On a problem of Steinhaus

by

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To F.

1. On the plane there is a circle S. Each pair of antipodal points of S is connected by an arc lying in the domain bounded by S. These arcs depend continuously on their ends. Obviously, every two different arcs intersect. Do there exist everywhere *three* different arcs having a point in common?

This problem has been raised by H. Steinhaus in 1953. We shall prove a theorem from which a positive answer follows. In fact, our theorem will be more general:

a. the circle S will be replaced by an n-sphere in the Euclidean $(n+1)$-space;

b. the arcs will be replaced by arbitrary acyclic continua;

c. continuity will be meant in a less restrictive sense, namely that of the upper semi-continuity.

Under these assumptions we will prove that there exist again three continua having a point in common. (Observe that for $n > 1$ even the existence of two intersecting arcs is not obvious.)

A closely related question may be formulated as follows: Consider a mapping f of a 2-dimensional Möbius band M into the 2-cell Q, and suppose that f maps homeomorphically the boundary of M onto the boundary S of Q. Do there exist three different points of M which have a common image under f?

Theorem 2 gives a positive answer to that question, even generalized to higher dimensions. Let us observe that the positive solution of the first problem does not imply directly that of the second. In fact, both theorems are consequences of a lemma.

In section 2 we establish the terminology, in 3 we give the exact formulations of both theorems and show how they can be deduced from the "main lemma", which is proved in 4. In 5 we give some problems and remarks; some auxiliary notions from algebraic topology are gathered in 6.

2. Let X be a compact space and let $\Phi\colon X\to 2^Y$ be an upper semi-continuous mapping of X in the space 2^Y of all non-empty compact subsets of a (metric, separable) space Y. The triple $\mathcal{F}=\{X, Y, \Phi\}$ will be called a *family*. The set X will be called the *basis* of $\mathcal{F}$, the sets $\Phi(x)$ — the *elements* of $\mathcal{F}$, the set $\bigcup_{x\in X}\Phi(x)\subset Y$ — the *field* of $\mathcal{F}$. The field will be also denoted by $\Phi(X)$. The family $\mathcal{F}$ will be called *simple* if its elements are disjoint; it will be said to be *acyclic* if all its elements are such. (We use the Čech homology theory with coefficients mod 2; a set X is said to be *acyclic* if it is connected and if $H_k(X)=0$ for $k\geqslant 2$.)

The subset $M=\underset{(x,y)}{E}[y\in\Phi(x)]$ of the Cartesian product $X\times Y$ will be said to be the *graph* of $\mathcal{F}=\{X, Y, \Phi\}$. The mappings $(x,y)\to x$ and $(x,y)\to y$ will be called *projections*.

It is known that the upper semi-continuity of Φ implies that the graph and the field of a family are compact sets.

Let $\mathcal{F}_i=\{X, Y, \Phi_i\}$, $i=1,2$, be two families. We shall call the family $\mathcal{F}_1$ a *prolongation* of the family $\mathcal{F}_2$ if $\Phi_1(x)\supset\Phi_2(x)$ for every $x\in X$.

Q^n will always denote the Euclidean unit n-cell, S^{n-1} its boundary. Let φ be any involution on S^{n-1} (i. e. the mapping $\varphi\colon S^{n-1}\to S^{n-1}$ such that $\varphi\varphi(x)=x$ for every $x\in S^{n-1}$). Let P be the space obtained from S^{n-1} by identification of pairs $(x,\varphi(x))$ and let $k\colon S^{n-1}\to P$ be the identification mapping. Obviously $\{P, Q^n, k^{-1}\}$ is a family: it will be called the *family induced by the involution* φ. If φ is an involution without fixed points then k is a covering mapping (see [7], p. 67) and P is a manifold.

The family induced by the antipodal involution will be called the *antipodal family*. Its basis is the projective $(n-1)$-space P^{n-1}.

3. Theorem 1. *Let $\mathcal{F}=\{P^{n-1}, Q^n, \Phi\}$ be a prolongation of the antipodal family $\mathcal{F}_1$. If $\mathcal{F}$ is acyclic then there exist three distinct elements of $\mathcal{F}$ having a point in common.*

Theorem 2. *Let f be a continuous mapping of an n-dimensional Möbius band M^n into the cell Q^n such that f maps the boundary of M^n homeomorphically onto the boundary of Q^n. Then f is the mapping onto Q^n and at least three distinct points of M^n are mapped into one point of the cell.*

Theorems 1 and 2 are consequences of the following

Main Lemma. *Let φ be an involution without fixed points on S^{n-1} and let $\mathcal{F}_1=\{P, Q^n, k^{-1}\}$ be the family induced by φ. Let $\mathcal{F}=\{P, Q^n, \Phi\}$ be a prolongation of $\mathcal{F}_1$ and let $M\subset R=P\times Q^n$, $M_1\subset R$ be the graphs of $\mathcal{F}, \mathcal{F}_1$ respectively.*

Suppose that $i_*\colon H_{n-1}(M_1)\to H_{n-1}(M)$ is trivial, where $i\colon M_1\to M$ is the inclusion mapping. Then every point of Q^n belongs to at least one element of the family $\mathcal{F}$ and there exists a point which belongs to three different elements of $\mathcal{F}$.

We will denote by $\bar{q}\colon R\to Q^n$, $\bar{p}\colon R\to P$ the projections; the mappings $q\colon M\to Q^n$, $q_1\colon M_1\to S^{n-1}$, and $p\colon M\to P$ will be defined by $q=\bar{q}|M$, $p=\bar{p}|M$, $q_1=\bar{q}|M_1$.

We defer the proof of the lemma to 4. Now we shall prove that it implies theorems 1 and 2.

The main lemma implies theorem 1. Suppose that the families $\mathcal{F}, \mathcal{F}_1$ satisfy the conditions of theorem 1. Consider the diagram

$$\begin{array}{ccc} H_{n-1}(M_1) & \xrightarrow{i_*} & H_{n-1}(M) \\ \downarrow{\scriptstyle q_{1*}} & & \downarrow{\scriptstyle p_*} \\ H_{n-1}(S^{n-1}) & \xrightarrow{k_*} & H_{n-1}(P^{n-1}) \end{array}$$

where M and M_1 are graphs of $\mathcal{F}$ and $\mathcal{F}_1$, and the mappings are as above. By definition $(x,y)\in M_1$ if and only if $x=(y,-y)$, where $-y$ is the antipode of y. Hence $kq_1(x,y)=k(y)=(y,-y)=x=pi(x,y)$, which proves that the diagram is commutative.

Now, since we use the homology theory mod 2 we infer that k_* is trivial, whence so is p_*i_*. But the inverse-images by p of any point of P^{n-1}, being elements of $\mathcal{F}$, are acyclic. Thus by the Vietoris mapping theorem (see [1]) p_* is an isomorphism and it follows that i_* is trivial. Therefore theorem 1 follows from the lemma.

The main lemma implies theorem 2. We recall that the n-dimensional Möbius band is obtained from the n-dimensional annulus

$$A^n\colon\ \tfrac{1}{2}\leqslant\sum_{i=1}^{n}x_i^2\leqslant 1$$

by identification of antipodal points of the sphere $\sum_{i=1}^{n}x_i^2=1$. Therefore M^n is a fibre space with the segment as the fibre and P^{n-1} as the base space. We shall identify P^{n-1} with the pairs $(x,-x)$ of antipodal points of B (— the boundary of M^n). The fibre over $(x,-x)$ will be denoted by $L(x,-x)$; it contains x and $-x$.

Let $f\colon M^n\to Q^n$ be the mapping satisfying the conditions of theorem 2. If $a\in S^{n-1}$ there is only one point $x\in B$ such that $a=f(x)$. We define the involution φ on S^{n-1} by $\varphi(a)=f(-x)$ where $a\in S^{n-1}$, $x\in B$, $f(x)=a$. Let $\mathcal{F}_1$ be the family induced by φ. We define elements of a family $\mathcal{F}$ with basis P^{n-1} by $\Phi(x,-x)=f(L(x,-x))$. Then $\mathcal{F}$ is a prolongation of $\mathcal{F}_1$. Let M, M_1 be the graphs of $\mathcal{F}$ and $\mathcal{F}_1$. We shall prove that

(i) $i_*\colon H_{n-1}(M_1)\to H_{n-1}(M)$ is trivial.

Define first $g\colon M^n\to M$ by

$$g(a)=\big((f(x),\varphi f(x)),f(a)\big)\quad\text{for}\quad a\in L(x,-x).$$

Obviously, g is a continuous mapping; if $a \in B$ and $a \in L(x, -x)$ we have, say, $a = x$, whence $g(a) = \big(\big(f(x), \varphi f(x)\big), f(x)\big)$, which proves that $g_1 = g|B$ maps B homeomorphically onto M_1.

Now let j: $B \to M^n$ be the inclusion mapping. Of course j_*: $H_{n-1}(B) \to H_{n-1}(M^n)$ is trivial. Therefore so is $i_* g_{1*} = g_* j_*$, which proves (i), since g_* is an isomorphism.

Since (i) is verified, the main lemma applies and it follows that there is a point in Q^n which belongs to three different elements of $\mathcal{F}$. The elements of $\mathcal{F}$ being the images by f of fibres of M^n, it follows that there exist three different fibres whose images intersect at one point. This is a little more than was demanded by theorem 2.

4. We proceed now to the proof of the main lemma. We shall suppose that the families $\mathcal{F}_1, \mathcal{F}$ satisfy the conditions of the lemma. If $a \in Q^n$, the subset A of R defined by

$$A = \mathop{E}_{(x,y) \in R} [y = a]$$

will be called an *axis through the point a*. The main lemma is equivalent to the statement that there exists an axis which intersects M at three points at least, and that every axis intersects M. The second statement will be proved in 4.1. In 4.2 we shall suppose that no axis intersects M at more than two points and with the aid of this hypothesis we shall prove that if the axis through a intersects M at two points, there exists a neighbourhood U of a such that $q^{-1}(U)$ consists of two sheets, each homeomorphically mapped by q onto U. (This is the essential part of the proof.) Having done this, we easily infer that such a situation is impossible, i. e. that every axis intersects M at exactly one point, which corresponds to the statement that q is a homeomorphism. But this will give at once a contradiction proving the lemma.

4.1. We will prove first that without loss of generality we may assume that the family $\mathcal{F}$ satisfies the following condition

4.1.1. *There exists an open neighbourhood U of S^{n-1} in Q^n such that any point of U belongs to only one element of $\mathcal{F}$.*

The cell Q^n is given by $\sum_{i=1}^{n} x_i^2 \leqslant 1$. Let Q_1^n be the cell $\sum_{i=1}^{n} x_i^2 \leqslant 2$. If a is a point of S^{n-1} (— the boundary of Q^n), let a' denote the point on the boundary S_1^{n-1} of Q_1^n which lies on the same radius as a. Define the involution φ' on S_1^{n-1} by $\varphi'(x') = (\varphi(x))'$. Let $\mathcal{F}_1$ be the family induced by φ' and M_1' its graph. Put

$$\Phi'(a', \varphi'(a')) = \Phi(a) \cup aa' \cup \varphi(a)\varphi'(a') .$$

This defines a family $\mathcal{F}'$ which is a prolongation of $\mathcal{F}_1'$. The graph of $\mathcal{F}'$ will be M'. But obviously

(i) i_*': $H_{n-1}(M_1') \to H_{n-1}(M')$ is trivial if and only if i_*: $H_{n-1}(M_1) \to H_{n-1}(M)$ is such;

(ii) $\mathcal{F}'$ satisfies the assertion of the main lemma if and only if $\mathcal{F}$ does so.

This shows that nothing will change if instead of $\mathcal{F}, \mathcal{F}_1$ we consider the families $\mathcal{F}', \mathcal{F}_1'$. But $\mathcal{F}'$ satisfies 4.1.1 (with respect to Q_1^n, S_1^{n-1}; as U we take $Q_1^n - Q^n$), so that the proof is complete.

Now we will prove that

4.1.2. *The mapping q_1: $M_1 \to S^{n-1}$ is a homeomorphism.*

Since M_1 is compact and $q_1(M_1) = S^{n-1}$ we have only to prove that $q(x_1, y) = q(x_2, y) = y \in S^{n-1}$ implies $x_1 = x_2$. But $(x_i, y) \in M_1$ implies $(x_i, y) = \big(\big(y, \varphi(y)\big), y\big)$ $(i = 1, 2)$, proving therefore 4.1.2.

Now let α be the fundamental class of M_1, i. e. the generator of $H_{n-1}(M_1)$.

4.1.3. *Let M' be a compact subset of R containing M_1 and i: $M_1 \to M'$ the inclusion mapping. If $i_*\alpha = 0 \in H_{n-1}(M')$ then $\overline{q}(M') = Q^n$.*

Let q': $M' \to \overline{q}(M')$ be the mapping induced by $\overline{q}$, let j: $S^{n-1} \to \overline{q}(M')$ be the inclusion mapping, let γ be the fundamental class of S^{n-1}. By 4.1.2 $q_{1*}\alpha = \gamma$ and we have

$$j_*\gamma = j_* q_{1*}\alpha = q_*' i_*\alpha = 0 ,$$

which proves 4.1.3.

Now let M_0 be a compact subset of M, containing M_1, in which α bounds irreducibly (see 6.1). We choose M_0 once and for all; by 4.1.3 we have

4.1.4. *Every axis intersects M_0 at one point at least.*

This proves in particular the first part of the main lemma. Observe that it follows also from [2], theorem 1.

4.2. In 4.2 a will be such a point of Q^n that the axis through a intersects M_0 at two points b_1, b_2. By 4.1.1 a is an interior point of Q^n. Let U_1, U_2 be the open neighbourhoods in R of b_1, b_2 respectively. We assume that $\overline{U}_1 \cap \overline{U}_2 = 0$ and that each U_i is the Cartesian product of two open cells which will be denoted by $U_i' = \overline{p}(U_i) \subset P$ and $U_i'' = \overline{q}(U_i) \subset Q^n$. Let $S_i = M_0 \cap U_i$, $i = 1, 2$; thus S_i is a neighbourhood of b_i in M_0. We shall prove that

4.2.1. *a belongs to the interior of at least one of the sets $q(S_1), q(S_2)$.*

First, remark that there exists a neighbourhood V_i of a which is an open cell, satisfies $\overline{V} \subset U_i''$ and

(i) $$\big(\mathrm{Fr}(U_i') \times V_i\big) \cap M_0 = 0 .$$

For if (i) were untrue for every sufficiently small neighbourhood of a, we should have $\big(\mathrm{Fr}(U_i')\times a\big)\cap M_0\neq 0$, which implies that A intersects M_0 at other points than b_1 and b_2.

Suppose now that $a\in \mathrm{Fr}\big(q(S_1)\big)\cap \mathrm{Fr}\big(q(S_2)\big)$. It follows that there exist points a_i, $i=1,2$, such that $a_i\in V_i$ and if A_i is the axis through a_i

(ii) $$A_i\cap U_i\cap M_0=0\,.$$

Let $r_i\colon \overline{V}_i-a_i\to \mathrm{Fr}(V_i)$ be a retraction and define $r\colon M_0\to R$ by

$$r(x,y)=\begin{cases}(x,r_i(y)) & \text{if}\quad (x,y)\in \overline{U}_i'\times\overline{V}_i\,,\\ (x,y) & \text{otherwise}\,.\end{cases}$$

By (ii) r is well-defined on the whole M_0; obviously

(iii) r is an identity on M_1,

and

(iv) $\overline{q}\big(r(M_0)\big)$ does not contain the point a.

We will prove that

(v) r is continuous on M_0.

For assume that $(x,y)\in \mathrm{Fr}(\overline{U}_i'\times\overline{V}_i)\cap M_0$. This implies by (i) $(x,y)\in\big(\overline{U}_i'\times \mathrm{Fr}(V_i)\big)\cap M_0$ but on $\overline{U}_i'\cap \mathrm{Fr}(V_i)$ both definitions of r agree.

Now (iii) and (v) imply that $i_*\colon H_{n-1}(M_1)\to H_{n-1}\big(r(M_0)\big)$ is trivial, therefore 4.1.3 apply and yield a contradiction of (iv). This proves 4.2.1.

We will assume henceforth that

4.2.2. *Every axis intersects M_0 at two points at most.*

(Remark that this is a negation of a statement a little stronger than that demanded by the main lemma. In fact, the main lemma is equivalent to the statement that there exists an axis intersecting M at three points at least. We will prove this for the intersection with M_0 which is a subset of M.)

Since we have proved that a belongs to the interior of at least one of the sets $q(S_1)$, $q(S_2)$ we may assume that $a\in \mathrm{Int}\big(q(S_1)\big)$. We will show that

4.2.3. *Mapping q considered on a sufficiently small neighbourhood of p_2 in M_0 is a homeomorphism.*

Let U be an open neighbourhood of p_2 in M_0 such that $q(U)\subset \mathrm{Int}\big(q(S_1)\big)$. We shall prove that U has the desired property. Since each axis that intersects $\overline{U}$ intersects also S_1, it follows from 4.2.2 that it intersects $\overline{U}$ exactly at one point. Therefore the mapping q considered on $\overline{U}$ is a 1-1 continuous mapping of $\overline{U}$ onto $q(\overline{U})$. Since $\overline{U}$ is compact it is a homeomorphism, therefore it is a homeomorphism also on U.

With the aid of 4.2.3 we may strenghten 4.2.1. Namely:

4.2.4. *a belongs to the interior of both sets $q(S_1)$, $q(S_2)$.*

By 6.2 if V is a sufficiently small neighbourhood of p_2 in M_0, then there exists an element β of $H_{n-1}\big(\mathrm{Fr}(V)\big)$ which bounds irreducibly in V. Thus, by 4.2.3, a certain element of $H_{n-1}\big(q(\mathrm{Fr}(V))\big)$ bounds irreducibly in $q(V)$ and $q(V)$ is a neighbourhood of a in $q(S_2)$ (if V was chosen sufficiently small, which is what we suppose). But it is known (e. g. [5], p. 117) that this implies that a is an interior point of $q(V)$, whence also of $q(S_2)$.

By 4.2.4 the points p_1, p_2 in the proof of 4.2.3 may be interchanged and we obtain the following result

4.2.5. *If S_i is a sufficiently small neighbourhood of p_i in M_0, $i=1,2$, then the mapping q considered on S_i is a homeomorphism.*

4.3. In 4.2 we have established some local properties of the mapping q considered on M_0. Now we shall prove that

4.3.1. *The mapping q maps homeomorphically M_0 onto Q^n.*

The fact that q maps M_0 onto Q^n was proved in 4.1.3. Since M_0 is compact we have only to prove that q is 1-1. Suppose to the contrary that there exists a point $a\in Q^n$ such that the axis through a intersects M_0 at two different points b_1, b_2. Let V be an open cell, neighbourhood of a in Q^n (we recall that by 4.1.1 $a\in \mathrm{Int}(Q^n)$) which is so small that $q^{-1}(V)=U_1\cup U_2$ where U_1, U_2 are two disjoint open cells, mapped by q homeomorphically onto U and disjoint also from M_1. By 4.2.5 such a neighbourhood V does exist. Write $U=U_1\cup U_2$. Observe that $q(U)=V$, $\mathrm{Fr}\big(q(U)\big)=q\big(\mathrm{Fr}(U)\big)$, and that $q(M_0-U)=Q^n-V$. Let $F=\mathrm{Fr}(U)$. This leads to the following commutative diagram

$$\begin{array}{ccccc} H_{n-1}(M_1) & \xrightarrow{i_*'} & H_{n-1}(M_0-U) & \xleftarrow{j_*'} & H_{n-1}(F)\\ \downarrow{\scriptstyle q_{1*}} & & \downarrow{\scriptstyle q_*'} & & \downarrow{\scriptstyle q_*''}\\ H_{n-1}(S^{n-1}) & \xrightarrow{i_*} & H_{n-1}(Q^n-V) & \xleftarrow{j_*} & H_{n-1}\big(\mathrm{Fr}(V)\big)\end{array}$$

where the horizontal homomorphisms are induced by inclusions and the vertical homomorphisms are induced by the mappings induced by the mapping q.

By 6.2 there exists a class $\beta\in H_{n-1}(F)$ which bounds irreducibly on $\overline{U}$. Since we use coefficients mod 2 and since $\overline{U}$ is the sum of two disjoint n-cells, we easily infer that β is the sum of fundamental classes β_1 and β_2 of $\mathrm{Fr}(U_1)$ and $\mathrm{Fr}(U_2)$ respectively. Since $\mathrm{Fr}(U_1)$ and $\mathrm{Fr}(U_2)$ are homeomorphically mapped by q onto $\mathrm{Fr}(V)$ we have

(i) $$q_*''\beta=q_*''\beta_1+q_*''\beta_2=0\,.$$

Now let α be the fundamental class of M_1 and γ that of S^{n-1}. We have by 4.1.2

(ii) $$q_{1*}\alpha = \gamma\,.$$

Again by 6.2 we have

(iii) $$i'_*\alpha - j'_*\beta = 0\,.$$

The relations (i)-(iii) give, by the use of commutativity relations,

$$0 = q'_*(i'_*\alpha - j'_*\beta) = i_* q_{1*}\alpha - j_* q''_*\beta = i_*\gamma\,.$$

But this is impossible, since the fundamental class γ of S^{n-1} does not bound in a true compact subset $Q^n - V$ of Q^n. This proves 4.3.1.

The final step before arriving at a contradiction will be the following:

4.3.2. *The identification mapping k: $S^{n-1}\to P^{n-1}$ is homotopic to a constant mapping.*

By 4.3.1 the mapping pq^{-1}: $Q^n\to P^{n-1}$ is well-defined and continuous. Let $x \in S^{n-1}$. Then

$$pq^{-1}(x) = p\big((x, \varphi(x)), x\big) = \big(x, \varphi(x)\big) = k(x)\,,$$

which proves that pq^{-1} is an extension over Q^n of the mapping k.

Now, since involution φ has no fixed points, the identification mapping k: $S^{n-1}\to P^{n-1}$ is a covering mapping, and we may apply the covering homotopy theorem to the identity mapping i: $S^{n-1}\to S^{n-1}$ and we infer from 4.3.2 that i may be extended over Q^n. This being false, the assumption 4.2.2 must also be false, which proves the main lemma.

5. 5.1. Let $\mathcal{F} = \{X, Y, \Phi\}$ be a family. We will say that the *point $a \in Y$ is of order N* (N being any cardinal), $\operatorname{ord}_{\mathcal{F}} a = N$, if a belongs to exactly N different elements of $\mathcal{F}$. Let $N(\mathcal{F}) = \max\limits_{a\in Y} \operatorname{ord}_{\mathcal{F}}(a)$.

Now let $\mathcal{F}_1 = \{P^{n-1}, Q^n, k^{-1}\}$ be the antipodal family. Let $N(n) = \min N(\mathcal{F})$ where min is taken with respect to all acyclic prolongations of $\mathcal{F}_1$. Theorem 1 shows that

5.1.1. $N(n) \geqslant 3\,.$

Consider now the family $\mathcal{F}_1$ as "imbedded" in E^n; more exactly, let S^{n-1} be as usual the unit sphere in E^n; we will consider the family $\mathcal{F}_2 = \{P^{n-1}, E^n, k^{-1}\}$ where k: $S^{n-1}\to P^{n-1}$ is the identification mapping. Let $N_0(n) = \min N(\mathcal{F})$, where min is taken with respect to all acyclic prolongations $\mathcal{F}$ of $\mathcal{F}_2$. We will show that

5.1.2. $N_0(n) \geqslant 2\,.$

Let $\mathcal{F}$ be any acyclic prolongation of $\mathcal{F}_2$ and suppose that the field of $\mathcal{F}$ is contained in the interior of the cell Q_1^n: $\sum x_i^2 \leqslant 2$. On the boundary

of that cell we consider the family $\mathcal{F}'_1$ induced by the antipodal involution. By a construction analogous to that applied in 4.1.1 we construct the prolongation $\mathcal{F}'$ of $\mathcal{F}'_1$: the prolongation of the pair $(a, -a)$ will consist of two straight segments joining $a, -a$ with points on the same radius but on S^{n-1}, and of the corresponding element of $\mathcal{F}$.

In general the family $\mathcal{F}'_1$ will not be acyclic, but the pair $\mathcal{F}'_1, \mathcal{F}'$ will satisfy the conditions of the main lemma provided $\mathcal{F}, \mathcal{F}_2$ satisfy them. Hence there exists a point a of order three with respect to $\mathcal{F}'$. Since

$$\operatorname{ord}_{\mathcal{F}'}(a) = \begin{cases} \operatorname{ord}_{\mathcal{F}}(a)+1 & \text{if} \quad a \in Q_1^n - Q^n\,, \\ \operatorname{ord}_{\mathcal{F}}(a) & \text{if} \quad a \in Q^n\,, \end{cases}$$

this completes the proof.

5.2. Consider now the question of the exact value of $N(n)$ and $N_0(n)$. Obviously, to prove that $N(n) \leqslant k$ it suffices to give an example of a prolongation of $\mathcal{F}_1$ in which no more than k elements intersect at one point. The following shows that

5.2.1. $N(2) = 3\,.$

In view of 5.1.1 we have only to construct an example of a family in the plane such that no more than three of its elements intersect at one point. Let S be given by $x_1^2 + x_2^2 = 1$. Let $(a, -a)$ be the pair of antipodal points of S, and assume that writing $a = (x_1, x_2)$ we have $x_1 \geqslant 0$. Then we define

$$\Phi(a, -a) = (x_1, x_2)(-x_1, x_2) \cup (-x_1, x_2)(-x_1, -x_2)$$

(see fig. 1).

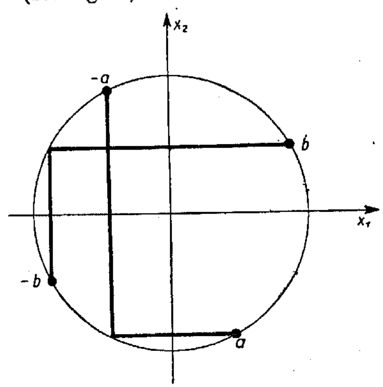

Fig. 1

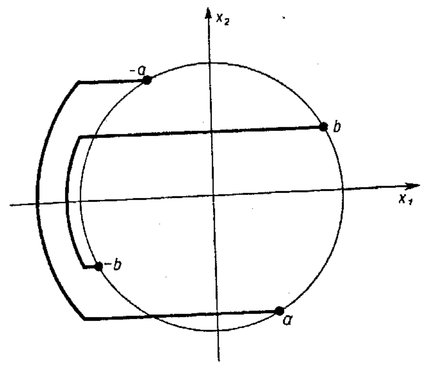

Fig. 2

It is easy to see that no more than three different Φ's intersect at one point. This proves 5.2.1.

In case $n = 2$ we also know the exact value of $N_0(n)$. Namely

5.2.2. $N_0(2) = 2$.

For let S and $(a, -a)$ be as in 5.2.1. We define

$$\Phi(a, -a) = (x_1, x_2)(-1, x_2) \cup \operatorname{Arc}\big((-1, x_2), (-1, -x_2)\big) \cup (-1, -x_2)(-x_1, -$$

where $\operatorname{Arc}(a, b)$ is that one of the arcs in which the pair (a, b) divid the circle with centre in O, which intersects x_1-axis in the negative part (fig. 2). Obviously no more than two different Φ's intersect, which proves 5.2.2.

The generalization of the above examples to higher dimensions is possible, but gives only $N(n) \leqslant 2n-1$ and $N_0(n) \leqslant 2n-2$. The question of the exact value of $N(n)$ and $N_0(n)$ for $n > 2$ is thus open. (It seems possible that $N_0(n) = N(n)-1$.)

5.3. Questions analogous to those from 5.1 and 5.2 may be asked in connection with theorem 2. Observe that since the Möbius band cannot be imbedded in the Euclidean space, the inequality analogous to 5.1.2 is trivial.

In theorem 2 we have supposed that the mapping f maps the boundary B of the Möbius band homeomorphically onto S^{n-1}. If we suppose only that f induces an arbitrary mapping $B \to S^{n-1}$ the theorem is no longer true, as may be shown by easy examples. But it seems that theorem 2 is still true if we suppose only that $B \to S^{n-1}$ has the degree 1.

5.4. In the example of 5.2.1 all points of Q^n were of finite order, but the set of points of order $\geqslant 3$ was infinite. The family of diameters of a circle give an example of a family where only one point of the field is of the order $\geqslant 3$, but in that case it is of infinite order. These facts are not accidental, as is shown by the following theorem:

5.4.1. *Let $\mathcal{F}$ be an acyclic prolongation of the antipodal family $\mathcal{F}_1$ and suppose that all points of Q^n are of finite order with respect to $\mathcal{F}$. Then the set of points which are of order $\geqslant 3$ is infinite, moreover, dimension of its closure is not smaller than $n-1$.*

(It seems that, in fact, it always contains interior points, i. e. is of dimension n).

5.4.1 is a strengthening of theorem 1. A corresponding strengthening of theorem 2 is also true. The proof of both may be obtained by small changes in the proof of the main lemma. We will outline them.

Instead of 4.2.2 we suppose that every axis intersects M at a finite number of points and that there exists a subset $A = \bar{A} \subset Q^n$ such that

$\dim A \leqslant n-2$ and for every $a \in Q^n - A$ the axis through a intersects M_0 at two points at most. This will be the "new version" of 4.2.2.

In 4.2.3 - 4.2.5 no change is needed. The argument given to prove 4.3.1 remains valid but only for points of $Q^n - A$. Suppose now that a point $a \in A$ and that the axis through a intersects M_0 at two points at least, say b_1, b_2.

We prove first that there are two sequences $\{b_1^m\}$ and $\{b_2^m\}$ such that $b_i^m \in M_0$, $\lim b_i^m = b_i$ and $a_i^m = q(b_i^m) \in Q^n - A$. It may easily be shown, using 6.2, that $\dim_p M_0 = n$ for every $p \in M_0$. Then let $\{U_1^m\}$, $\{U_2^m\}$ be two sequences of neighbourhoods in M_0 of b_1, b_2 respectively, such that $\delta[U_i^m] \to 0$. By the remark above and by the Hurewicz theorem ([4], p. 91) we have

$$\dim q(U_i^m) \geqslant \dim U_i^m = n,$$

whence $q(U_i^m) - A \neq 0$. Then choose $a_i^m \in q(U_i^m) - A$ and let b_i^m be any point of the intersection of the axis through a_i^m with U_i^m. Obviously the sequences $\{b_i^m\}$ $(i = 1, 2)$ satisfy the conditions imposed.

Now since $\dim A \leqslant n-2$ and $\lim a_1^m = a = \lim a_2^m$, there exists a sequence of arcs $L^m \subset Q^n - A$ such that $a_1^m, a_2^m \in L^m$ and $\delta[L^m] \to 0$. But the mapping q is 1-1 on $q^{-1}(Q^n - A)$; therefore the arcs L^m may be "lifted", i. e. there exist arcs $N^m \subset M_0$ such that $q(N^m) = L^m$. It is easy to see that $b_i^m \in N^m$ for $i = 1, 2$ and $m = 1, 2, 3, \ldots$ Now the sequence N^m contains a convergent subsequence, its limit N is then contained in M_0, contains b_1 and b_2, is connected and $q(N) = a$. But this implies that N is contained in the axis through a; therefore the intersection of this axis with M_0 is infinite, which contradicts the "new version" of 4.2.2 and completes the proof of 4.3.1 in the "old version".

Having 4.3.1 we prove 4.3.2 as before, the contradiction which follows proves that the "new version" of 4.2.2 is false, i. e. that if every axis intersects M_0 at a finite number of points, the closure of the set of points through which pass the axes intersecting M_0 at more than two points is of dimension at least $n-1$. This is the generalization of the main lemma which is needed to prove 5.4.1 and the corresponding generalization of theorem 2.

5.5. Let us remark that while in the main lemma the prolongations of an arbitrary family induced by an involution without fixed points are considered, theorem 1 relates only to prolongations of the antipodal family. Nevertheless it is true also in the general case, as shown in [6].

6. We recall in a form suitable for our purposes some notions from algebraical topology. The homology theory is that of Čech. For further details see [3].

Let (X, M) be a compact pair. Let $a \in H_k(M)$. We will say that a *bounds in* X if $i_* a = 0 \in H_k(X)$, where $i\colon M \to X$ is the inclusion mapping. We will say that a *bounds irreducibly in* X if a bounds in X and for every compact set M' such that $M \subset M' \subset X$ if a bounds in M' then $M' = X$.

6.1. *If a bounds in X there exists a compact space $M' \subset X$ in which a bounds irreducibly.*

The continuity of the Čech theory implies that the Brouwer reduction theorem (see [4], p. 161 and [3], Chap. X) may be applied. This proves 6.1.

6.2. *Suppose that $a \in H_{k-1}(M)$ bounds irreducibly in X. Let U be an open subset of X disjoint with M and let $F = \mathrm{Fr}(U)$. Then there exists an element $\beta \in H_{k-1}(F)$ which bounds irreducibly in $\overline{U}$ and for which $m_1 a - m_2 \beta = 0 \in H_{k-1}(X-U)$, where m_1, m_2 are homomorphisms induced by inclusions $M \to X-U$ and $F \to X-U$ respectively.*

Consider the following diagram

$$\begin{array}{ccccc}
H_k(X, M) & \xrightarrow{k_1} & H_k(X, X-U) & \xleftarrow{k_2} & H_k(\overline{U}, F) \\
\downarrow \partial_1 & & \downarrow \partial_2 & & \downarrow \partial_3 \\
H_{k-1}(M) & \xrightarrow{m_1} & H_{k-1}(X-U) & \xleftarrow{m_2} & H_{k-1}(F) \\
& \searrow l_1 & \swarrow l_2 & & \downarrow l_3 \\
& & H_{k-1}\big((X-U) \cup \overline{V}\big) & \xleftarrow{l_4} & H_{k-1}(\overline{V})
\end{array}$$

where V is a subset of U such that $\overline{V} \supset F$, the homomorphisms ∂_i, $i = 1, 2, 3$, are boundary operators and all other homomorphisms are induced by inclusions.

The diagram is obviously commutative. Observe that k_2, being induced by a relative homeomorphism, is an isomorphism ([3], p. 266).

Let $a \in H_{k-1}(M)$. If a bounds in X then there exists $\xi \in H_k(X, M)$ such that $\partial_1 \xi = a$. Define $\beta \in H_{k-1}(F)$ by

$$\beta = \partial_3 k_2^{-1} k_1 \xi .$$

We say that β *satisfies* 6.2.

Observe first that, by commutativity, we have

$$m_1 a - m_2 \beta = m_1 \partial_1 \xi - m_2 \partial_3 k_2^{-1} k_1 \xi = 0 .$$

Suppose now that β bounds in $\overline{V}$, i. e. $l_3 \beta = 0$. Then

$$l_1 a = l_2 m_1 a = l_2 m_2 \beta = l_4 l_3 \beta = 0 \in H_{k-1}\big((X-U) \cup \overline{V}\big) .$$

Since a bounds irreducibly it follows that $(X-U) \cup \overline{V} = X$, which implies $\overline{U} = \overline{V}$. This completes the proof.

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