equivalent elements there is exactly one element, that is \( a_j^i N_j V_j \cap a_j^i N_j V_j = \emptyset \) if only \( j \neq i \). Since \( G = \bigcup_j a_j^i V_j \) and \( a_j^i V_j \subseteq a_j^i N_j V_j \) we have \( a_j^i V_j = a_j^i N_j V_j \). This proves that, in the case \( k_1 = 1 \), \( V_j \in B_j \) and hence \( B_j = B_j + 1 \). Consider the case \( k_1 \geq 2 \). Denote by \( a_j^i V_j = 1 \leq i \leq k_1 \), elements of \( j \)th class of equivalent sets of the sequence (2). Put

\[
\mathcal{B}_j = \bigcup_{j=1}^{k_1} a_j^i V_j \quad (i = 1, 2, ..., k_1).
\]

We are going to show that the elements \( a_j^i \) have all the properties mentioned in (3) of Lemma 1. Indeed:

1° \( a_j^i \cap a_j^i \neq \emptyset \) if only \( r \neq i \),

2° all \( a_j^i \) are independent of \( B_j \),

3° \( [B_j, a_j^i V_j, ..., a_j^i V_j] = [B_j, a_j^i, ..., a_j^i] \).

1° is evident. To prove 2° take an arbitrary element \( b \in B_j \). We have to prove \( a_j^i \cap b \neq \emptyset \) and \( b \cap (a_j^i)^{\prime} \neq \emptyset \). By Lemma 2, \( b = N_j V_j \). Since \( \bigcup_{j=1}^{k_1} a_j^i N_j V_j = G_j \), we have \( b \cap a_j^i N_j V_j = \emptyset \) or, which means the same, \( N_j V_j \cap a_j^i N_j V_j = \emptyset \), and hence we obtain \( N_j V_j \cap a_j^i V_j = \emptyset \). In order to obtain \( b \cap (a_j^i)^{\prime} \neq \emptyset \) note that \( (a_j^i)^{\prime} = \bigcup_{j=1}^{k_1} a_j^i V_j \). Hence, since \( B_j \geq 2 \), \( b \cap (a_j^i)^{\prime} \neq \emptyset \).

3° It is evident that \([B_j, a_j^i V_j, ..., a_j^i V_j] \supset [B_j, a_j^i, ..., a_j^i] \). In order to obtain the converse inclusion note that \( a_j^i V_j = a_j^i N_j V_j \cap a_j^i \).

References


Bye par la Réduction le 4. 5. 1968

Concerning the classification of topological spaces from the stand-point of the theory of retracts

by

K. Borsuk (Warszawa)

We consider in topology two spaces as different if they are not homeomorphic and we identify all homeomorphic spaces. It is clear, from the intuitive notion of point of view, that difference between two not homeomorphic spaces may be more or less essential. But the concepts which allow us to state precisely how far one space differs topologically from another are not numerous. To such notions belong the notion of homotopy (type of dimension) due to Fréchet [2], the notion of type of homology due to Borsuk [3], the notion of domination due to J. H. C. Whitehead [2] and also the notion of R-type [1].

In this note I introduce some notions intimately related to the notion of R-type. Those notions allow us in some cases to formulate precisely the sense of the statement that one space is topologically more complicated than another, and also to formulate precisely in some cases the degree of diversity between two topological spaces. Moreover, I give some examples, determine the number of all R-types among the AR-sets and pose some problems.

1. Basic definitions. By an R-mapping of a space \( X \) onto a space \( Y \) we understand here a function \( \varphi \) satisfying the following three conditions:

1° \( X \) is the set of arguments and \( Y \) the set of values for \( \varphi \),

2° \( \varphi \) is continuous,

3° There exists a continuous mapping \( \psi \) for which \( Y \) is the set of arguments and the values belong to \( X \) and for which we have \( \varphi \psi(y) = y \) for every \( y \in Y \).

In particular every retraction, i.e. every continuous mapping of \( X \) onto a subset \( Y \) of \( X \), identical on \( Y \), is an R-mapping. It is easy to show [1] that the R-mappings are the same as the mappings of the form \( r \circ h \), where \( r \) is a retraction and \( h \) a homeomorphism.

Two spaces \( X \) and \( Y \) are said to be R-equal, symbolically

\[
X = Y, \tag{1}
\]
provided that there exist an $R$-mapping of $X$ onto $Y$ and also an $R$-mapping of $Y$ onto $X$. Evidently the relation $\leq_R$ is an equivalence relation. Hence every class $\mathcal{X}$ of spaces can be split up into a family of subclasses, called $R$-types, two spaces $X, Y \in \mathcal{X}$ belonging to the same $R$-type if and only if $X \equiv_R Y$.

If there exists an $R$-mapping of $X$ onto $Y$, but an $R$-mapping of $Y$ onto $X$ does not exist, then $X$ will be said to be $R$-larger than $Y$, or $Y$ to be $R$-smaller than $X$, symbolically

$$X \geq_R Y \quad \text{or} \quad Y \leq_R X.$$

Manifesterly the relation $\leq$ (hence also the inverse relation $\geq$) is transitive, not reflexive and not symmetric. Every class $\mathcal{X}$ of spaces is partially ordered by this relation.

$X \leq_R Y$ means that $X < Y$ or $X = Y$, and $X \geq_R Y$ means that $X > Y$ or $X = Y$.

If between two spaces $X$ and $Y$ none of the relations $X \leq_R Y$ or $X \geq_R Y$ holds, then the spaces $X$ and $Y$ are said to be $R$-uncomparable. In particular the empty space and every non-empty space are always $R$-uncomparable.

2. Elementary properties and examples. Recalling the known properties of retraction (see, for instance [3]) we have:

(3) If $X \geq_R Y$ then $Y$ is homeomorphic with a closed subset of $X$.

It follows in particular that:

(4) If $X \geq_R Y$ then $\dim X \geq \dim Y$.

(5) If $X \geq_R Y$ then the number of points of order $\geq n$ (in the sense of Menger and Urysohn [5], [7]) is for the space $X$ not smaller than the corresponding number for the space $Y$.

It follows, for instance, that if $X$ and $Y$ are two dendrites and if there exist two integers $m$ and $n$ such that the number of points of order $\geq m$ is for $X$ larger than the corresponding number for $Y$ and the number of points of order $\geq n$ is for $X$ smaller than the corresponding number for $Y$, then $X$ and $Y$ are $R$-uncomparable. This remark allows us to construct a family of $R$-uncomparable dendrites (even polytopes) containing an arbitrarily given finite number of elements. The problem of constructing an infinite family of $R$-uncomparable dendrites is notably more difficult (comp. the following paper by K. Sieklucki [6]).

(6) If $X \geq_R Y$ then every group of homology $H_n(X, \mathbb{Z})$ (where $n$ denotes the dimension and $\mathbb{Z}$ the group of coefficients) is isomorphic with some direct summand of the group $H_n(X, \mathbb{Z})$.

(7) If $X \geq_R Y$ then every Betti-number of $X$ is not smaller than the corresponding Betti-number of $Y$.

The properties (4), (5), and (7) constitute singular cases of the following statement:

(8) If $X \geq_R Y$ then every property of the space $X$ invariant under all $R$-mappings holds also for the space $Y$.

To the invariants of all $R$-mappings belong in particular the following properties: normality, separability, metrizability, connectedness, compactness, local compactness, local connectedness, arcwise connectedness, local connectedness in the dimension $n$, contractibility, local contractibility, property ANR, property AR, the existence of a fixed point for continuous mappings into itself.

3. $R$-minorant and $R$-majorant. The space $X$, is said to be an $R$-minorant (respectively, an $R$-majorant) for a class $\mathcal{X}$ of spaces provided $X \leq_R X_{\in \mathcal{X}}$ for every $X \in \mathcal{X}$ (respectively, $X \geq_R X_{\in \mathcal{X}}$ for every $X \in \mathcal{X}$).

Let us observe that

(9) For every class $\mathcal{X}$ of non-empty spaces there exist an $R$-minorant and an $R$-majorant.

For, every space containing only one point is an $R$-minorant for $\mathcal{X}$. On the other hand, the Cartesian product of all spaces belonging to the class $\mathcal{X}$ is evidently an $R$-majorant for $\mathcal{X}$.

If the class $\mathcal{X}$ is countable or finite and if its elements are metrizable and separable spaces, then the Cartesian product of all these spaces is also metrizable and separable. Hence

(10) For a countable or finite class of non-empty, metrizable and separable spaces there exist a metrizable and separable $R$-minorant and a metrizable and separable $R$-majorant.

If all spaces of the family $\mathcal{X}$ are compact, then their Cartesian product is also compact. Hence

(11) For a class of not empty, compact spaces there exist an compact $R$-minorant and a compact $R$-majorant.

Let us observe the following:
Let $X$ denote the class of all subsets of a metric, separable and complete space $M$ of power $c$. Then a metrizable and separable $\mathcal{R}$-minorant for $X$ does not exist.

In fact, in $M$ there exists a family of $2^c$ subsets of which no two are homeomorphic. On the other hand in every metric separable space there exist only $c$ closed subsets and, by (3) every space $Y \subseteq X$ is homeomorphic with a closed subset of $X$.

4. $\mathcal{R}$-closed classes of spaces. A class $\mathcal{X}$ of spaces is said to be $\mathcal{R}$-closed on the left (respectively on the right) provided it contains an $\mathcal{R}$-minorant (respectively an $\mathcal{R}$-majorant) of $X$.

Henceforth we shall always understand by a space a separable, metric space. Since every such space is homeomorphic to a subset of the Hilbert space $\mathbb{R}^n$, we can always assume that the considered spaces are subsets of $\mathbb{R}^n$. Thus we can speak about the class of all separable metric spaces and about various subclasses of it. For instance, we can speak about the class of all (compact) AR-sets or about the class of all (compact) ANR-sets. Let us observe that

(13) The class of all compact 0-dimensional spaces is $\mathcal{R}$-closed on the left and on the right.

For the set containing only one point constitutes an $\mathcal{R}$-minorant of this class and the Cantor discontinuum constitutes its $\mathcal{R}$-majorant.

However

(14) For $n > 0$ the class of all compact $n$-dimensional spaces is not $\mathcal{R}$-closed on the left.

For an $n$-dimensional compactum $X$ which is an $\mathcal{R}$-minorant in the class of all compact $n$-dimensional spaces would be homeomorphic to a subset of the $n$-dimensional Euclidean cube $Q_n$. It follows, by $\dim X = n$, that $X$ contains $Q_n$ topologically. Hence every $n$-dimensional compactum would contain the $n$-dimensional cube $Q_n$, which is manifestly not true. By the same reasoning we find also that:

(15) For $n > 0$ the class of all $n$-dimensional AR-sets (respectively ANR-sets) is not $\mathcal{R}$-closed on the left.

However

(16) The class of all AR-sets is $\mathcal{R}$-closed on the left and on the right.

For the set containing only a single point constitutes its $\mathcal{R}$-minorant and the Hilbert cube $Q_n$ constitutes its $\mathcal{R}$-majorant.

Moreover

(17) The class of all non-empty ANR-sets is $\mathcal{R}$-closed on the left, but not $\mathcal{R}$-closed on the right.
It follows by 6° and (3) that
7° If \( m < n \) then \( A_m \) is not \( \leq A_n \).

But by 5° we have for \( m < n \) the relation \( A_n \leq A_m \). Consequently
(19) If \( m < n \) then \( A_n \leq A_m \).

Hence we can say that the sequence \( \{A_n\} \) is \( \mathbb{R} \)-decreasing.

6. A sequence of \( \mathbb{R} \)-uncomparable polytopes. Now let us construct a sequence of 2-dimensional polytopes, which are \( \mathbb{R} \)-uncomparable \( AR \)-sets. Let \( E_{n+2} \) denote the \((n+2)\)-dimensional Euclidean space with the points \((x_1, x_2, \ldots, x_n, y, z)\). Setting for \( k = 1, 2, \ldots, n+1\)
\[ g_k(x, y, z) = (x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, y, z) \] with \( n=0 \) for \( i \neq k \) and \( x_k = x \),
we obtain a homeomorphism \( g_k \) mapping \( E_0 \) into \( E_{n+2} \). It is readily seen that the set
\[ B_n = \bigcup_{k=1}^{n+1} g_k(A_n) \]
is a 2-dimensional polytope which is an \( AR \)-set. It may also be defined as the space obtained from \( n+1 \) copies of the polytope \( A_n \) by the identification of the corresponding points belonging to the discs \( P_k \). For the sake of arithmetic simplicity, we have constructed the polytope \( B_n \) in the space \( E_{n+2} \). But it is readily seen that \( B_n \) is homeomorphic with a subset of the space \( E_0 \).

Let us observe that
(20) For \( n \neq m \) neither of the sets \( B_n \), \( B_m \) is homeomorphic with a subset of the other.

To prove it, let us assume that there exists a homeomorphism \( h \) mapping \( B_n \) into \( B_m \). It is easily seen that then \( h(0, 0, \ldots, 0) = (0, 0, \ldots, 0) \) and that for every index \( i = 1, 2, \ldots, m+1 \) there exists an index \( j(i) \leq n+1 \) such that
\[ h_i[\varphi(A_n)] \supset \varphi(\varphi_i(A_m)) \] for \( i = 1, 2, \ldots, m+1 \).

Moreover it is clear that for different indices \( i \) the indices \( j(i) \) are different. It follows \( m+1 \leq n+1 \) whence \( m \leq n \). On the other hand, we infer by (21) and (19) that \( m \neq n \). Consequently \( m = n \), i.e. statement (20) is proved.

7. The number of \( \mathbb{R} \)-types. It is known (see for instance [4], p. 335) that in every separable and complete space \( Z \) of the power \( \mathfrak{c} \) there exists a family of \( 2^\mathfrak{c} \) subsets none of which is a continuous image of another. It follows that under subsets of \( Z \) there exists \( 2^\mathfrak{c} \) of different (even uncomparable) \( \mathbb{R} \)-types. On the other hand, it is evident that the number of \( \mathbb{R} \)-types under compact (metric) spaces cannot be larger than \( \mathfrak{c} \), because the number of all compact subsets of the Hilbert space is equal to \( \mathfrak{c} \). Let us prove that already the number of different \( \mathbb{R} \)-types under 2-dimensional compacta is \( \mathfrak{c} \). More exactly, let us prove the following statement:
(22) There exists in the space \( E_2 \) a family of \( \mathfrak{c} \) two-dimensional \( \mathbb{R} \)-uncomparable \( AR \)-sets.

Let us denote by \( W_n \), the 3-dimensional ball defined in \( E_3 \) by the inequality
\[ (x_1 - \frac{1}{n})^2 + y^2 + z^2 \leq \frac{1}{2n^2}. \]

Let \( L \) denote the segment with endpoints \((0, 0, 0)\) and \((1, 0, 0)\). It is clear that there exists a homeomorphism \( h_n \) mapping the set \( B_{2n-1} \) (defined in Nr. 6) onto a subset of \( W_{2n-1} \), satisfying the condition
\[ L \cap h_n(B_{2n-1}) = \left( \frac{1}{2n-1}, 0, 0 \right) \]
\[ L \cap g_n(B_{2n}) = \left( \frac{1}{2n}, 0, 0 \right). \]

Let \( \kappa = (k_n) \) be an arbitrary sequence of natural numbers. For every \( n = 1, 2, \ldots \), let us denote by \( g_n \) a homeomorphic mapping of the set \( B_{2n} \) onto a subset of \( W_{2n} \) satisfying the condition
\[ L \cap g_n(B_{2n}) = \left( \frac{1}{2n+1}, 0, 0 \right) \]
\[ L \cap h_n(B_{2n-1}) = \left( \frac{1}{2n-1}, 0, 0 \right). \]

It is easily seen that the set
(23)
\[ C_\kappa = \bigcup_{n=1}^{\infty} [h_n(B_{2n-1}) \cup g_n(B_{2n})] \cup L \]
is an \( AR \)-set. Since there exist \( \mathfrak{c} \) different sequences of natural numbers, \( \kappa \), it follows from that for every two sequences \( \kappa = (k_n) \) and \( \kappa' = (k'_n) \) the relation \( C_\kappa \leq C_{\kappa'} \) implies \( \kappa = \kappa' \). By (3) it suffices to show that
(24) If there exists a homeomorphism \( h \) mapping \( C_\kappa \) into \( C_{\kappa'} \), then \( \kappa = \kappa' \).

Let the set \( C_\kappa \) be given by the formula
(25)
\[ C_\kappa = \bigcup_{n=1}^{\infty} [h_n(B_{2n-1}) \cup g_n(B_{2n})] \cup L \],
where \( h_n, g_n \) are homeomorphisms of \( B_{2n-1} \) into \( W_{2n-1} \) and of \( B_{2n} \) into \( W_{2n} \) such that
\[ L \cap h_n'(B_{2n-1}) = \left( \frac{1}{2n-1}, 0, 0 \right), \]
\[ L \cap g_n'(B_{2n}) = \left( \frac{1}{2n}, 0, 0 \right). \]
It is clear that each of the sets \( h_n(B_{2n+1}) \) is maximal with respect to the property of being a subcontinuum of \( C \), which is 2-dimensional at each of its points. Analogously, each of the sets \( g_n(B_{2n+1}) \) is maximal with respect to the property of being a subcontinuum of \( C \), which is 2-dimensional at each of its points. It follows that the homomorphism \( h \) maps each of the sets \( h_n(B_{2n+1}) \) and \( g_n(B_{2n+1}) \) onto a subcontinuum of \( C \). By (20) we infer that

\[
\varrho[h_n(B_{2n+1})] \subset h_n(B_{2n+1}).
\]

It follows that \( h(n) \) sends the points of the segment \( L \) lying between the points \((1/(2n+1), 0, 0)\) and \((1/(2n+1), 0, 0)\) onto points lying also between the same points. We infer that \( h(n) \) sends the set \( g_n(B_{2n+1}) \) into the set \( g_n(B_{2n+1}) \). Again applying (20), we conclude that \( h_n = h_n' \) for every \( n = 1, 2, \ldots \), i.e., the sequences \( n \) and \( n' \) are identical. Thus (24), and consequently also (22) is proved.

Let us observe that

(25) The number of different \( \mathfrak{R} \)-types between all polytopes is countable.

In fact, every \( n \)-dimensional polytope is homeomorphic with a polytope lying in the Euclidean space \( E_{n+1} \), and every polytope lying in \( E_{n+1} \) is evidently homeomorphic with a polytope for which there exists a triangulation all vertices of which have all coordinates rational.

It follows that among the \( \mathfrak{R} \)-types of \( \mathfrak{ANR} \)-sets the \( \mathfrak{R} \)-types containing polytopes are rather exceptional. It seems interesting to study the topological properties of \( \mathfrak{ANR} \)-sets belonging to those \( \mathfrak{R} \)-types, i.e., the \( \mathfrak{ANR} \)-sets \( \mathfrak{R} \)-equal to the polytopes.

8. \( \mathfrak{R} \)-neighbours. Suppose that \( X, Y \) are two spaces such that

\[
X < Y,
\]

but no space \( Z \) satisfying the condition

\[
X < Z < Y
\]

does not exist. Then the sets \( X \) and \( Y \) are said to be \( \mathfrak{R} \)-neighbours. More exactly, \( X \) is said to be an \( \mathfrak{R} \)-neighbour of \( Y \) on the left and \( Y \) an \( \mathfrak{R} \)-neighbour of \( X \) on the right.

Examples: 1. It is easy to see that for the segment there exists only one \( \mathfrak{R} \)-neighbour on the left, namely the set containing only one point, but many \( \mathfrak{R} \)-neighbours on the right, for instance the circle, the dendrite which is the sum of three segments with one common endpoint, the set composed of the segment and of one isolated point, the ray, the closure of the diagram of the function \( y = \sin(1/x) \), with \( 0 < x < 1 \) and so on.

2. The disc is the \( \mathfrak{R} \)-neighbour on the left for the sphere and also for the projective plane.

3. The set which is the sum of a circle and one of its rays has two \( \mathfrak{R} \)-neighbours on the left; the circle and the sum of three segments having one endpoint in common.

The notion of the \( \mathfrak{R} \)-neighbour leads in a natural way to the notion of the \textit{index of \( \mathfrak{R} \)-proximity} \( I(X, Y) \) of two spaces \( X \) and \( Y \), which we define as follows:

A finite sequence \( X = X_0, X_1, \ldots, X_n, X_{n+1} = Y \) of spaces is said to be a \textit{transition} from \( X \) to \( Y \) of index \( n \), if for every \( i = 0, 1, \ldots, n \) the spaces \( X_i \) and \( X_{i+1} \) are \( \mathfrak{R} \)-neighbours. If there exists a transition from \( X \) to \( Y \), then we define \( I(X, Y) \) as the minimum of indices of all transitions from \( X \) to \( Y \). If, however, a transition from \( X \) to \( Y \) does not exist, then we set \( I(X, Y) = \infty \). For instance, the index of \( \mathfrak{R} \)-proximity for two \( \mathfrak{R} \)-neighbours is equal to 0. If \( X \) is the 2-dimensional sphere and \( Y \) is the projective plane, then, by 2, we have \( I(X, Y) = 1 \). But we are able to calculate the index of \( \mathfrak{R} \)-proximity only in very special cases. Already in the case where \( X \) is the 2-dimensional sphere and \( Y \) is the surface of the torus, we do not know whether \( I(X, Y) \) is finite or infinite.

9. Problems concerning families of spaces ordered by the relation \( \mathfrak{R} \). In Nr. 5 we have constructed a sequence of 2-dimensional polytopes \( (A_n) \) such that

\[
A_n < A_{n+1} < A_n \quad \text{for} \quad n = 2, 3, \ldots,
\]

where \( A_\infty = P_\infty \cup P_1 \cup L \) (we use here the notation of Nr. 5 and 7). It is easily seen that \( A_{n+1} \) is an \( \mathfrak{R} \)-neighbour (on the left) of \( A_n \), i.e., there exists no space between \( A_n \) and \( A_{n+1} \) (in the sense of the relation \( < \)).

On the other hand, it is easy to define a space lying between \( A_n \) and all sets \( A_1, A_2, \ldots \). To obtain such a space, let us denote by \( Q_n \) the disc defined in the space \( E_3 \) by the conditions

\[
(x - 2\pi)^2 + y^2 = \frac{1}{4n^2(n+1)}; \quad z = 0.
\]

It suffices to set

\[
A_n = \bigcup_{n=1}^\infty Q_n \cup A_\infty.
\]
to obtain a 2-dimensional AR-set satisfying the relation
\[ A_m < A_n < A_m \quad \text{for every} \quad m, n = 1, 2, \ldots \]

It is easy to formulate numerous problems concerning the families of spaces ordered by the relation \( < \). Let us mention some of them:

1° Does there exist an uncountable family of spaces ordered by the relation \( < \)?

2° Does there exist a family of spaces ordered by the relation \( < \) in a dense manner (i.e., such that for every two spaces \( X < Y \) of this family there exists in it a space \( Z \) satisfying the relation \( X < Z < Y \))? 

3° Does there exist a family of spaces ordered by the relation \( < \) similarly to the set of all real numbers?

All those problems concern quite arbitrary spaces, but each of them may also be formulated for a special class of spaces, for instance for AR-sets or ANR-sets. For the class of polytopes, problems 1° and 3° have manifestly negative answers, but problem 2° remains open.

References

On a family of power \( c \) consisting of \( \mathbb{R} \)-uncomparable dendrites

by

K. Siekluki (Warszawa)

K. Borsuk has introduced the notion of \( \mathbb{R} \)-type ([1], p. 322) and proved that the set of different \( \mathbb{R} \)-types among 2-dimensional compacta has power \( c \). It is a consequence of the theorem, proved in that paper ([1], p. 327), on the existence of the family of power \( c \) consisting of \( \mathbb{R} \)-uncomparable, 2-dimensional AR-sets. The purpose of the present paper is to prove that even among 1-dimensional AR-sets (i.e., among dendrites) there exist \( c \) different \( \mathbb{R} \)-types. More strictly we shall prove that

On the plane \( \mathbb{R}^2 \) there exists a family of power \( c \) consisting of \( \mathbb{R} \)-uncomparable dendrites.

It is easy to construct such a family having power \( n \), where \( n \) is a natural number. (See [1], p. 322.) Beginning the construction of the family of power \( c \) we shall prove first that there exists an

1. \( \mathbb{R} \)-decreasing sequence of dendrites. Let \( d_n \) be a closed segment on the plane \( \mathbb{R}^2 \) with the endpoints \((0, 0), (0, 1)\) and let \( T(d_n) \) denote the dendrite (Fig. 1) consisting of the points of \( d_n \) and those points \((x, y) \in \mathbb{R}^2 \) for which

\[
x = \frac{2^n - 1}{2^j}, \quad 0 < y \leq \frac{1}{2^j+1} \quad \text{where} \quad i = 1, 2, \ldots, 2^{j-1}; j = 1, 2, \ldots
\]

while \( K(d_n) \) denotes the dendrite (Fig. 2) consisting of the points of \( d_n \) and those points \((x, y) \in \mathbb{R}^2 \) for which

\[
x = \frac{2^n - 1}{2^i}, \quad 0 < |y| \leq \frac{1}{2^j+1} \quad \text{where} \quad i = 1, 2, \ldots, 2^{j-1}; j = 1, 2, \ldots
\]

If \( d \subset \mathbb{R}^2 \) is a closed segment parallel either to the axis of abscissae or to the axis of ordinates and \( \phi \) denotes the affine mapping preserving

\[(^c) \text{ A dendrite is a locally connected continuum containing no simple closed curve. See for example [2], p. 224.}\]