have $(a\to\neg b)\preccurlyeq(b\to\neg a)$. Consequently the axiom A9 is satisfied.

It follows from $(**)$ that $a\cdot(a\to 0)\preccurlyeq b$ for any $a, b\in A$. Hence by (11) and (10) $\neg a\preccurlyeq a\to b$. Thus the axiom A10 is satisfied. Using (9) and (10) we infer that the axiom A11 is satisfied. From (5) and (6) we find that the axiom A12 is satisfied. By (2) we infer that the axioms A13, A14 and A16 are satisfied. Using (7) and (8) we infer that the axiom A15 is satisfied. This completes the proof of 2.3.

We shall prove that

2.4. *The following conditions are fulfilled in every regular matrix* $\mathfrak{M}=\langle A, e, +, \cdot, \sim, \to, \neg\rangle$ *of the system* $\mathcal{S}$:

(a) $\sim a\subset\neg a$,

(b) $a\cdot\sim a=a\cdot\neg a$,

(c) $\neg a+b\subset a\to b$,

(d) $\sim a+b\subset a\to b$,

(e) *if* $a\cdot b=0$ *then* $a\subset\sim b$,

(f) *if* $a+b=e$ *then* $\neg a\subset b$,

(g) *if* $a+c=e$ *then* $a\to b\subset c+b$,

(h) $\neg a=a\to\sim a$.

Obviously (h) follows from P1. It follows from A1 and (h) that $\sim a\preccurlyeq\neg a$. By (8) and (12) we have $\sim\neg a\preccurlyeq\sim\sim a$. Hence (a) holds. On account of (a) we obtain $a\cdot\sim a\subset a\cdot\neg a$. On the other hand $a\cdot\neg a\subset a$. By A10 and (10) we have $a\cdot\neg a\preccurlyeq\sim a$. From (1), (7), (17) and (12) we obtain $\sim\sim a\preccurlyeq\sim(a\cdot\neg a)$. Hence $a\cdot\neg a\subset\sim a$. Consequently (b) holds. It follows from P12 that $\neg a+b\preccurlyeq a\to b$. By (5) $\sim(a\to b)\preccurlyeq a\cdot\sim b$. Making use of (7), and (15) we obtain $\sim(a\to b)\preccurlyeq\sim(\neg a+b)$. Hence (c) holds. The condition (d) follows from (a) and (c). To prove (e) let us suppose that $a\cdot b=0$. Hence $\sim a+\sim b=e$. But by (d), $\sim a+\sim b\subset a\to\sim b$. Thus $a\preccurlyeq\sim b$. Analogously we obtain $b\preccurlyeq\sim a$, i. e. $\sim\sim b\preccurlyeq\sim a$. Hence $a\subset\sim b$. To prove (f) let us suppose that $a+b=e$. It follows from A1, A10, P6 and (3) that $a+b\preccurlyeq\neg a\to b$. Hence $\neg a\to b=e$, and consequently $\neg a\preccurlyeq b$. By P21 we obtain $a+b\preccurlyeq\sim b\to\sim\neg a$. Thus $\sim b\to\sim\neg a=e$, and $\sim b\preccurlyeq\sim\neg a$. Hence $\neg a\subset b$. To prove (g) let us suppose that $a+c=e$. Then by P14 $a\to b\preccurlyeq c+b$ and by P22 $\sim(c+b)\preccurlyeq\sim(a\to b)$. Hence $a\to b\subset c+b$.

The well known method of Lindenbaum enables us to prove very easily the existence of a characteristic matrix of $\mathcal{S}$. Indeed, given arbitrary formulas α, β of $\mathcal{S}$ we shall write $\alpha\simeq\beta$ provided that $\vdash\alpha\equiv\beta$. The relation $\simeq$ is a congruence relation in the sense of modern algebra. In fact this follows from P3, P4 and P2. For every formula α of $\mathcal{S}$ let $|\alpha|$ denote the class of all formulas β of $\mathcal{S}$ such that $\alpha\simeq\beta$. Let A_0 be the set of all cosets $|\alpha|$ where α is arbitrary formula of $\mathcal{S}$. We define in A_0 the algebraical operations $+, \cdot, \sim, \to, \neg$ as follows: $|\alpha|\circ|\beta|=|\alpha\circ\beta|$ if $\circ$ is one of the binary logical operations of $\mathcal{S}$, and $\circ|\alpha|=|\circ\alpha|$ if $\circ$ is one of the unary operations of $\mathcal{S}$. It follows from A1, A11 and P6 that if α and β are arbitrary provable formulas then $\alpha\simeq\beta$. The element $|\alpha|$ where α is a provable formula will be denoted by e_0. It is easy to verify that

2.5. *The matrix* $\mathfrak{M}_0=\langle A_0, e_0, +, \cdot, \sim, \to, \neg\rangle$ *is a characteristic matrix of the system* $\mathcal{S}$.

It is easy to see, by making use of a similar method to that of Gödel for the intuitionistic sentential calculus [1] that there is no finite characteristic matrix of $\mathcal{S}$.

[1] See K. Gödel, *Zum intuitionistischen Aussagenkalküls*, Ak. der Wiss. in Wien, Math.-natur. Kl. Anzeiger 69 (1932), p. 65-66.

§ 3. $\mathcal{N}$-lattices and their representation. We shall say that an abstract algebra $\mathfrak{A} = \langle A, e, +, \cdot, \sim, \to, \neg\rangle$ is a *$\mathcal{N}$-lattice* when it fulfils the conditions (1)-(11) (§ 2). On account of 2.2 and 2.3 $\mathcal{N}$-lattices are regular matrices of $\mathcal{S}$ and conversely. The aim of this section is to give the representation theorem for these lattices.

A class $\boldsymbol{H}(\mathfrak{X}_1)$ of open subsets of a topological space $\mathfrak{X}_1$ is said to be a *Heyting algebra of sets* (cf. [1]) provided that it is closed under the set-theoretical operations of sum and product as well as under the operations of *pseudocodifference* $\to_1$ and *pseudocomplement* $\neg_1$ defined as follows

(24) $X \to_1 Y = \mathrm{Int}_1((\mathfrak{X}_1 - X) + Y), \quad \neg_1 X = \mathrm{Int}_1(\mathfrak{X}_1 - X) = X \to_1 A,$

for any $X, Y \in \boldsymbol{H}(\mathfrak{X}_1)$, where Int_1 is the operation of interior in the space $\mathfrak{X}_1$.

Now let $\mathfrak{X}$ be a non-empty set and let g be a one-to-one mapping of $\mathfrak{X}$ onto $\mathfrak{X}$ which is an involution, i. e., $g(g(x)) = x$ for every $x \in \mathfrak{X}$. Setting

(25) $$\sim X = \mathfrak{X} - g(X) \quad \text{for every} \quad X \subset \mathfrak{X}$$

we find that every family of subsets of the set $\mathfrak{X}$ which is closed under this operation as well as under the set-theoretical operations of sum and product is a quasi-Boolean algebra (see [1]). Every quasi-Boolean algebra of this kind is said to be a *quasi-field of sets*. Every quasi-Boolean algebra is isomorphic with a quasi-field of sets (cf. [1]). Moreover

3.1. *Every quasi-Boolean algebra is isomorphic with a quasi-field $\boldsymbol{B}(\mathfrak{X})$ of open subsets of a bicompact topological T_0-space $\mathfrak{X}$, which contains $\mathfrak{X}$.*

This remark easily follows from the proof of the theorem on the representation of quasi-Boolean algebra as quasi-fields of sets using the method of Stone [4]. In fact, let $\langle A, +, \cdot, \sim\rangle$ be a quasi-Boolean algebra. For every subset $A_0 \subset A$, let $\widetilde{A}_0$ be the set of all elements $\sim x$ such that $x \in A_0$. It is easy to see that if $\mathfrak{q}$ is a prime filter then $\widetilde{\mathfrak{q}}$ is a prime ideal. Let $\mathfrak{X}$ be the set of all prime filters of $\langle A, +, \cdot, \sim\rangle$. For every $\mathfrak{q} \in \mathfrak{X}$ let

(26) $$g(\mathfrak{q}) = A - \widetilde{\mathfrak{q}}.$$

It is easy to see that g is a one-to-one mapping of $\mathfrak{X}$ onto $\mathfrak{X}$. Moreover, g is an involution of $\mathfrak{X}$. For every $a \in A$ let

(27) $$h(a) = \mathop{E}_{\mathfrak{q} \in \mathfrak{X}} (a \in \mathfrak{q}).$$

Every quasi-Boolean algebra being a distributive lattice, it is well known that

(28) $a \subset b$ *if and only if* $h(a) \subset h(b)$ *for any* $a, b \in A$,

i. e., the mapping h is one-to-one. Moreover,

(29) $$h(a \cdot b) = h(a) \cdot h(b), \quad h(a+b) = h(a) + h(b).$$

We shall consider $\mathfrak{X}$ as a topological space with the class $\boldsymbol{B}(\mathfrak{X})$ of all sets $h(a)$, $a \in A$ as the class of neighbourhoods. Then $\mathfrak{X}$ is a T_0-space. We have

(30) $$h(\sim a) = \mathfrak{X} - g(h(a)) = \sim h(a).$$

It follows from (28), (29) and (30) that $\boldsymbol{B}(\mathfrak{X})$ is a quasi-field of open subsets of $\mathfrak{X}$ and h is the isomorphism of $\langle A, +, \cdot, \sim\rangle$ onto $\boldsymbol{B}(\mathfrak{X})$. Since the unit element $e \in A$, $\mathfrak{X} \in \boldsymbol{B}(\mathfrak{X})$. It remains to show that $\mathfrak{X}$ is a bicompact space. For this aim let us suppose that there exists a class $\{a_\iota\}$, $\iota \in I$, of elements of A such that $\mathfrak{X} = \sum_{\iota \in I} h(a_\iota)$, but for every finite subset $I_0 \subset I$ we have $\mathfrak{X} \neq \sum_{\iota \in I_0} h(a_\iota)$. Consequently $e \neq \sum_{\iota \in I_0} a_\iota$. Hence we obtain $0 \neq \sim \sum_{\iota \in I_0} a_\iota$. Thus $0 \neq \prod_{\iota \in I_0} \sim a_\iota$. It follows from this inequality that the filter $\mathfrak{q}^*$ generated by all the elements $\sim a_\iota$, $\iota \in I$ is a proper filter. Consequently, there exists a prime filter $\mathfrak{q}$ containing $\mathfrak{q}^*$. Obviously $\mathfrak{q} \in h(\sim a_\iota) = \mathfrak{X} - g(h(a_\iota))$ for every $\iota \in I$, i. e., $\mathfrak{q} \notin \sum_{\iota \in I} g(h(a_\iota))$. Since g is a one-to-one mapping of $\mathfrak{X}$ onto $\mathfrak{X}$, we obtain $\mathfrak{q} \notin g \sum_{\iota \in I} h(a_\iota) = \mathfrak{X}$, which contradicts our hypothesis.

Let $\mathfrak{X}$ be a topological space, g — an involution of $\mathfrak{X}$, and $\boldsymbol{B}(\mathfrak{X})$ a quasi-field of open subsets of $\mathfrak{X}$, with the operation of quasi-complement $\sim$ defined by (25). Let $\mathfrak{X}_1$ be a non-empty subset of $\mathfrak{X}$ fulfilling the following conditions, where $\mathfrak{X}_2 = g(\mathfrak{X}_1)$:

(i) $\mathfrak{X}_1 \subset \mathfrak{X} - \sum_{X \in \boldsymbol{B}(\mathfrak{X})} X \cdot \sim X$,

(ii) *if* $X, Y \in \boldsymbol{B}(\mathfrak{X})$ *and* $X - Y \neq A$, *then* $(\mathfrak{X}_1 + \mathfrak{X}_2) \cdot (X - Y) \neq A$,

(iii) *the family* $\boldsymbol{H}(\mathfrak{X}_1)$ *of all open subsets relatively* $\mathfrak{X}_1$ *of the form* $X \cdot \mathfrak{X}_1$, *where* $X \in \boldsymbol{B}(\mathfrak{X})$, *is a Heyting algebra of sets, the operations of pseudo-codifference* $\to_1$ *and of pseudocomplement* $\neg_1$ *being defined by* (24),

(iv) *if* $X, Y \in \boldsymbol{B}(\mathfrak{X})$, *then there exists* $Z \in \boldsymbol{B}(\mathfrak{X})$ *such that*

(31) $$\mathfrak{X}_1 \cdot X \to_1 \mathfrak{X}_1 \cdot Y = \mathfrak{X}_1 \cdot Z \quad \textit{and} \quad (\mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X)) + \mathfrak{X}_2 \cdot Y = \mathfrak{X}_2 \cdot Z.$$

It is easy to see that the set Z is uniquely determined. Indeed this follows from the condition (ii).

We shall show that $\boldsymbol{B}(\mathfrak{X})$ is a $\mathcal{N}$-lattice under the set-theoretical operations of sum and product, the operation $\sim$ of quasi-complement defined by (25), and the operations $\to$ of $\mathcal{N}$-codifference and $\neg$ of $\mathcal{N}$-complement defined as follows:

(32) $X \to Y = Z$ if and only if $\mathfrak{X}_1 \cdot X \to_1 \mathfrak{X}_1 \cdot Y = \mathfrak{X}_1 \cdot Z$ and
$$\big(\mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X)\big) + \mathfrak{X}_2 \cdot Y = \mathfrak{X}_2 \cdot Z .$$

(33) $\neg X = X \to \Lambda$ (i. e. $\neg X = Z$ if and only if $\neg_1(\mathfrak{X}_1 \cdot X) = \mathfrak{X}_1 \cdot Z$
$$\text{and } \mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X) = \mathfrak{X}_2 \cdot Z) .$$

Every $\mathcal{N}$-lattice of this kind is said to be a *$\mathcal{N}$-lattice of sets*, or, more precisely, a *$\mathcal{N}$-lattice of open subsets of* $\mathfrak{X}$.

It follows from (i) that $\mathfrak{X}_1 \cdot X \cdot \sim X = \Lambda$ for every $X \in \boldsymbol{B}(\mathfrak{X})$. Hence $\mathfrak{X}_1 \cdot X \cdot \big(\mathfrak{X} - g(X)\big) = \Lambda$. In consequence

(34) $$\mathfrak{X}_1 \cdot X \subset \mathfrak{X}_1 \cdot g(X) \quad \text{for every} \quad X \in \boldsymbol{B}(\mathfrak{X}) .$$

Now we shall prove that for arbitrary $X, Y \in \boldsymbol{B}(\mathfrak{X})$

(35) $$X \prec Y \quad \text{if and only if} \quad \mathfrak{X}_1 \cdot X \subset \mathfrak{X}_1 \cdot Y .$$

Indeed, let us suppose that $X \prec Y$. By definition of the relation $\prec$ we have $X \to Y = \mathfrak{X}$. Hence $\mathfrak{X}_1 \cdot (X \to Y) = \mathfrak{X}_1 \cdot X \to_1 \mathfrak{X}_1 \cdot Y = \mathfrak{X}_1$. Consequently $\mathfrak{X}_1 \cdot X \subset \mathfrak{X}_1 \cdot Y$. Conversely, let us suppose that non $X \prec Y$. Thus $X \to Y \neq \mathfrak{X}$. By (ii) either $\mathfrak{X}_1 \cdot (X \to Y) \neq \mathfrak{X}_1$ or $\mathfrak{X}_2 \cdot (X \to Y) \neq \mathfrak{X}_2$. In the first case we obviously have $\mathfrak{X}_1 \cdot X \not\subset \mathfrak{X}_1 \cdot Y$. If the second case holds, then $\mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X) + {}$ ${} + \mathfrak{X}_2 \cdot Y \neq \mathfrak{X}_2$. Hence $g(\mathfrak{X}_1 \cdot X) \not\subset \mathfrak{X}_2 \cdot Y$. Consequently $\mathfrak{X}_1 \cdot X \not\subset g(\mathfrak{X}_2 \cdot Y)$ $= \mathfrak{X}_1 \cdot g(Y)$. By (34) we obtain $\mathfrak{X}_1 \cdot X \not\subset \mathfrak{X}_1 \cdot Y$.

It follows immediately from (35) that the relation $\prec$ is a quasi-ordering one. Thus the condition (1) holds.

We shall show that

(36) $$\sim Y \prec \sim X \quad \text{if and only if} \quad \mathfrak{X}_2 \cdot X \subset \mathfrak{X}_2 \cdot Y .$$

Indeed, on account of (35), $\sim Y \prec \sim X$ if and only if $\mathfrak{X}_2 \cdot \big(\mathfrak{X} - g(Y)\big)$ $\subset \mathfrak{X}_1 \cdot \big(\mathfrak{X} - g(X)\big)$. This is equivalent to $g\big(\mathfrak{X}_1 \cdot g(X)\big) \subset g\big(\mathfrak{X}_1 \cdot g(Y)\big)$ and consequently to $\mathfrak{X}_2 \cdot X \subset \mathfrak{X}_2 \cdot Y$.

If $X \subset Y$ holds for any $X, Y \in \boldsymbol{B}(\mathfrak{X})$, then $\mathfrak{X}_1 \cdot X \subset \mathfrak{X}_1 \cdot Y$ and $\mathfrak{X}_2 \cdot X$ $\subset \mathfrak{X}_2 \cdot Y$. Consequently, by (35) and (36), $X \prec Y$ and $\sim Y \prec \sim X$. Conversely, if $X \not\subset Y$, then $X - Y \neq \Lambda$. Hence by (ii), $\mathfrak{X}_1 \cdot X \not\subset \mathfrak{X}_1 \cdot Y$ or $\mathfrak{X}_2 \cdot X \not\subset \mathfrak{X}_2 \cdot Y$. Consequently by (35) and (36) either non $X \prec Y$ or non $\sim Y \prec \sim X$. Since $\boldsymbol{B}(\mathfrak{X})$ is a quasi-field of sets containing $\mathfrak{X}$ and Λ, we infer that the condition (2) is also satisfied.

The conditions (3) and (4) follow immediately from (35).

To prove (5) and (6) consider the equality
$$\mathfrak{X}_2 \cdot \big(\mathfrak{X} - g(X) + Y\big) = \mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X) + \mathfrak{X}_2 \cdot Y ,$$

which holds for any $X, Y \in \boldsymbol{B}(\mathfrak{X})$. Hence we obtain
$$\mathfrak{X}_2 \cdot (\sim X + Y) = \mathfrak{X}_2 \cdot (X \to Y) .$$

Consequently we have $\sim(\sim X + Y) \prec \sim(X \to Y)$ and $\sim(X \to Y) \prec \sim(\sim X + Y)$. Hence $X \cdot \sim Y \prec \sim(X \to Y)$ and $\sim(X \to Y) \prec X \cdot \sim Y$.

To prove (7) and (8), let us consider the identity $\mathfrak{X}_2 \cdot \big(\mathfrak{X} - g(X)\big) = \mathfrak{X}_2 -$ $- g(\mathfrak{X}_1 \cdot X)$. Hence $\mathfrak{X}_2 \cdot \sim X = \mathfrak{X}_2 \cdot \neg X$ for every $X \in \boldsymbol{B}(\mathfrak{X})$. Consequently, by (36) $\sim\sim X \prec \sim\neg X$ and $\sim\neg X \prec \sim\sim X$, which proves (7) and (8).

On account of (i) we have $\mathfrak{X}_1 \cdot X \cdot \sim X \subset \mathfrak{X}_1 \cdot Y$ for any $X, Y \in \boldsymbol{B}(\mathfrak{X})$. Thus the condition (9) is fulfilled.

The condition (10) follows from (iii). In fact, $X \prec Y \to Z$ is equivalent to $\mathfrak{X}_1 \cdot X \subset \mathfrak{X}_1 \cdot Y \to_1 \mathfrak{X}_1 \cdot Z$. Since by (iii) $\mathfrak{X}_1 \cdot X$, $\mathfrak{X}_1 \cdot Y$, $\mathfrak{X}_1 \cdot Z$ are elements of a Heyting algebra of sets it is well known that the conditions given above is equivalent to the following one: $\mathfrak{X}_1 \cdot X \cdot Y \subset \mathfrak{X}_1 \cdot Z$, which holds if and only if $X \cdot Y \prec Z$.

The condition (11) follows immediately from the definition of this operation in the considered lattice.

We shall prove that every $\mathcal{N}$-lattice is isomorphic with a $\mathcal{N}$-lattice of sets. For this purpose we shall use a method similar to that of Stone.

In the sequel let $\mathfrak{A} = \langle A, e, +, \cdot, \sim, \to, \neg \rangle$ be a $\mathcal{N}$-lattice.

A non-empty subset $\mathfrak{q}$ of A is said to be a *special filter of the first kind* (s. f. f. k.) provided that:

(I) *if* $a, b \in \mathfrak{q}$, *then* $a \cdot b \in \mathfrak{q}$,

(II) *if* $a \in \mathfrak{q}$, $b \in A$, *and* $a \prec b$, *then* $b \in \mathfrak{q}$.

Moreover, if $\mathfrak{q} \neq A$, it is said to be a *proper* s. f. f. k.

Let a be an element of A and let non $a \prec 0$. It follows from (1) and (4) that the set $\mathfrak{q}_1(a) = \underset{x \in A}{E}(a \prec x)$ is an example of a proper s. f. f. k. It will be called the s. f. f. k. *generated by* a and denoted as above by $\mathfrak{q}_1(a)$.

A proper special filter $\mathfrak{q}$ of the first kind is said to be *prime* provided that

(III) *if* $a + b \in \mathfrak{q}$, *then either* $a \in \mathfrak{q}$ *or* $b \in \mathfrak{q}$.

Obviously every prime s. f. f. k. is a prime filter.

A s. f. f. k. is said to be *multiplically irreducible* if it is not a product of two s. f. f. k.'s different from it.

A non-empty subset $\dot{\mathfrak{q}}$ of A is said to be a *special filter of the second kind* (s. f. s. k.) provided that:

(I*) *if* $a, b \in \mathfrak{q}$, *then* $a \cdot b \in \mathfrak{q}$,

(II*) *if* $a \in \mathfrak{q}$ *and* $\sim b \prec \sim a$, *then* $b \in \mathfrak{q}$.

Moreover, if $\mathfrak{q} \neq A$, then $\mathfrak{q}$ will be called a *proper* s. f. s. k.

Let $a \neq 0$ be an element of A. It follows from (1), (2) and (3) that the set $\mathfrak{q}_2(a) = \underset{x\in A}{E}(\sim x \prec \sim a)$ is an example of a proper s. f. s. k. It will be called the s. f. s. k. *generated by* a and will be denoted as above by $\mathfrak{q}_2(a)$.

A proper special filter $\mathfrak{q}$ of the second kind is said to be *prime* provided that it fulfils the condition (III). Clearly, every s. f. s. k. which is prime is also a prime filter.

A s. f. s. k. is said to be *multiplically irreducible* provided that it is not a product of two s. f. s. k.'s different from it.

Let $\mathfrak{q}'$ be a s. f. f. k. and let a be an element of A, $a \neq 0$. We shall denote by $\mathfrak{q}' \cup_1 \mathfrak{q}_1(a)$ the smallest s. f. f. k. containing $\mathfrak{q}'$ and $\mathfrak{q}_1(a)$.

3.2. *The* s. f. f. k. $\mathfrak{q}' \cup_1 \mathfrak{q}_1(a)$ *is the set of all elements* $x \in A$ *fulfilling the condition*: $c \cdot a \prec x$ *for some* $c \in \mathfrak{q}'$.

Let Q' be the set of all $x \in A$ such that $c \cdot a \prec x$ for some $c \in \mathfrak{q}'$. Then Q' is a s. f. f. k. Indeed, if $x, y \in Q'$, then $c_1 \cdot a \prec x$ and $c_2 \cdot a \prec y$, where $c_1, c_2 \in \mathfrak{q}'$. Hence $(c_1 \cdot c_2) \cdot a \prec x \cdot y$, where $c_1 \cdot c_2 \in \mathfrak{q}'$. Consequently $x \cdot y \in Q'$. If $x \in Q'$ and $x \prec y$, then $c \cdot a \prec x \prec y$ for some $c \in \mathfrak{q}'$. Thus $y \in Q'$. It is easy to see, that $\mathfrak{q}' \subset Q'$ and $\mathfrak{q}_1(a) \subset Q'$. We shall show that Q' is the smallest s. f. f. k. containing $\mathfrak{q}'$ and $\mathfrak{q}_1(a)$. In fact, let us suppose that Q_1 is a s. f. f. k. containing $\mathfrak{q}$ and $\mathfrak{q}_1(a)$. If $x \in Q'$, then $c \cdot a \prec x$ for some $c \in \mathfrak{q}'$. Obviously $a, c \in Q_1$. Hence $a \cdot c \in Q_1$ and hence $x \in Q_1$. Thus $Q' \subset Q_1$.

Let $\mathfrak{q}''$ be a s. f. s. k. and let a be an element of A, $a \neq 0$. We shall denote by $\mathfrak{q}'' \cup_2 \mathfrak{q}_2(a)$ the smallest s. f. s. k. containing $\mathfrak{q}''$ and $\mathfrak{q}_2(a)$.

3.3. *The* s. f. s. k. $\mathfrak{q}'' \cup_2 \mathfrak{q}_2(a)$ *is the set of all elements* $x \in A$ *such that* $\sim x \prec \sim(c \cdot a)$ *for some* $c \in \mathfrak{q}''$.

Let Q'' be the set of all elements $x \in A$ fulfilling the condition $\sim x \prec \sim(c \cdot a)$ for some $c \in \mathfrak{q}''$. Then Q'' is a s. f. s. k. In fact, if $x \cdot y \in Q''$, then $\sim x \prec \sim c_1 + \sim a$ and $\sim y \prec \sim c_2 + \sim a$, where $c_1, c_2 \in \mathfrak{q}''$. By (3) $\sim x + \sim y \prec \sim c_1 + \sim c_2 + \sim a$. Hence $\sim(x \cdot y) \prec \sim((c_1 \cdot c_2) \cdot a)$, where $c_1 \cdot c_2 \in \mathfrak{q}''$. Thus $x \cdot y \in Q''$. If $x \in Q''$ and $\sim y \prec \sim x$, then $\sim y \prec \sim x \prec \sim(c \cdot a)$ for some $c \in \mathfrak{q}''$. Hence $y \in Q''$. It is easy to see that Q'' contains $\mathfrak{q}''$ and $\mathfrak{q}_2(a)$. We shall show that Q'' is the smallest s. f. s. k. with this property. In fact, let us suppose that Q_2 is a s. f. s. k. containing $\mathfrak{q}''$ and $\mathfrak{q}_2(a)$. If $x \in Q''$, then $\sim x \prec \sim(c \cdot a)$ for some $c \in \mathfrak{q}''$. Obviously $c \cdot a \in Q_2$ and thus $x \in Q_2$. Consequently, $Q'' \subset Q_2$.

3.4. *Every proper* s. f. f. k. *multiplically irreducible is prime.*

Let us suppose that $\mathfrak{q}$ is a proper s. f. f. k. and $a+b \in \mathfrak{q}$, but $a \notin \mathfrak{q}$ and $b \notin \mathfrak{q}$. Then obviously $a \neq 0$ and $b \neq 0$. Moreover, we shall show that $\mathfrak{q} = (\mathfrak{q} \cup_1 \mathfrak{q}_1(a)) \cdot (\mathfrak{q} \cup_1 \mathfrak{q}_1(b))$. Indeed, $\mathfrak{q} \subset (\mathfrak{q} \cup_1 \mathfrak{q}_1(a)) \cdot (\mathfrak{q} \cup_1 \mathfrak{q}_1(b))$. If $x \in \mathfrak{q} \cup_1 \mathfrak{q}_1(a)$ and $x \in \mathfrak{q} \cup_1 \mathfrak{q}_1(b)$, then by 3.2 there exist $c_1, c_2 \in \mathfrak{q}$, such that $c_1 \cdot a \prec x$ and $c_2 \cdot b \prec x$. Hence $(c_1 \cdot c_2) \cdot a + (c_1 \cdot c_2) \cdot b \prec x$. Consequently, $(c_1 \cdot c_2) \cdot (a+b) \prec x$, where $(c_1 \cdot c_2) \cdot (a+b) \in \mathfrak{q}$. Thus $x \in \mathfrak{q}$. In consequence, $\mathfrak{q}$ is not multiplically irreducible.

3.5. *Every proper* s. f. s. k. *multiplically irreducible is prime.*

Let us suppose that $\mathfrak{q}$ is a proper s. f. s. k. and $a+b \in \mathfrak{q}$, but $a \notin \mathfrak{q}$ and $b \notin \mathfrak{q}$. We shall show that $\mathfrak{q} = (\mathfrak{q} \cup_2 \mathfrak{q}_2(a)) \cdot (\mathfrak{q} \cup_2 \mathfrak{q}_2(b))$. Obviously $a \neq 0$ and $b \neq 0$. It is sufficient to show that $(\mathfrak{q} \cup_2 \mathfrak{q}_2(a)) \cdot (\mathfrak{q} \cup_2 \mathfrak{q}_2(b)) \subset \mathfrak{q}$. If $x \in \mathfrak{q} \cup_2 \mathfrak{q}_2(a)$ and $x \in \mathfrak{q} \cup_2 \mathfrak{q}_2(b)$, then by 3.3 there exist $c_1, c_2 \in \mathfrak{q}$ such that $\sim x \prec \sim(c_1 \cdot a)$ and $\sim x \prec \sim(c_2 \cdot b)$. Hence by (4) $\sim x \prec \sim(c_1 \cdot a) \cdot \sim(c_2 \cdot b) = \sim(c_1 \cdot a + c_2 \cdot b) = \sim((c_1 + c_2) \cdot (c_1 + b) \cdot (c_2 + a) \cdot (a+b))$, where $(c_1 + c_2) \cdot (c_1 + b) \cdot (c_2 + a) \cdot (a+b) \in \mathfrak{q}$. Hence $x \in \mathfrak{q}$. Consequently, $\mathfrak{q}$ is not multiplically irreducible.

3.6. *If* $x, y \in A$ *and non* $x \prec y$, *then there exists a prime* s. f. f. k. $\mathfrak{q}$ *such that* $x \in \mathfrak{q}$ *and* $y \notin \mathfrak{q}$.

Let $\boldsymbol{P}$ be the class of all s. f. f. k. containing x and not containing y. The class $\boldsymbol{P}$ is not empty, since $\mathfrak{q}_1(x) \in \boldsymbol{P}$. Obviously $\boldsymbol{P}$ is partly ordered by the relation of inclusion. Let $\boldsymbol{C} \subset \boldsymbol{P}$ be an arbitrary chain. Since the set-theoretical sum of all s. f. f. k. belonging to $\boldsymbol{C}$ is also a s. f. f. k. containing x and not containing y, we infer that $\boldsymbol{C}$ has in $\boldsymbol{P}$ an upper bound. By the lemma of Zorn, there exists in $\boldsymbol{P}$ a maximal element $\mathfrak{q}$, i. e., a proper s. f. f. k. $\mathfrak{q}$ which is contained in no s. f. f. k. belonging to $\boldsymbol{P}$. Obviously, $\mathfrak{q}$ is multiplically irreducible. Hence, by 3.4, it is prime.

3.7. *If* $x, y \in A$ *and non* $\sim y \prec \sim x$, *then there exists a prime* s. f. s. k. $\mathfrak{q}$ *such that* $x \in \mathfrak{q}$ *and* $y \notin \mathfrak{q}$.

A similar proof to that of 3.6 making use of 3.5 is omitted.

3.8. *Every* $\mathcal{N}$*-lattice* $\mathfrak{A} = \langle A, e, +, \cdot, \sim, \rightarrow, \neg\rangle$ *is isomorphic with a* $\mathcal{N}$*-lattice of open subsets of a bicompact* T_0*-space.*

Let $\mathfrak{X}$ be the set of all prime filters of $\mathfrak{A}$, g the mapping of $\mathfrak{X}$ onto $\mathfrak{X}$ defined by (26) and let for every $a \in A$

$$h(a) = \underset{\mathfrak{q}\in\mathfrak{X}}{E}(a \in \mathfrak{q})\,.$$

Since $\langle A, +, \cdot, \sim\rangle$ is a quasi-Boolean algebra, we infer that the conditions (28), (29) and (30) are satisfied. Moreover, the set $\mathfrak{X}$ considered as a topological space with the class $\boldsymbol{B}(\mathfrak{X})$ of all subsets $h(a)$, $a \in A$ as the class of neighbourhoods is a bicompact T_0-space and $\boldsymbol{B}(\mathfrak{X})$ is a quasi-field of open subsets of $\mathfrak{X}$ isomorphic with $\langle A, +, \cdot, \sim\rangle$, the mapping h constituting that isomorphism. The elements $h(0)$ and $h(e)$ are the zero element and the unit element of $\boldsymbol{B}(\mathfrak{X})$, respectively.

Let $\mathfrak{X}_1$ be the set of all prime s. f. f. k.'s and let $\mathfrak{X}_2$ be the set of all prime s. f. s. k.'s. We shall show that

$$g(\mathfrak{X}_1) = \mathfrak{X}_2 . \tag{37}$$

Indeed, let us suppose that $\mathfrak{q}_1$ is a prime s. f. f. k. Then by (26) $g(\mathfrak{q}_1)$ is a prime filter. Let us suppose that $a \in g(\mathfrak{q}_1)$ and $\sim b \prec \sim a$. Then $a \in A - \widetilde{\mathfrak{q}}_1$, i. e. $\sim a \notin \mathfrak{q}_1$. Hence $\sim b \notin \mathfrak{q}_1$. Consequently $b \in g(\mathfrak{q}_1)$. Thus $g(\mathfrak{q}_1) \in \mathfrak{X}_2$, and we have $g(\mathfrak{X}_1) \subset \mathfrak{X}_2$. Now let us suppose that $\mathfrak{q}_2$ is a prime s. f. s. k. Then $g(\mathfrak{q}_2)$ is by (26) a prime filter. If $a \in g(\mathfrak{q}_2)$ and $a \prec b$, i. e. $\sim\sim a \prec \sim\sim b$, then $a \in A - \widetilde{\mathfrak{q}}_2$. Hence $\sim a \notin \mathfrak{q}_2$. Consequently $\sim b \notin \mathfrak{q}_2$. Thus $b \in A - \widetilde{\mathfrak{q}}_2$, i. e. $b \in g(\mathfrak{q}_2)$. We have proved that $g(\mathfrak{X}_2) \subset \mathfrak{X}_1$. Thus $\mathfrak{X}_2 \subset g(\mathfrak{X}_1)$.

Let us set for every $a \in A$

$$h_1(a) = \underset{\mathfrak{q} \in \mathfrak{X}_1}{E} (a \in \mathfrak{q}) \quad \text{and} \quad h_2(a) = \underset{\mathfrak{q} \in \mathfrak{X}_2}{E} (a \in \mathfrak{q}) . \tag{38}$$

Obviously,

$$h_1(a) = \mathfrak{X}_1 \cdot h(a) \quad \text{and} \quad h_2(a) = \mathfrak{X}_2 \cdot h(a) . \tag{39}$$

It is easy to see, making use of 3.6 and 3.7, that

$$h_1(a) \subset h_1(b) \text{ if and only if } a \prec b \tag{40}$$

and

$$h_2(a) \subset h_2(b) \text{ if and only if } \sim b \prec \sim a. \tag{41}$$

Now we shall prove that the conditions (i)-(iv) (p. 71) are satisfied for $\mathfrak{X}_1$ and $\mathfrak{X}_2$, i. e., that $\boldsymbol{B}(\mathfrak{X})$ is a $\mathfrak{N}$-lattice of open subsets of $\mathfrak{X}$.

Proof of (i). If $\mathfrak{q} \in \mathfrak{X}_1$, then by (9) $a \cdot \sim a \notin \mathfrak{q}$ for every $a \in A$. Consequently, $\mathfrak{q} \notin h(a) \cdot \sim h(a)$. Hence (i) holds.

Proof of (ii). Let us suppose that $h(a) - h(b) \neq \Lambda$, $a, b \in A$. Thus $h(a) \not\subset h(b)$. Hence, by (28), $a \not\subset b$. Consequently we have either non $a \prec b$ or non $\sim b \prec \sim a$. By (40) and (41) we infer that either $h_1(a) \not\subset h_1(b)$ or $h_2(a) \not\subset h_2(b)$. Hence $(\mathfrak{X}_1 + \mathfrak{X}_2) \cdot \big(h(a) - h(b)\big) \neq \Lambda$.

Proof of (iii). Let $\boldsymbol{H}(\mathfrak{X}_1)$ be the class of all sets $h_1(a)$ where $a \in A$. It is easy to see that

$$h_1(a) + h_1(b) = h_1(a+b), \quad h_1(a) \cdot h_1(b) = h_1(a \cdot b) . \tag{42}$$

Now we shall prove that

$$h_1(a \to b) = \mathrm{Int}_1 \Big(\big(\mathfrak{X}_1 - h_1(a)\big) + h_1(b)\Big) , \tag{43}$$

where Int_1 denotes the operation of interior relatively $\mathfrak{X}_1$. We have

$$h_1(a \to b) \subset \big(\mathfrak{X}_1 - h_1(a)\big) + h_1(b) . \tag{44}$$

In fact, if $\mathfrak{q} \in h_1(a \to b)$ and $\mathfrak{q} \in h_1(a)$, then $a \cdot (a \to b) \in \mathfrak{q}$. But $a \cdot (a \to b) \prec b$. Hence $b \in \mathfrak{q}$, and consequently $\mathfrak{q} \in h_1(b)$. Thus $h_1(a \to b) \cdot h_1(a) \subset h_1(b)$, which proves (44).

Let us suppose that for some $x \in A$

$$h_1(x) \subset \big(\mathfrak{X}_1 - h_1(a)\big) + h_1(b) .$$

Hence $h_1(x) \cdot h_1(a) \subset h_1(b)$. By (42) and (40) $x \cdot a \prec b$. Thus by (10) $x \prec a \to b$. Using (40) we obtain $h_1(x) \subset h_1(a \to b)$. Consequently (43) holds. By (24) we obtain

$$h_1(a \to b) = h_1(a) \to_1 h_1(b) . \tag{45}$$

It follows from (11) and (43) that

$$h_1(\neg a) = \mathrm{Int}_1 \big(\mathfrak{X}_1 - h_1(a)\big) . \tag{46}$$

By (24) we obtain

$$h_1(\neg a) = \neg_1 h_1(a) . \tag{47}$$

It follows from (42), (43) and (46) that $\boldsymbol{H}(\mathfrak{X}_1)$ is a Heyting algebra of sets, the operations $\to_1$ and $\neg_1$ being defined by (24). Hence (iii) is fulfilled.

Proof of (iv). It follows from (39), (45) that

$$\mathfrak{X}_1 \cdot h(a \to b) = \mathfrak{X}_1 \cdot h(a) \to_1 \mathfrak{X}_1 \cdot h(b) \quad \text{for any} \quad h(a), h(b) \in \boldsymbol{B}(\mathfrak{X}). \tag{48}$$

Moreover, for every $a \in A$ the following equality holds:

$$h_2(\sim a) = \mathfrak{X}_2 - g\big(h_1(a)\big) . \tag{49}$$

Indeed, $\mathfrak{q} \in \mathfrak{X}_2 - g\big(h_1(a)\big)$ if and only if $\mathfrak{q} \in \mathfrak{X}_2$ and $\mathfrak{q} \notin g\big(h_1(a)\big)$. These conditions are equivalent to the conditions $\mathfrak{q} \in \mathfrak{X}_2$ and $\mathfrak{q} \notin g\big(h(a)\big)$, which holds if and only if $\mathfrak{q} \in \mathfrak{X}_2$ and $\sim a \in \mathfrak{q}$. Thus (49) holds.

Now we shall prove that

$$\mathfrak{X}_2 \cdot h(a \to b) = \Big(\mathfrak{X}_2 - g\big(\mathfrak{X}_1 \cdot h(a)\big)\Big) + \mathfrak{X}_2 \cdot h(b) . \tag{50}$$

On account of (39) it is sufficient to show that $h_2(a \to b) = \Big(\mathfrak{X}_2 - g\big(h_1(a)\big)\Big) + h_2(b)$. We have $\mathfrak{q} \in h_2(a \to b)$ if and only if $\mathfrak{q} \in \mathfrak{X}_2$ and $a \to b \in \mathfrak{q}$. It follows from (6) and (5) that $a \to b \in \mathfrak{q}$ if and only if $\sim(a \cdot \sim b) \in \mathfrak{q}$, and consequently if and only if either $\sim a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. Thus $\mathfrak{q} \in h_2(a \to b)$ if and only if $\mathfrak{q} \in h_2(\sim a) + h_2(b)$. Using (49) we obtain (50). The conditions (i)-(iv) being satisfied, we infer that $\boldsymbol{B}(\mathfrak{X})$ is a $\mathfrak{N}$-lattice of open subsets of $\mathfrak{X}$. Moreover, let us suppose that $h(a) \to h(b) = h(c)$. Then by (32)

$$h_1(a) \to_1 h_1(b) = h_1(c), \quad \text{and} \quad \Big(\mathfrak{X}_2 - g\big(h_1(a)\big)\Big) + h_2(b) = h_2(c) .$$

Hence by (48) and (50) we infer that $h_1(c) = h_1(a \to b)$ and $h_2(c) = h_2(a \to b)$. Making use of (40) and (41) we infer that $c = a \to b$. Thus

(51) $$h(a) \to h(b) = h(a \to b) .$$

Moreover we find from (11), (51) that

(52) $$\neg h(a) = h(\neg a) .$$

We infer from (28), (29), (51) and (52) that h is an isomorphism of $\mathfrak{A}$ onto $\boldsymbol{B}(\mathfrak{X})$, which completes the proof of 3.8.

A. Białynicki-Birula, who read the manuscript of this paper, has given the following method of the construction of $\mathfrak{N}$-lattices of sets.

Let $\mathfrak{X}_1$ be a topological space and let $\boldsymbol{H}(\mathfrak{X}_1)$ be a Heyting algebra of open subsets of $\mathfrak{X}_1$ constituting the class of neighbourhoods of $\mathfrak{X}_1$. Let f be a one-to-one mapping of $\mathfrak{X}_1$ onto a set $\mathfrak{X}_2$ such that $f(x) = x$ for each $x \in \mathfrak{X}_1 \cdot \mathfrak{X}_2$. We assume also that

(53) $$\mathfrak{X}_1 \cdot \mathfrak{X}_2 \subset \prod_{X \in \boldsymbol{H}(\mathfrak{X}_1)} \Big(X + f\big(\mathrm{Int}_1(\mathfrak{X}_1 - X)\big)\Big) .$$

Let us set $\mathfrak{X} = \mathfrak{X}_1 + f(\mathfrak{X}_1) = \mathfrak{X}_1 + \mathfrak{X}_2$. Then the mapping g of $\mathfrak{X}$ onto $\mathfrak{X}$ defined as follows

(54) $$g(x) = \begin{cases} f(x) & \text{for every} \quad x \in \mathfrak{X}_1 , \\ f^{-1}(x) & \text{for every} \quad x \in \mathfrak{X}_2 \end{cases}$$

is an involution of $\mathfrak{X}$.

Let $\boldsymbol{B}(\mathfrak{X})$ be the class of all subsets of $\mathfrak{X}$ defined as follows: a subset $X \subset \mathfrak{X}$ belongs to $\boldsymbol{B}(\mathfrak{X})$ if and only if it fulfils the following conditions:

(55) $$X \cdot \mathfrak{X}_1 \in \boldsymbol{H}(\mathfrak{X}_1) ,$$

(56) $$X \cdot \mathfrak{X}_2 = \mathfrak{X}_2 - g(Y), \quad \text{where} \quad Y \in \boldsymbol{H}(\mathfrak{X}_1) ,$$

(57) $$X \cdot \mathfrak{X}_1 \subset g(X) \cdot \mathfrak{X}_1 .$$

It is easy to see that $\boldsymbol{B}(\mathfrak{X})$ is a ring of subsets of $\mathfrak{X}$. Thus $\mathfrak{X}$ can be considered as a topological space with the class $\boldsymbol{B}(\mathfrak{X})$ of neighbourhoods. It is also easy to verify that for every $X \in \boldsymbol{B}(\mathfrak{X})$ the set $\sim X = \mathfrak{X} - g(X) \in \boldsymbol{B}(\mathfrak{X})$. Consequently $\boldsymbol{B}(\mathfrak{X})$ is a quasi-field of open subsets of $\mathfrak{X}$.

We shall show that for any $X, Y \in \boldsymbol{B}(\mathfrak{X})$ the set

$$Z = (\mathfrak{X}_1 \cdot X \to_1 \mathfrak{X}_1 \cdot Y) + \Big(\big(\mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X)\big) + \mathfrak{X}_2 \cdot Y\Big) \in \boldsymbol{B}(\mathfrak{X}) .$$

We obviously have, by (53), $Z \cdot \mathfrak{X}_1 \in \boldsymbol{H}(\mathfrak{X}_1)$. Moreover, $Z \cdot \mathfrak{X}_2 = \big(\mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X)\big) + \big(\mathfrak{X}_2 - g(Y)\big)$ where $\mathfrak{X}_1 \cdot X$, $Y \in \boldsymbol{H}(\mathfrak{X}_1)$. Thus $Z \cdot \mathfrak{X}_2 = \mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X \cdot Y)$ where $\mathfrak{X}_1 \cdot X \cdot Y \in \boldsymbol{H}(\mathfrak{X}_1)$. Since $Z \cdot \mathfrak{X}_1 = \mathrm{Int}_1\big((\mathfrak{X}_1 - \mathfrak{X}_1 \cdot X) + \mathfrak{X}_1 \cdot Y\big) \subset (\mathfrak{X}_1 - \mathfrak{X}_1 \cdot X) + \mathfrak{X}_1 \cdot Y$ and $g(Z) \cdot \mathfrak{X}_1 = g(Z \cdot \mathfrak{X}_2) = g\big((\mathfrak{X}_2 - g(\mathfrak{X}_1 \cdot X) + \mathfrak{X}_2 \cdot Y\big) = (\mathfrak{X}_1 - \mathfrak{X}_1 \cdot X) + \mathfrak{X}_1 \cdot g(Y)$ we infer on account of $\mathfrak{X}_1 \cdot Y \subset \mathfrak{X}_1 \cdot g(Y)$ that $Z \cdot \mathfrak{X}_1 \subset g(Z) \cdot \mathfrak{X}_1$. Thus $Z \in \boldsymbol{B}(\mathfrak{X})$.

Now let us suppose that $X \in \boldsymbol{B}(\mathfrak{X})$ and $x \in \mathfrak{X}_1 \cdot X$. Thus by (57) $x \in g(X) \cdot \mathfrak{X}_1$. Hence $x \notin \mathfrak{X} - g(X) = \sim X$. Consequently

$$\mathfrak{X}_1 \subset \mathfrak{X} - \sum_{X \in \boldsymbol{B}(\mathfrak{X})} X \cdot \sim X .$$

Since the conditions (i), (ii), (iii) and (iv) (p. 71) are satisfied, we infer that $\boldsymbol{B}(\mathfrak{X})$ is a $\mathfrak{N}$-lattice of sets.

A. Białynicki-Birula has also remarked that every $\mathfrak{N}$-lattice is isomorphic to a $\mathfrak{N}$-lattice of sets of this kind. Indeed, this statement follows immediately from the proof of 3.8 and from the following theorem

3.9. *Every prime filter $\mathfrak{q}$ of a $\mathfrak{N}$-lattice is either a prime filter of the first kind or a prime filter of the second kind.*

To prove 3.9 we shall show that for every prime filter $\mathfrak{q}$ either $\mathfrak{q} \supset g(\mathfrak{q})$ or $g(\mathfrak{q}) \supset \mathfrak{q}$. In fact, let us suppose that non $g(\mathfrak{q}) \supset \mathfrak{q}$. Hence there exists an a such that $a \in \mathfrak{q}$ and $a \notin g(\mathfrak{q})$. It follows from $a \notin g(\mathfrak{q})$ that $\sim a \in \mathfrak{q}$. Thus $a \cdot \sim a \in \mathfrak{q}$. We shall show that $\mathfrak{q} \supset g(\mathfrak{q})$. Let us suppose that $b \in g(\mathfrak{q})$. It is easy to see that $\sim b \notin \mathfrak{q}$. But it follows from P18, A14 and A16 that in every $\mathfrak{N}$-lattice for any its elements a, b holds

$$a \cdot \sim a \subset b + \sim b .$$

Consequently, $b + \sim b \in \mathfrak{q}$. Since $\mathfrak{q}$ is a prime filter and $\sim b \notin \mathfrak{q}$ we infer that $b \in \mathfrak{q}$.

Now we shall demonstrate that if $\mathfrak{q} \supset g(\mathfrak{q})$ then $\mathfrak{q}$ is a prime filter of the second kind. Indeed, if $\mathfrak{q} \supset g(\mathfrak{q})$, then for any element a either $a \in \mathfrak{q}$ or $\sim a \in \mathfrak{q}$. For, if $a \notin \mathfrak{q}$ we obtain $\sim a \in g(\mathfrak{q})$ and consequently $\sim a \in \mathfrak{q}$. Suppose now that $a \in \mathfrak{q}$, $\sim b \prec \sim a$ and $b \notin \mathfrak{q}$. Hence $\sim b \in \mathfrak{q}$ and in consequence $a \cdot \sim b \in \mathfrak{q}$. Since $\sim b \prec \sim a$ we obtain $a \cdot \sim b \prec a \cdot \sim a \prec b$. On the other hand we obtain $\sim b \prec \sim a + b = \sim(a \cdot \sim b)$. Hence $a \cdot \sim b \subset b$. Thus $b \in \mathfrak{q}$, which contradicts our assumption. Consequently, $\mathfrak{q}$ is a prime filter of the second kind. Now let us suppose that $g(\mathfrak{q}) \supset \mathfrak{q}$, $a \in \mathfrak{q}$, $a \prec b$ but $b \notin \mathfrak{q}$. Hence $\sim b \in g(\mathfrak{q})$. On the other hand $a \in g(\mathfrak{q})$. Thus $a \cdot \sim b \in g(\mathfrak{q})$. Since $a \prec b$, we infer that $a \cdot \sim b \prec b \cdot \sim b$ and hence $a \cdot \sim b \prec \sim a$. We find also that $a \prec \sim a + b = \sim(a \cdot \sim b)$. Hence $a \cdot \sim b \subset \sim a$ and $\sim a \in g(\mathfrak{q})$. In consequence $a \notin \mathfrak{q}$. Thus if $g(\mathfrak{q}) \supset \mathfrak{q}$, then $\mathfrak{q}$ is a prime filter of the first kind. This completes the proof of 3.9.

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Functionals on uniformly closed rings of continuous functions

by

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In this paper we are concerned with the following problem: Suppose X is a completely regular space and let R be a linear ring of continuous real-valued functions defined on X which satisfies the following conditions:

1° *All constant functions belong to* R.

2° R *is closed with respect to the uniform convergence (i. e. if* $\{f_n\}$ *uniformly converges to* f *and* $f_n \in R$ $(n = 1, 2, \ldots)$, *then* $f \in R$*)*.

Under what conditions imposed on X and R each non-trivial linear multiplicative functional φ (¹) defined on R is of the form

$$\varphi(f) = f(p_0) \tag{$*$}$$

where p_0 is a fixed point of X?

We note some results related to this problem:

If R *is the ring of all bounded continuous functions on* X, *then the answer to our problem is positive if and only if* X *is a compact space* (Stone [4]).

If R *is the ring of all continuous functions on* X *then the answer to the problem is positive if and only if* X *is a* Q*-space* (Hewitt [1], [2]).

The main role in our considerations is played by the evaluation mapping of X into the Tihonov cube build up by means of all members f of R which satisfy the inequality $0 \leqslant f(p) \leqslant 1$ (i. e. denote by R^* the set of all members f of R which satisfy the above inequality and agree that the coordinates of points of the Tihonov cube $I^{\mathfrak{m}}$ $(\mathfrak{m} = \overline{\overline{R}})$ are enumerated by means of members of R^*. Then the evaluation mapping can be described as a mapping which carries a point $p \in X$ into the point $a \in I^{\mathfrak{m}}$ whose fth coordinate is equal to $f(p)$). We denote this evaluation mapping by F_R.

(¹) A functional φ is said to be *non-trivial* provided that φ does not vanish identically.