On a family of power $\mathfrak{c}$ consisting of $\mathfrak{R}$-uncomparable dendrites

by

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K. Borsuk has introduced the notion of $\mathfrak{R}$-type ([1], p. 322) and proved that the set of different $\mathfrak{R}$-types among 2-dimensional compacta has power $\mathfrak{c}$. It is a consequence of the theorem, proved in that paper ([1], p. 327), on the existence of the family of power $\mathfrak{c}$ consisting of $\mathfrak{R}$-uncomparable 2-dimensional AR-sets. The purpose of the present paper is to prove that even among 1-dimensional AR-sets (i. e. among dendrites (¹)) there exist $\mathfrak{c}$ different $\mathfrak{R}$-types. More strictly we shall prove that

On the plane E^2 there exists a family of power $\mathfrak{c}$ consisting of $\mathfrak{R}$-uncomparable dendrites.

It is easy to construct such a family having power n, where n is a natural number. (See [1], p. 322.) Beginning the construction of the family of power $\mathfrak{c}$ we shall prove first that there exists an

1. $\mathfrak{R}$-decreasing sequence of dendrites. Let d_0 be a closed segment on the plane E^2 with the endpoints $(0, 0)$, $(0, 1)$ and let $T(d_0)$ denote the dendrite (Fig. 1) consisting of the points of d_0 and those points $(x, y) \in E^2$ for which

$$x = \frac{2i-1}{2^j}, \quad 0 < y \leqslant \frac{1}{2^{j+1}} \quad \text{where} \quad i = 1, 2, \ldots, 2^{j-1};\ j = 1, 2, \ldots$$

while $K(d_0)$ denotes the dendrite (Fig. 2) consisting of the points of d_0 and those points $(x, y) \in E^2$ for which

$$x = \frac{2i-1}{2^j}, \quad 0 < |y| \leqslant \frac{1}{2^{j+1}} \quad \text{where} \quad i = 1, 2, \ldots, 2^{j-1};\ j = 1, 2, \ldots$$

If $d \subset E^2$ is a closed segment parallel either to the axis of abscissae or to the axis of ordinates and φ denotes the affine mapping preserving

(¹) A dendrite is a locally conected continuum containing no simple closed curve. See for example [2], p. 224.

the orientation of E^2 and setting d on d_0 in such a manner that the angle between the vectors $[(0,0),(0,1)]$ and $[\varphi^{-1}(0,0),\varphi^{-1}(0,1)]$ belongs to the interval $\langle -\frac{1}{2}\pi, 0\rangle$, then we define $T(d)=\varphi^{-1}\big(T(d_0)\big)$, $K(d)=\varphi^{-1}\big(K(d_0)\big)$. Now let A be a plane dendrite for which the interior (with respect to A) of the set of points having the order equal to 2 splits into a family of

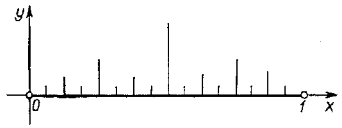

Fig. 1

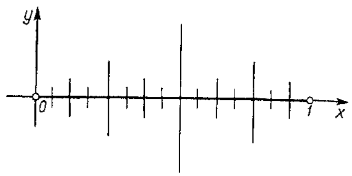

Fig. 2

components $\{d_\alpha\}$ where d_α is an open segment parallel either to the axis of abscissae or to the axis of ordinates. Then we define $T(A)=\bigcup_\alpha T(\bar{d}_\alpha)$, $K(A)=\bigcup_\alpha K(\bar{d}_\alpha)$. It follows from this definition that

1° The sets $T^n(d_0)=\underbrace{T\circ\ldots\circ T}_{n}(d_0)$ and $K^n(d_0)=\underbrace{K\circ\ldots\circ K}_{n}(d_0)$ are dendrites for $n=1,2,\ldots$

2° The sets $T^\infty(d_0)=\operatorname*{Lim}_{n=\infty}T^n(d_0)$, $K^\infty(d_0)=\operatorname*{Lim}_{n=\infty}K^n(d_0)$ are dendrites.

3° The set $K^\infty\circ T^m(d_0)=\operatorname*{Lim}_{m=\infty}K^m\circ T^m(d_0)$ is a dendrite for $n=1,2,\ldots$

Let B_n denote the dendrite obtained from $K^\infty\circ T^m(d_0)$ by adding the segments

$$\overline{(0,0),(-1,1)},\quad \overline{(0,0),(-1,\tfrac{1}{2})},\quad \overline{(0,0),(-1,-\tfrac{1}{2})},\quad \overline{(0,0),(-1,-1)}\,.$$

In Figure 3 the first three dendrites of the sequence $\{B_n\}$ are only sketched. It is easy to see that for $n'<n''$, $B_{n'}\underset{\mathfrak{R}}{\geqslant}B_{n''}$ holds. We shall prove that $B_{n'}\underset{\mathfrak{R}}{>}B_{n''}$ also holds.

We shall say that a point $p\in B_n$ such that $\mathrm{Ord}_pB_n=3$ is *a point of the k-th grade* $(1\leqslant k\leqslant n)$ if

$$p\in\begin{cases} d_0 & \text{for}\quad k=1\,,\\ T(d_0)-d_0 & \text{for}\quad k=2\,,\\ T^{k-1}(d_0)-T^{k-2}(d_0) & \text{for}\quad 3\leqslant k\leqslant n\,.\end{cases}$$

If the point $p\in B_n$ has order 3 and grade k, then that component of the set $B_n-\{p\}$ which contains no point of order 3 and grade k we shall call *the branch over p*. It can easily be seen that for each point $p\in B_n$ such

that $\mathrm{Ord}_pB_n=3$ there exists exactly one branch over p, which we shall denote by $G(p)$.

Let us assume that for $n'<n''$, $B_{n'}\underset{\mathfrak{R}}{\leqslant}B_{n''}$ holds. Then there exists a homeomorphism h mapping $B_{n'}$ into $B_{n''}$. Since the origin of the coordinates is the only point of order 5, we have $h\big((0,0)\big)=(0,0)$. Looking at the image $h(d_0)$ we see at once that there exists a segment d_1 such that $(0,0)\in d_1\subset d_0$ and $d_1\subset h(d_0)$. Hence we infer that there exist points of order 3 and grade 1: p_1', p_1'' such that $h(p_1')=p_1''$ and $h\big(G(p_1')\big)\subset G(p_1'')$.

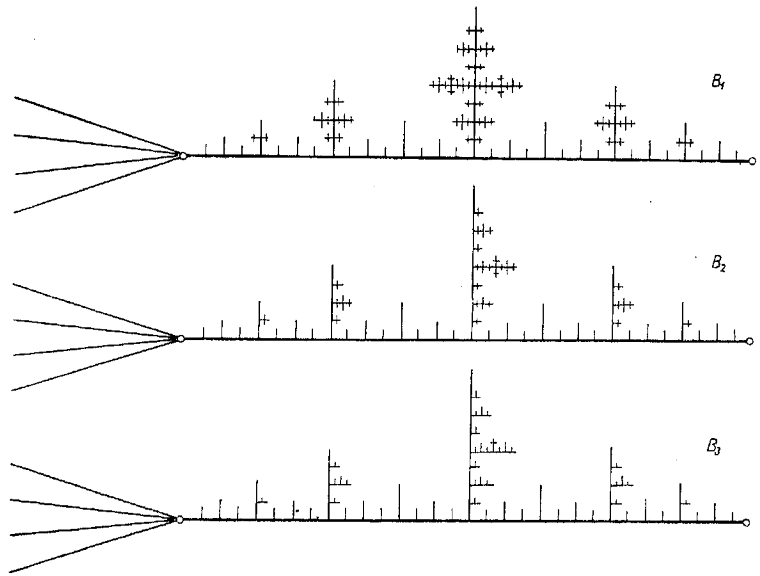

Fig. 3

Similarly we infer (if $n'>1$) that there exist points of order 3 and grade 2: p_2', p_2'' such that $h(p_2')=p_2''$ and $h\big(G(p_2')\big)\subset G(p_2'')$. We can repeat this reasoning n' times till we obtain points of order 3 and grade n': $p_{n'}', p_{n'}''$ such that $h(p_{n'}')=p_{n'}''$ and $h\big(G(p_{n'}')\big)\subset G(p_{n'}'')$ which is impossible because in the set $\overline{G(p_{n'}')}$ each arc with the endpoint $p_{n'}'$ contains a dense subset of points of order 4, while in the set $\overline{G(p_{n'}'')}$ each arc with the endpoint $p_{n'}''$ contains a neighbourhood of $p_{n'}''$ consisting of points of order not exceeding 3.

2. $\mathfrak{R}$-uncomparable sequence of dendrites. Let C_n denote a dendrite obtained from B_n by adding $n+6$ segments starting from the point $(-1, 1)$ and not cutting B_n beyond that point. Let us suppose that $n' < n''$. Since $C_{n'}$ is the dendrite of order $n'+6$ and $C_{n''}$ is the dendrite of order $n''+6$, $C_{n''}$ is not homeomorphic with a subset of $C_{n'}$. But if there exists a homeomorphism h mapping $C_{n'}$ into $C_{n''}$, then, the point $(-1, 1)$ being the only point of order $\geqslant 6$, we have $h\big((-1,1)\big) = (-1, 1)$ and $h(B_{n'}) \subset B_{n''}$, contrary to the results of part 1.

3. The family of power $\mathfrak{c}$ consisting of $\mathfrak{R}$-uncomparable dendrites. Let us consider the plane dendrite D (Fig. 4) consisting of the segment $\overline{(0,0),(0,1)}$ and those point $(x, y) \in E^2$ for which

$$x = r\cos\pi\left(1-\frac{1}{2i}\right), \quad y = r\sin\pi\left(1-\frac{1}{2i}\right) \quad \text{and} \quad 0 < r \leqslant \frac{1}{i} \quad \text{for} \quad i = 1, 2, \ldots$$

or

$$x = r\cos\frac{\pi}{2i}+1, \quad y = r\sin\frac{\pi}{2i} \quad \text{and} \quad 0 < r \leqslant \frac{1}{i} \quad \text{for} \quad i = 1, 2, \ldots$$

Let us take the sequence $p_m \in D$ where $p_m = (0, 1-1/2^m)$ for $m = 1, 2, \ldots$ Denoting by $\mathfrak{A}$ the family of all natural sequences $\{n_m\}$ such that every

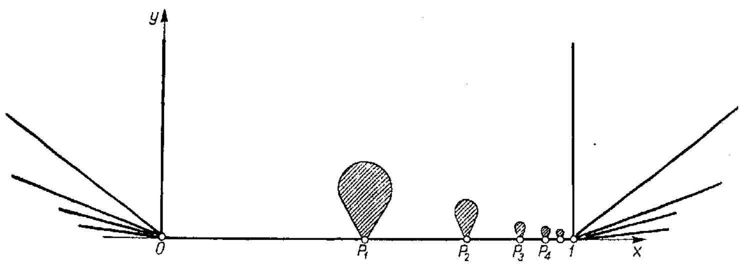

Fig. 4

natural number appears exactly once in the sequence $\{n_m\}$ we have $\overline{\overline{\mathfrak{A}}} = \mathfrak{c}$. Let $\tau = \{n_m\} \in \mathfrak{A}$ and let h_m denote a homeomorphism mapping C_{n_m} into E^2 in such a manner that $h_m\big((0,1)\big) = p_m$, $D \cap h_m(C_{n_m}) = \{p_m\}$ and $\delta\big(h_m(C_{n_m})\big) \leqslant 1/2^{m+2}$. Then it is easy to see that the set $D_\tau = C \cup \bigcup_{m=1}^{\infty} h_m(C_{n_m})$ (sketchy in Figure 4) is a dendrite and the family $\{D_\tau\}$ is of the power $\mathfrak{c}$.

Let us suppose that for $\tau' \neq \tau''$ there exists a homeomorphism h mapping for example $D_{\tau'}$ into $D_{\tau''}$. Since the points $(0, 0)$ and $(0, 1)$ are the only points of order ω, and in the presence of the order type of points $p_1, p_2, \ldots$ we infer that $h\big((0,0)\big) = (0, 0)$, $h\big((0,1)\big) = (0, 1)$ and

$h\big(\overline{(0,0),(0,1)}\big) = \overline{(0,0),(0,1)}$. Hence by the definition of sequences $\{n_m\}$ we conclude that there exist different indices n', n'' such that $C_{n'}$ is homeomorphic with a subset of $C_{n''}$, which is impossible in view of the results of the 2nd part. This proves the family $\{D_\tau\}$ of power $\mathfrak{c}$, that constructed above, consists of $\mathfrak{R}$-uncomparable dendrites.

References

[1] K. Borsuk, *Concerning the classification of topological spaces from the standpoint of the theory of retracts*, Fund. Math. this volume, p. 321-330.

[2] C. Kuratowski, *Topologie II*, Warszawa 1952.

Reçu par la Rédaction le 24. 6. 1958