Equivalence of singular and Čech homology for ANR-s
Application to unicoherence

by

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In § 1 of this paper we consider (non-compact) absolute neighborhood retracts for metric spaces (abbreviated as ANR-s) and define a natural isomorphism between singular homology (cohomology) and Čech homology (cohomology) groups based on arbitrary open coverings [1].

In § 2 we state a necessary and sufficient condition (inferred from literature) for unicoherence of normal paracompact connected and locally connected spaces in terms of 1-dimensional Čech cohomology. In the case of connected ANR-s we obtain then a criterion for unicoherence in terms of singular homology. The application of this criterion to a class of function spaces gives (among other things) the answer to a problem of K. Borsuk.

All groups in the paper are assumed to be discrete.

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§ 1. Equivalence of singular and Čech homology for ANR-s

1. Let X be a topological space (no separation axiom is assumed), G a discrete Abelian group and p an integer, $p \geqslant 0$. We denote by $H_p(X; G; S)$ and $H^p(X; G; S)$ the singular homology and cohomology groups (over the coefficient group G) respectively, both taken with

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(1) The equivalence of both theories was well-known for finite polyhedra. An extension to locally finite polyhedra can be found in the paper [5] by C. H. Dowker (8.1, p. 565 and 8.2, p. 566). For the definition of locally finite polyhedra see for instance [13], p. 390; for a proof that these polyhedra are ANR-s see Corollary 3.5, p. 394 of the same paper. Recently J. Dugundji in [8] (Theorem 2.1, p. 41) and Y. Kodama in [14] found proofs of the equivalence in question for arbitrary ANR-s. In distinction from our proof, which is direct, the proofs of these authors reduce the general case to the case of infinite polyhedra. In the case of cohomology (not homology) the equivalence can be considered also as a consequence of Cartan's uniqueness theorem for cohomology with coefficients in sheaves (H. Cartan, Séminaire 1950/51, Cohomologie des groupes, suite spectrale, faisceaux, N° 16).

discrete topology. $H_p(X; G; \check{C})$ and $H^p(X; G; \check{C})$ will denote Čech homology and cohomology groups (over G) based on *arbitrary open coverings* (not necessarily finite) and taken discrete (for a definition see [11], p. 237 or [6], p. 90-91). We establish an isomorphism between S and $\check{C}$ groups by interference of Vietoris homology groups $H_p(X; G; V)$ and cohomology groups $H^p(X; G; V)$, taken in the sense of [6] (p. 90-91). For the convenience of the reader we reproduce here the definition of these groups.

Let $\omega = \{U_i\}$ be an arbitrary open covering of X. A family $\mu = \{M_j\}$ of subsets of X is said to be *inscribed* in ω if each M_j is contained in a U_i. The Vietoris complex V_ω is a simplicial complex whose vertices are all points of X; a finite set of vertices forms a simplex if and only if it is contained in a $U_i \in \omega$. If $\omega < \omega'$, i. e. if ω' is a refinement of ω, $\pi_{\omega'\omega}$: $V_{\omega'} \to V_\omega$ is the injection induced by $V_{\omega'} \subseteq V_\omega$. $\pi_{\omega'\omega}$ induces the homomorphism $\pi_{*\omega'\omega}$: $H_p(V_{\omega'}; G) \to H_p(V_\omega; G)$ of homology groups and the homomorphism $\pi^*_{\omega'\omega}$: $H^p(V_\omega; G) \to H^p(V_{\omega'}; G)$ of cohomology groups. These groups and homomorphisms form an inverse and a direct system of groups when ω runs through the set Ω of all open coverings of X ordered by $<$. The groups $H_p(X; G; V)$ and $H^p(X; G; V)$ are defined as the inverse and the direct limit of these systems [2].

We state now the main result of this section.

THEOREM 1. *Let A be an* ANR *and G a discrete Abelian group. There exist natural isomorphies*

$$H_p(A; G; S) \approx H_p(A; G; V) \approx H_p(A; G; \check{C}) , \tag{1}$$

$$H^p(A; G; S) \approx H^p(A; G; V) \approx H^p(A; G; \check{C}) . \tag{2}$$

The proof is needed only for the first isomorphy of (1) and (2), since the second isomorphy has already been proved by C. H. Dowker for all topological spaces ([6], Theorem 2 and 2a, p. 91, Lemma 7, p. 91 and Lemma 7a, p. 93).

2. In the proof of Theorem 1 we shall need a notion of barycentric subdivision of a Euclidean complex K *with respect to a closed subcomplex* L; the resulting complex will be denoted by $K^1 \bmod L$. Its vertices are, by definition, all vertices of L and all barycenters of the simplexes belonging to $K \setminus L$. Let $\sigma_0, \sigma_1, \ldots, \sigma_n$ be a sequence of simplexes from K, σ_i being a face of σ_{i+1}, and let σ_m be the last member of the sequence still belonging to L. Then the vertices of σ_m and the barycenters of $\sigma_{m+1}, \ldots, \sigma_n$ form, by definition, a simplex of $K^1 \bmod L$ [3]. Only the sets of vertices

[2] $H_p(X; G; V)$ is a suitable generalization of the classic Vietoris definition; $H^p(X; G; V)$ is the Alexander cohomology group.

[3] In the case when all σ_i, $i = 0, \ldots, n$, belong to $K \setminus L$, their barycenters form a simplex of $K^1 \bmod L$ too.

obtained in this way are considered as simplexes of $K^1 \bmod L$. L is obviously a closed subcomplex of $K^1 \bmod L$ and the process can be iterated, yielding thus complexes $K^k \bmod L$.

LEMMA 1. *Let K be a finite Euclidean complex, L a closed subcomplex of K and ω an open covering of the polyhedron $|K|$. If the closed simplexes of L are inscribed in ω, there exists an integer k such that the closed simplexes of $K^k \bmod L$ are also inscribed in ω.*

The proof rests upon the following statement:

(a) *Let σ be a Euclidean simplex, τ a face of σ and U an open set of $\|\sigma\|$ containing the (closed) carrier $\|\tau\|$ of τ. For a sufficiently large k the closure of the star* $\mathrm{St}(\tau; |\sigma|^k \bmod |\tau|)$ [4] *belongs to U.*

Proof of (a). Let $\sigma = (a_0, \ldots, a_p, b_0, \ldots, b_q)$, $\tau = (a_0, \ldots, a_p)$, $n = p+q$. If a and c are barycenters of τ and σ respectively, then an elementary computation (in vector notations) yields

$$c - a = \frac{1}{n+1} \sum_{j=1}^{q} (b_j - a) . \tag{1}$$

Hence, if d denotes the maximum of the distances $\|b_j - a\|$, $j \in \{1, \ldots, q\}$, we obtain

$$\|c - a\| \leqslant \frac{q}{n+1}\, d \leqslant \frac{n}{n+1}\, d . \tag{2}$$

(2) is a fortiori true for barycenters c of simplexes $\sigma' \in \mathrm{St}(\tau; |\sigma|)$. Since the simplexes of $\mathrm{St}(\tau; |\sigma|^k \bmod |\tau|)$ are of the form

$$\sigma_r^k = (a_0, \ldots, a_p, c_{r1}^k, \ldots, c_{rq(r,k)}^k) , \qquad q(r,k) \leqslant n , \tag{3}$$

c_{ri}^k being barycenters of simplexes $\sigma_r^{k-1} \in \mathrm{St}(\tau; |\sigma|^{k-1} \bmod |\tau|)$, the application of (2) yields

$$\|c_{ri}^k - a\| \leqslant \frac{n}{n+1} \max_{ri} \|c_{ri}^{k-1} - a\| \leqslant \left(\frac{n}{n+1}\right)^2 \max_{ri} \|c_{ri}^{k-2} - a\| \leqslant \ldots \leqslant \left(\frac{n}{n+1}\right)^k d . \tag{4}$$

We infer from (4) that, for k sufficiently large, all c_{ri}^k will be arbitrarily close to $a \in \|\tau\|$. Thus there is a k for which all (closed) simplexes (3) belong to $U \supseteq \|\tau\|$. This proves the assertion (a).

Proof of Lemma 1 is now carried through by induction following the integer $\dim L$. For $\dim L = -1$, i. e. in the case when L is empty,

[4] If σ is a simplex of K, then $\mathrm{St}(\sigma; K)$ denotes the open star of σ with respect to K; $|\sigma|$ denotes the closure of σ.

the statement coincides with the well-known fact that iterated (absolute) barycentric subdivisions reduce the mesh of a complex under any given positive number, hence also under the Lebesgue number belonging to ω (see for instance [11], Lemma 6.5, p. 63 and Lemma 7.5, p. 65). We now assume Lemma 1 true for $\dim L < p$, $p \geqslant 0$, and prove it for $\dim L = p$. An arbitrary p-dimensional simplex $\tau \in L$ is the only p-dimensional simplex of $L \cap |\sigma_r|$ for any $\sigma_r \in \mathrm{St}(\tau; K^1 \mathrm{mod} L)$. Therefore, the effect of further subdivisions of K modulo L coincides in $|\sigma_r|$ with the effect of the subdivisions of $|\sigma_r|$ modulo $|\tau|$. This fact enables us to apply the statement (a) to $|\sigma_r|$, $|\tau|$ and $U \cap \|\sigma_r\|$, where $U \in \omega$, $U \supseteq \|\tau\|$. If $k_r(\tau)$ is chosen in accordance with (a), then for $k(\tau) = 1 + \max_r k_r(\tau)$ we obviously get

$$|\mathrm{Cl}\,\mathrm{St}(\tau; K^{k(\tau)} \mathrm{mod} L)| \subseteq U \in \omega . \tag{5}$$

Finally, for $k = \max_\tau k(\tau)$, τ running through the set $L^p \setminus L^{p-1}$ of all p-dimensional simplexes of L, we find that the sets $|\mathrm{Cl}\,\mathrm{St}(\tau; K^k \mathrm{mod} L)|$, $\tau \in L^p \setminus L^{p-1}$, are inscribed in ω. Removing all the simplexes belonging to these (open) stars, we get a closed complex K_1 with $\dim(K_1 \cap L) < p$. From the assumption of induction it follows now that continuing the process of repeated subdivisions modulo L we shall obtain, after a finite number of steps, a complex inscribed in ω. This ends the proof of Lemma 1.

The subdivision $K^1 \mathrm{mod}\, L$ induces a homomorphism Sd of the group $\mathrm{C}(K)$ of (ordered) chains of K into the group $\mathrm{C}(K^1 \mathrm{mod} L)$ of chains of $K^1 \mathrm{mod} L$; Sd is defined by induction as follows:

$$\mathrm{Sd}(\sigma^0) = \sigma^0 . \tag{6}$$

If $\sigma^p \in L$, then

$$\mathrm{Sd}(\sigma^p) = \sigma^p , \tag{7}$$

otherwise

$$\mathrm{Sd}(\sigma^p) = b_\sigma(\mathrm{Sd}\,\partial\sigma^p) , \tag{8}$$

$b_\sigma(\mathrm{Sd}\,\partial\sigma^p)$ denoting the join of $\mathrm{Sd}\,\partial\sigma^p$ with the barycenter b_σ of σ^p. Obviously Sd commutes with ∂. Applying Sd subsequently we get Sd^k: $\mathrm{C}(K) \to \mathrm{C}(K^k \mathrm{mod} L)$.

Remark 1. If $x^p = \sum g_i \sigma_i^p$ is a chain (over G) belonging to the complex K and $L = |\partial x^p|$, then

$$\partial(\mathrm{Sd}\,x^p) = \mathrm{Sd}(\partial x^p) = \partial x^p . \tag{9}$$

3. Let X be a topological space, ω an open covering of X, S the singular complex of X, $S_\omega \subseteq S$ the subcomplex consisting only of singular simplexes whose (closed) carriers are inscribed in ω and η_ω: $\mathrm{C}(S_\omega) \to \mathrm{C}(S)$, the induced injection.

LEMMA 2. *There exists a chain mapping* ε_ω: $\mathrm{C}(S) \to \mathrm{C}(S_\omega)$ *with the following properties:*

$$\|\varepsilon_\omega s^p\| \subseteq \|s^p\|, \quad s^p \in S , \tag{1}$$

$$\varepsilon_\omega \eta_\omega = 1 , \tag{2}$$

$$\eta_\omega \varepsilon_\omega \simeq 1 , \tag{3}$$

the sign $\simeq$ *denoting chain homotopy.*

If ω *and* ω' *are two open coverings of* X, *there exists a chain transformation* $\Delta_{\omega\omega'}$: $\mathrm{C}(S) \to \mathrm{C}(S_{\omega \cup \omega'})$ *such that*

$$\partial\Delta_{\omega\omega'} + \Delta_{\omega\omega'}\partial = \eta^{\omega}_{\omega\cup\omega'}\varepsilon_\omega - \eta^{\omega'}_{\omega\cup\omega'}\varepsilon_{\omega'} , \tag{4}$$

$\eta^{\omega}_{\omega\cup\omega'}$ *denoting the injection induced by* $S_\omega \subseteq S_{\omega\cup\omega'}$ (5).

Proof. We define ε_ω by means of a chain mapping α_ω, assigning to every singular simplex $s^p = (\sigma^p, \varphi)$ a chain of a simplicial subdivision of $|\sigma^p|$, such that $|\alpha_\omega s^p|$ coincides with this subdivision (σ^p is a Euclidean ordered simplex and φ a map of $\|\sigma^p\|$ into X). ε_ω is then given by

$$\varepsilon_\omega s^p = \varepsilon_\omega(\sigma^p, \varphi) = \varphi(\alpha_\omega s^p) . \tag{5}$$

As to α_ω, it is defined by induction starting with

$$\alpha_\omega s^0 = \alpha_\omega(\sigma^0, \varphi) = \sigma^0 . \tag{6}$$

If $s^p = (\sigma^p, \varphi)$ and b is the barycenter of σ^p, we consider the join $b(\alpha_\omega\partial s^p)$; $\alpha_\omega\partial s^p$ is by supposition a chain of a subdivision of $|\partial\sigma^p|$ and obviously

$$\partial b(\alpha_\omega\partial s^p) = \alpha_\omega\partial s^p . \tag{7}$$

We take now the kth barycentric subdivision of $|b\alpha_\omega\partial s^p| \bmod |\alpha_\omega\partial s^p|$, k being the least integer for which the subdivision is inscribed in the covering $\varphi^{-1}(\omega)$. According to Lemma 1, k exists. We put finally

$$\alpha_\omega s^p = \mathrm{Sd}^k b(\alpha_\omega\partial s^p) . \tag{8}$$

According to Remark 1 and (7) we have

$$\partial\alpha_\omega s^p = \partial b(\alpha_\omega\partial s^p) = \alpha_\omega\partial s^p . \tag{9}$$

From (5) and (9) we infer that

$$\partial\varepsilon_\omega s^p = \varepsilon_\omega\partial s^p \tag{10}$$

holds, showing that ε_ω is a chain mapping. $|\varepsilon_\omega s^p|$ is obviously inscribed in ω. Furthermore, it is clear that ε_ω satisfies (1) and (2). As to (3), it follows easily from the last statement of the lemma. Indeed, if $\omega < \omega'$,

(5) Compare this lemma with Theorem 8.2, p. 197 of [11].

then $\omega < \omega \cup \omega'$ and thus $S_{\omega\cup\omega'} = S_\omega$. Therefore, $\eta^{\omega}_{\omega\cup\omega'}$ reduces to identity 1 and $\eta^{\omega'}_{\omega\cup\omega'}$ to the injection $\eta_{\omega'\omega}$: $C(S_{\omega'}) \to C(S_\omega)$. Thus (4) goes over into

(11) $$\partial\Delta_{\omega\omega'} + \Delta_{\omega\omega'}\partial = \varepsilon_\omega - \eta_{\omega'\omega}\varepsilon_{\omega'} .$$

In the special case when $X \in \omega$, $\eta_{\omega'\omega}$ reduces to $\eta_{\omega'}$ and ε_ω to 1. Leaving out the index ω we get

(12) $$\partial\Delta_{\omega'} + \Delta_{\omega'}\partial = 1 - \eta_{\omega'}\varepsilon_{\omega'} .$$

Proof of (4). To every $s^p = (\sigma^p, \varphi) \in S$ we assign a chain $\beta_{\omega\omega'}s^p$ of the following simplicial subdivision of the prism $|\sigma^p| \times I$, $I = [0, 1]$. On the lower base the subdivision coincides with $|\alpha_{\omega'}s^p|$, on the upper base it coincides with $|\alpha_\omega s^p|$; we assume further that the other faces of the boundary of the prism are already subdivided as prisms of lower dimensions. Finally, we take the join of the barycenter b of the prism with the given subdivision of the boundary of the prism. Let $\Phi(\tau, t)$: $\|\sigma^p\| \times I \to X$ be the mapping defined by

(13) $$\Phi(\tau, t) = \varphi(\tau) .$$

We now take the kth barycentric subdivision of the described simplicial subdivision of $|\sigma^p| \times I$ modulo the (already subdivided) boundary, k being the least integer for which the obtained complex is inscribed in $\Phi^{-1}(\omega \cup \omega')$. k exists according to Lemma 1.

Finally, we set

(14) $$\beta_{\omega\omega'}s^p = \mathrm{Sd}^k b(\alpha_\omega s^p - \alpha_{\omega'}s^p - \beta_{\omega\omega'}\partial s^p) .$$

It is not difficult to verify that

(15) $$\partial\beta_{\omega\omega'}s^p = \alpha_\omega s^p - \alpha_{\omega'}s^p - \beta_{\omega\omega'}\partial s^p .$$

$\Delta_{\omega\omega'}$ is now defined by

(16) $$\Delta_{\omega\omega'}s^p = \Phi\beta_{\omega\omega'}s^p$$

and (4) follows easily from (15) and (16). This ends the proof of Lemma 2.

Corollary 1. *The injection η_ω: $C(S_\omega) \to C(S)$ induces an isomorphism (onto) $\eta_{*\omega}$ of the homology groups $H_p(S_\omega; G)$ and $H_p(S; G)$ and an isomorphism (onto) η^*_ω of the cohomology groups $H^p(S; G)$ and $H^p(S_\omega; G)$.*

4. ANR-s for metrizable spaces can be defined as neighborhood retracts of convex sets of real normed vector spaces. The equivalence of this definition and the usual one follows from the results of M. Wojdysławski ([19], 7, p. 186) and J. Dugundji ([7], p. 363). In the sequel A will always denote an ANR, imbedded in a convex set K of a vector

space L, N will denote a neighborhood of retraction of A in K and ϑ: $N \to A$ the retraction map.

Special open coverings ω' of A, designated for brevity as "convex" coverings, will play an important role in the subsequent proofs. ω' is a "convex" covering if its members are of the form $U = A \cap W$, W being a convex (in L) open subset of N. The family of all "convex" coverings of A will be denoted by Ω' in distinction from the family Ω of all open coverings of A.

If M is a subset of A and $[M]$ denotes the convex hull of M (in L), then we have

Lemma 3. *Every open covering $\omega = \{U_i\} \in \Omega$ of A admits a "convex" refinement $\omega' = \{U'_j\} \in \Omega'$ such that for every set $M \subseteq A$, inscribed in ω', the convex hull $[M]$ is contained in N and $\vartheta[M]$ is inscribed in ω.*

Proof. Let $\{W_j\}$ be a refinement of the open covering $\vartheta^{-1}(\omega) = \{\vartheta^{-1}(U_i)\}$ of N, W_j being open spheroids [6] of K contained in N. Clearly, $\omega' = \{A \cap W_j\}$ belongs to Ω'. For every $A \cap W_j \in \omega'$ there is a $U_i \in \omega$ such that $W_j \subseteq \vartheta^{-1}(U_i)$. Thus $A \cap W_j \subseteq A \cap \vartheta^{-1}(U_i) = U_i$, showing that $\omega < \omega'$. If the set M is inscribed in ω', for instance if $M \subseteq A \cap W_j$, then $[M] \subseteq W_j \subseteq N$ and thus $\vartheta[M] \subseteq \vartheta(W_j) \subseteq U_i$, showing that $\vartheta[M]$ is inscribed in ω.

5. In this section we define several chain transformations and establish some of their properties to be used in section 6.

a. Let X be an arbitrary topological space and ω an open covering of X. The following relation defines a chain mapping μ_ω of S_ω into the Vietoris complex V_ω:

(1) $$\mu_\omega s^p = (\varphi(e_0), \ldots, \varphi(e_p)) ,$$

$s^p = (\sigma^p, \varphi)$, $\sigma^p = (e_0, \ldots, e_p)$.

b. If $\omega' \in \Omega'$ is a "convex" covering of A, we define a chain mapping $\lambda_{\omega'}$: $C(V_{\omega'}) \to C(S)$ as follows. Let v^p be a Vietoris simplex from $V_{\omega'}$. Choose $U' = A \cap W \in \omega'$ so that $\|v^p\| \subseteq U'$. Let $[v^p]$ denote the *linear* singular simplex spanned [7] by v^p. $[v^p]$ is contained in $W \subseteq N$ so that $\vartheta[v^p]$ is a well determined singular simplex of A. We now put

(2) $$\lambda_{\omega'}v^p = \vartheta[v^p] .$$

[6] A spheroid of K is a set W consisting of all $x \in K$ with $\|x - x_0\| < \varepsilon$, $x_0 \in K$; W is obviously a convex set.

[7] If $v^p = (a_0, \ldots, a_p)$ is an ordered Vietoris simplex and $\sigma^p = (e_0, \ldots, e_p)$ an ordered Euclidean simplex, $[v^p]$ is defined as (σ^p, α), where

$$\alpha\left(\sum_0^p t_i e_i\right) = \sum_0^p t_i a_i , \quad \sum_0^p t_i = 1 , \quad t_i \geqslant 0 .$$

Clearly

(3) $$\partial\lambda_{\omega'} = \lambda_{\omega'}\partial .$$

c. Every $\omega \in \Omega$ admits a refinement $\omega' \in \Omega'$ such that

(4) $$\mu_\omega \varepsilon_\omega \lambda_{\omega'} = \pi_{\omega'\omega} .$$

To prove this assertion we choose ω' in accordance with Lemma 3. If v^p belongs to $V_{\omega'}$ then $\lambda_{\omega'}v^p = \vartheta[v^p]$ is inscribed in ω, for the carrier of $[v^p]$ obviously coincides with the convex hull of $\|v^p\|$. (4) now follows from Lemma 2, (2).

d. We shall now define, for every $\omega' \in \Omega'$, a chain transformation $D_{\omega'}$: $C(S_{\omega'}) \to C(S)$ such that

(5) $$\partial D_{\omega'} + D_{\omega'}\partial = \lambda_{\omega'}\mu_{\omega'} - \eta_{\omega'} .$$

If $s^p = (\sigma^p, \varphi) \in S_{\omega'}$, consider the linear singular simplex $[\mu_{\omega'}s^p]$ spanned by $\mu_{\omega'}s^p$. Let

(6) $$[\mu_{\omega'}s^p] = (\sigma^p, \alpha) ,$$

α: $\|\sigma^p\| \to N$ and let

(7) $$\|s^p\| \subseteq U' = A \cap W \in \omega' ,$$

W being a convex set of N. If $\tau \in \|\sigma^p\|$, $t \in I$, $\varphi(\tau) + t\big(\alpha(\tau) - \varphi(\tau)\big)$ is a point of $W \subseteq N$, so that

(8) $$\Phi(\tau, t) = \vartheta\Big(\varphi(\tau) + t\big(\alpha(\tau) - \varphi(\tau)\big)\Big)$$

is a well determined mapping of $\|\sigma^p\| \times I$ into A. For $\sigma^p = (e_0, \ldots, e_p)$ denote by $\sigma^p \times I$ the Euclidean chain

$$\sum_{i=0}^{p}(-1)^i(e_0\times 0, \ldots, e_i\times 0, e_i\times 1, \ldots, e_p\times 1),$$

which obviously verifies

(9) $$\partial(\sigma^p \times I) = (\sigma^p \times 1) - (\sigma^p \times 0) - (\partial\sigma^p) \times I .$$

Finally, set

(10) $$D_{\omega'}s^p = (\sigma^p \times I, \Phi) .$$

From (10), (9), (8), (6), the definition of $\lambda_{\omega'}$ and the properties of ϑ it is easy to infer that (5) holds good.

Combining (5), (3.10) and (3.12) we obtain

(11) $$\partial(D_{\omega'}\varepsilon_{\omega'} - \Delta_{\omega'}) + (D_{\omega'}\varepsilon_{\omega'} - \Delta_{\omega'})\partial = \lambda_{\omega'}\mu_{\omega'}\varepsilon_{\omega'} - 1 ,$$

proving the important chain homotopy

(12) $$\lambda_{\omega'}\mu_{\omega'}\varepsilon_{\omega'} \simeq 1 .$$

Denoting generally, for a chain mapping τ, by $\tau_*(\tau^*)$ the induced homomorphisms between the homology (cohomology) groups, we can summarize the results of this section in the following

Lemma 4. *For every $\omega \in \Omega$ we have*

(13) $$\varepsilon_{*\omega}\eta_{*\omega} = \eta_{*\omega}\varepsilon_{*\omega} = 1 ,$$

(13′) $$\varepsilon^*_\omega\eta^*_\omega = \eta^*_\omega\varepsilon^*_\omega = 1 .$$

For every pair $\omega < \omega'$, $\omega \in \Omega$, $\omega' \in \Omega$, we have

(14) $$\mu_{*\omega}\varepsilon_{*\omega} = \pi_{*\omega'\omega}\mu_{*\omega'}\varepsilon_{*\omega'} ,$$

(14′) $$\varepsilon^*_\omega\mu^*_\omega = \varepsilon^*_{\omega'}\mu^*_{\omega'}\pi^*_{\omega'\omega} .$$

For every $\omega \in \Omega$ there exists an $\omega' \in \Omega'$, $\omega < \omega'$, such that

(15) $$\mu_{*\omega}\varepsilon_{*\omega}\lambda_{*\omega'} = \pi_{*\omega'\omega} ,$$

(15′) $$\lambda^*_{\omega'}\varepsilon^*_\omega\mu^*_\omega = \pi^*_{\omega'\omega} .$$

For every $\omega' \in \Omega'$ we have

(16) $$\lambda_{*\omega'}\mu_{*\omega'} = \eta_{*\omega'} ,$$

(16′) $$\mu^*_{\omega'}\lambda^*_{\omega'} = \eta^*_{\omega'} .$$

Proof. (13) and (13′) follow from (3.2) and (3.3). Applying μ_ω to (3.11) and using the obvious relation $\pi_{\omega'\omega}\mu_{\omega'} = \mu_\omega\eta_{\omega'\omega}$ we get (14) and (14′). (15) and (15′) are consequences of (4), while (16) and (16′) follow from (5).

6. Proof of Theorem 1. For reasons of brevity we denote in this section the groups $H_p(A; G; S)$, $H_p(A; G; V)$, $H_p(V_\omega; G)$, $H^p(A; G; S)$, $H^p(A; G; V)$ and $H^p(V_\omega; G)$ simply by H_S, H_V, H_ω, H^S, H^V and H^ω respectively.

a. Proof of (1.1). According to (5.14) the homomorphisms $\mu_{*\omega}\varepsilon_{*\omega}$: $H_S \to H_\omega$, $\omega \in \Omega$, induce a well-determined homomorphism ν_*: $H_S \to H_V = \varprojlim H_\omega$, defined by

(1) $$\nu_* h = \{\mu_{*\omega}\varepsilon_{*\omega}h\} ,$$

$h \in H_S$, while $\{\mu_{*\omega}\varepsilon_{*\omega}h\}$ is a "thread" of H_V. If

(2) $$\nu_* h = \{\mu_{*\omega}\varepsilon_{*\omega}h\} = 0 ,$$

then

(3) $$\mu_{*\omega}\varepsilon_{*\omega}h = 0 ,$$

for all $\omega' \in \Omega$, hence also for $\omega' \in \Omega'$. In this case (3) implies

(4) $$\lambda_{*\omega'}\mu_{*\omega'}\varepsilon_{*\omega'}h = 0 .$$

On the other hand, (16) and (13) imply $\lambda_{*\omega'}\mu_{*\omega'}\varepsilon_{*\omega'} = \eta_{*\omega'}\varepsilon_{*\omega'} = 1$, so that (4) goes into $h = 0$, proving the vanishing of the kernel of ν_*. To show that

$$\nu_*(H_S) = H_V\,, \tag{5}$$

take an arbitrary "thread" $\{h_\omega\} \in H_V$, $h_\omega \in H_\omega$, $\omega \in \Omega$. For each $\omega' \in \Omega'$ consider the class

$$h = \lambda_{*\omega'} h_{\omega'} \in H_S\,. \tag{6}$$

Any two "convex" coverings admit a common "convex" refinement (Lemma 3) and if $\omega' < \omega''$, $\omega' \in \Omega'$, $\omega'' \in \Omega'$, $\lambda_{\omega''} = \lambda_{\omega'}\pi_{\omega''\omega'}$. Therefore, all $\omega' \in \Omega'$ lead to the same element $h \in H_S$. To prove that $\nu_* h = \{h_\omega\}$ we have to show that

$$\mu_{*\omega}\varepsilon_{*\omega} h = h_\omega \tag{7}$$

for all $\omega \in \Omega$. According to Lemma 4 for any $\omega \in \Omega$ there is an $\omega' \in \Omega'$ such that (5.15) holds; (6) also holds good for this ω'. Hence, according to (15) and to the definition of a thread, $\mu_{*\omega}\varepsilon_{*\omega} h = \mu_{*\omega}\varepsilon_{*\omega}\lambda_{*\omega'} h_{\omega'} = \pi_{*\omega'\omega} h_{\omega'} = h_\omega$, proving (7) and the isomorphy $H_S \approx H_V$.

b. Proof of (1.2). $\varepsilon^*_\omega \mu^*_\omega$ is a homomorphism of H^ω into H^S. According to (5.14′), to all $h^\omega \in H^\omega$ belonging to the "bundle" $\{h^\omega\} \in H^V = \varinjlim H^\omega$, the same element

$$h = \nu^*\{h^\omega\} = \varepsilon^*_\omega \mu^*_\omega h^\omega \in H^S \tag{8}$$

is assigned. For an arbitrary $h \in H^S$ take an arbitrary $\omega' \in \Omega'$. $\lambda^*_{\omega'} h$ will be a well-defined element of $H^{\omega'}$. For the „bundle" $\{\lambda^*_{\omega'} h\} \in H^V$ we have by (8)

$$\nu^*\{\lambda^*_{\omega'} h\} = \varepsilon^*_{\omega'}\mu^*_{\omega'}\lambda^*_{\omega'} h\,, \tag{9}$$

while from (5.16′) and (5.13′) we infer that $\varepsilon^*_{\omega'}\mu^*_{\omega'}\lambda^*_{\omega'} = \varepsilon^*_{\omega'}\eta^*_{\omega'} = 1$. Thus (9) goes over into

$$\nu^*\{\lambda^*_{\omega'} h\} = h\,, \tag{10}$$

proving

$$\nu^*(H^V) = H^S\,. \tag{11}$$

Finally let

$$\nu^*\{h^\omega\} = \varepsilon^*_\omega \mu^*_\omega h^\omega = 0\,. \tag{12}$$

Choose an $\omega' \in \Omega'$ such that (5.15′) holds. Then (12) implies

$$0 = \lambda^*_{\omega'}\varepsilon^*_\omega \mu^*_\omega h^\omega = \pi^*_{\omega'\omega} h^\omega\,, \tag{13}$$

showing that $\{h^\omega\} = 0$. This proves the isomorphy $H^V \approx H^S$.

It is not difficult to verify that the homomorphisms ν_* and ν^* are natural.

Remark 2. Theorem 1 is not valid when the Čech homology and cohomology is based on *finite* open coverings. A well-known counterexample furnished by C. H. Dowker ([4], Theorem 9.6, p. 230) is the space C of real numbers. $H^1(C; J; S) = 0$, but the corresponding Čech group based on finite coverings is a group of power $2^{\aleph_0}$ (J is the group of integers). A reason for this fact consists in that finite open convex coverings fail to form a cofinal set in the set of all finite open coverings.

§ 2. Application to unicoherence

1. Let S_1 denote the unit circumference $|z| = 1$ in the plane of complex numbers. In the sequel (in accordance with [9], p. 63) we say for a space X that it has the property (b) if and only if every mapping $f\colon X \to S_1$ is of the form

$$f(x) = e^{i\varphi(x)}, \quad x \in X\,, \tag{1}$$

φ denoting a real single-valued mapping of X.

In 1936 S. Eilenberg found ([9], Theorem 3, p. 70) that for metrizable connected and locally connected spaces the property (b) is equivalent to unicoherence. Actually, his proof that unicoherence implies (b) is valid for arbitrary connected and locally connected spaces (no assumptions on metrizability). The proof of the other part of the equivalence ([9], Theorem 2, p. 69) can easily be modified (using the lemma of Urysohn) in order to become valid for connected normal spaces (no assumptions concerning separation of points is needed). The equivalence of unicoherence and the property (b) for connected locally connected normal T_1 spaces follows also from a more recent result of A. H. Stone concerning the degree of multicoherence ([17], Theorem 5, p. 472).

LEMMA 5. *Let X be a topological space. The mapping $f\colon X \to S_1$ is homotopic to the constant* 1 *if and only if f is of the form* (1).

Proof. If (1) holds, then

$$F(x, t) = e^{it\varphi(x)} \tag{2}$$

gives a continuous deformation of $f(x) = F(x, 1)$ into the constant 1, proving the sufficiency of the condition.

Suppose now that $f(x)$ is homotopic to the constant 1, i. e. that there is a mapping $F(x, t)\colon X \times I \to S_1$ with

$$F(x, 0) = 1, \quad F(x, 1) = f(x)\,. \tag{3}$$

For two non-diametral points $z_1, z_2 \in S_1$ we denote (following [9], p. 63) by $[z_1, z_2]$ the length of the shorter of the two arcs leading from z_1 to z_2, provided with the right sign. If for three points $z_1, z_2, z_3 \in S_1$

(4) $$\operatorname{diam}\{z_1, z_2, z_3\} < 1,$$

they belong to a half-circumference and thus

(5) $$[z_1, z_3] = [z_1, z_2] + [z_2, z_3].$$

Notice also the relation

(6) $$e^{i[z_1,z_2]} = z_2/z_1.$$

Resuming the proof, consider for a fixed $\xi \in X$ the mapping $F_\xi(t) = F(\xi, t)$ as a function of $t \in I$. I being compact, we can find a finite sequence

(7) $$0 = t_0 < t_1 < \ldots < t_n = 1$$

such that

(8) $$\operatorname{diam} F_\xi([t_j, t_{j+1}]) < 1, \quad j = 0, 1, 2, \ldots, n-1.$$

Let U_j be an open set of $X \times I$ containing all points (ξ, t) with $t \in [t_j, t_{j+1}]$ and such that

(9) $$\operatorname{diam} F(U_j) < 1.$$

Finally let U_ξ be an open set of X, $\xi \in U_\xi$, such that

(10) $$U_\xi \times [t_j, t_{j+1}] \subseteq U_j$$

and thus

(11) $$\operatorname{diam} F(U_\xi \times [t_j, t_{j+1}]) < 1.$$

We now define a real transformation Φ_ξ: $U_\xi \times I \to C$ by setting

(12) $$\Phi_\xi(x, t) = \sum_{j=0}^{p-1} [F(\xi, t_j), F(\xi, t_{j+1})] + [F(\xi, t_p), F(x, t)],$$

for $x \in U_\xi$, $t \in [t_p, t_{p+1}]$.

To prove that (12) is a single-valued function we need to show the uniqueness of the definition for $t = t_p$, i. e. to show that

(13) $$[F(\xi, t_{p-1}), F(x, t_p)] = [F(\xi, t_{p-1}), F(\xi, t_p)] + [F(\xi, t_p), F(x, t_p)].$$

Since (ξ, t_{p-1}), (ξ, t_p) and (x, t_p) belong to U_{p-1}, (9) yields $\operatorname{diam}\{F(\xi, t_{p-1}), F(\xi, t_p), F(x, t_p)\} < 1$, and thus (13) follows from (5). Furthermore, it is clear that $\Phi_\xi(x, t)$ is a mapping of $U_\xi \times I$, because it is continuous in the closed subsets $U_\xi \times [t_p, t_{p+1}]$. Applying (6) we see also that

(14) $$e^{i\Phi_\xi(x,t)} = \left(\prod_{j=0}^{p-1} \frac{F(\xi, t_{j+1})}{F(\xi, t_j)}\right) \cdot \frac{F(x, t)}{F(\xi, t_p)} = \frac{F(x, t)}{F(\xi, 0)} = F(x, t),$$

for $x \in U_\xi$, $t \in I$.

Finally,

(15) $$\Phi_\xi(x, 0) = [F(\xi, t_0), F(x, 0)] = [1, 1] = 0.$$

We shall show now that, for $x \in U_{\xi'} \cap U_{\xi''}$, $\xi', \xi'' \in X$,

(16) $$\Phi_{\xi'}(x, t) = \Phi_{\xi''}(x, t).$$

Indeed, by (14),

(17) $$e^{i\Phi_{\xi'}(x,t)} = F(x, t) = e^{i\Phi_{\xi''}(x,t)},$$

so that

(18) $$e^{i(\Phi_{\xi'} - \Phi_{\xi''})} = 1.$$

Since $I_x = \{(x, t) | t \in I\}$ is connected and $\Phi_{\xi'} - \Phi_{\xi''}$ is continuous on I_x, there is a constant k such that

(19) $$\Phi_{\xi'}(x, t) - \Phi_{\xi''}(x, t) = 2k\pi.$$

It follows from (15) that $k = 0$, proving thus (16).

We now define a (single-valued) transformation $\Phi(x, t)$: $X \times I \to C$ by relation

(20) $$\Phi(x, t) = \Phi_x(x, t).$$

According to (16), $\Phi(x, t)$ coincides with the continuous transformation $\Phi_\xi(x, t)$ in the open set $U_\xi \times I$, so that Φ is continuous at every point $(\xi, t) \in X \times I$. Moreover, we infer from (14) and (15) that

(21) $$e^{i\Phi(x,t)} = F(x, t),$$

(22) $$\Phi(x, 0) = 0.$$

The real mapping

(23) $$\varphi(x) = \Phi(x, 1)$$

obviously satisfies (1) and the proof of Lemma 5 is completed.

Remark 3. For metrizable X, Lemma 5 has already been proved by S. Eilenberg ([9], Theorem 1, p. 68); the present author is indebted to Professor A. H. Stone for calling his attention (in a letter) to the fact that Lemma 5 is valid for all X.

Finally, it follows from a theorem of C. H. Dowker ([4], Theorem 8.1, p. 226) that a paracompact normal space X admits a mapping f: $X \to S_1$, not homotopic to 1, if and only if $H^1(X; J; \check{C}) \neq 0$, J being the group of integers.

Combining all these results we obtain

THEOREM 2. *Let X be a normal paracompact connected and locally connected space. In order that X be unicoherent it is necessary and sufficient that*

(24) $$H^1(X; J; \check{C}) = 0.$$

In particular, the theorem can be applied to all connected locally connected metrizable spaces.

2. In the case of connected ANR-s the assumption of local connectedness is always fulfilled. Moreover, Theorem 1 enables us to replace $H^1(X; J; \check{C})$ in Theorem 2 by $H^1(X; J; S)$.

Finally, singular complexes belong to closure-finite complexes, which enables us to express cohomology groups over arbitrary coefficient groups in terms of integral homology groups. Specializing a general formula due to S. Eilenberg and S. MacLane ([10], p. 813; see also [11], p. 161) to dimension 1 we obtain (8) for arcwise connected spaces

$$H^1(X; J; S) \approx \operatorname{Hom}\big(H_1(X; J; S), J\big). \tag{1}$$

Thus for ANR-s Theorem 2 goes over into

THEOREM 3. *In order that a connected absolute neighborhood retract A be unicoherent it is necessary and sufficient that*

$$H^1(A; J; S) \approx \operatorname{Hom}\big(H_1(A; J; S), J\big) = 0. \tag{2}$$

In connection with this theorem there naturally arises the question of characterizing the Abelian groups G admitting non-trivial homomorphisms into J.

LEMMA 6. *An Abelian group G admits non-trivial homomorphisms into J if and only if J is a direct summand of G.*

Proof. The sufficiency of the condition is obvious. To prove the necessity suppose that $h: G \to J$ is a non-trivial homomorphism. We may suppose (without loss of generality) that $h(G) = J$, because all subgroups of J, excepting $\{0\}$, are isomorphic to J. Now choose $g \in G$ so that $h(g) = 1$. Let A be the subgroup of G generated by g and let B be the kernel of h: $G \to J$. It is easy to verify that $G = A \oplus B$ and that $A \approx J$.

We compare now the criterion of Theorem 3 with the classical criterion of K. Borsuk ([1], Korollar 1, p. 230) and E. Čech ([3], Theorem A, p. 232 and Theorem B, p. 233) asserting (in equivalent formulation) that a Peano continuum X is unicoherent if and only if

$$H_1(X; R; \check{C}) = 0, \tag{3}$$

R denoting the group of rationals.

For connected ANR-s (3) is equivalent to

$$H_1(A; R; S) = 0 \tag{4}$$

(8) Relation (1) can easily be proved directly. $\operatorname{Hom}(A, B)$ denotes the group of all homomorphisms of A into B.

and implies (2). Indeed, in the contrary case Lemma 6 would yield a decomposition in a direct sum of the form

$$H_1(A; J; S) \approx J \oplus B. \tag{5}$$

On the other hand, specializing another universal coefficient formula ([11], p. 161) to dimension 1, we obtain for arcwise connected spaces X and arbitrary coefficient group G the relation

$$H_1(X; G; S) \approx H_1(X; J; S) \otimes G. \tag{6}$$

Thus, for $G = R$ we would have

$$H_1(A; R; S) \approx H_1(A; J; S) \otimes R \approx R \oplus (B \otimes R) \neq 0, \tag{7}$$

in contradiction with (4). So (3) implies the unicoherence of connected ANR-s. That the converse is not true can be shown on a connected locally finite (2-dimensional) polyhedron Π, having the property

$$\pi_1(\Pi) \approx R \neq 0. \tag{8}$$

The existence of such a polyhedron (which is an ANR) follows from a result of J. H. C. Whitehead ([18], Theorem on p. 261). Since $\pi_1(\Pi) \approx H_1(\Pi; J; S)$ (π_1 is here Abelian), (8) implies

$$H_1(\Pi; J; S) \approx R. \tag{9}$$

Obviously

$$\operatorname{Hom}(R, J) = 0 \tag{10}$$

and Π is unicoherent (Theorem 3). Nevertheless, (6) yields

$$H_1(\Pi; R; S) \approx H_1(\Pi; J; S) \otimes R \approx R \otimes R \approx R \neq 0, \tag{11}$$

so that the Borsuk-Čech condition (3) is not fulfilled.

Remark 4. If C denotes the group of reals and P the group of reals mod 1, then

$$H_1(\Pi; P; \check{C}) \approx H_1(\Pi; P; S) \approx H_1(\Pi; J; S) \otimes P \approx R \otimes P \approx C/R \neq 0. \tag{12}$$

On the other hand

$$H^1(\Pi; J; \check{C}) \approx H^1(\Pi; J; S) \approx \operatorname{Hom}\big(H_1(\Pi; J; S), J\big) \approx \operatorname{Hom}(R, J) = 0. \tag{13}$$

So $H^1(X; J; \check{C}) = 0$ does not always imply $H_1(X; P; \check{C}) = 0$ (9).

3. In this section we consider function spaces $\langle X, S_m \rangle$ of all mappings f of the (metrizable) compactum X into the m-sphere S_m, provided with the usual topology of uniform convergence (see for instance [16],

(9) It can be shown that $H^1(X; P; \check{C}) = 0$ always implies $H_1(X; J; \check{C}) = 0$.

p. 171). $\langle X, S_m\rangle$ is a neighborhood retract of the real normed space $\langle X, E^{m+1}\rangle$, E^{m+1} denoting the $(m+1)$-dimensional Euclidean space (see for instance [16], Theorem 2′, p. 173). Therefore, $\langle X, S_m\rangle$ is an ANR (see § 1, 4) (10).

We assume further that $\dim X = k < m$. This assumption secures the connectedness of the space $\langle X, S_m\rangle$ (any two mappings $f \in \langle X, S_m\rangle$ are homotopic under these conditions and thus can be connected by an arc in $\langle X, S_m\rangle$).

The question of unicoherence of these spaces is answered by

THEOREM 4. *Let X be a (metrizable) compactum, $\dim X = k < m$. If $k < m-1$, $\langle X, S_m\rangle$ is unicoherent. If $k = m-1$, $\langle X, S_m\rangle$ is unicoherent if and only if*

$$\mathrm{Hom}\big(H^k(X; J; \check{C}), J\big) = 0\,, \tag{1}$$

i. e. if and only if $H^k(X; J; \check{C})$ does not admit J as a direct summand.

Proof. The present author has shown ([16], Theorem 23, p. 238 and Lemma 9, p. 230 or [15], Theorem 5, p. 2216) that

$$H_p(\langle X, S_m\rangle; J; S) = 0, \quad \text{for} \quad 0 < p < m-k\,, \tag{2}$$

$$H_p(\langle X, S_m\rangle; J; S) \approx H^k(X; J; \check{C}), \quad \text{for} \quad p = m-k\,. \tag{3}$$

Specializing to $p = 1$ we get

$$H_1(\langle X, S_m\rangle; J; S) = 0, \quad \text{for} \quad k < m-1\,, \tag{4}$$

$$H_1(\langle X, S_m\rangle; J; S) \approx H^k(X; J; \check{C}), \quad \text{for} \quad k = m-1\,. \tag{5}$$

Theorem 4 now follows from Theorem 3 and Lemma 6.

In the particular case when $X = S_1$, $m = 2$, we have $H^1(S_1; J; \check{C}) \approx J$ and thus $\mathrm{Hom}\big(H^1(S_1; J; \check{C}), J\big) \approx \mathrm{Hom}(J, J) \approx J \neq 0$. Therefore, the space $\langle S_1, S_2\rangle$ is not unicoherent. This answers a problem of K. Borsuk ([2], Problem 3, p. 37) (11).

(10) Generally, if X is a bicompactum and R an ANR, $\langle X, R\rangle$ is an ANR too. The author is indebted for this remark to Professor S. T. Hu.

(11) However, the problem was already settled in 1956 by T. Ganea [12], who found that the degree of multicoherence $r(\langle S_1, S_2\rangle) = 1$.

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Added remark. In a letter of August 12, 1957, Professor T. Ganea kindly pointed out to the author another method of proof for theorem 4, based on his results concerning the degree of multicoherence of a locally connected space [12], as well as on the fact that for locally simply connected spaces the group of covering maps coincides with the fundamental group π_1 and finally on the result of J. C. Moore (Fund. Math. 43 (1956), p. 196), according to which $\pi_i(\langle X, S_m\rangle) \approx H_i(\langle X, S_m\rangle)$ for $0 \leqslant i \leqslant m-k$. It follows in addition that the degree of multicoherence of $\langle X, S_m\rangle$ can be only 0 or 1, so that theorem 4 actually determines also the degree of multicoherence of $\langle X, S_m\rangle$.