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Concerning the classification of topological spaces from the stand-point of the theory of retracts

by

K. Borsuk (Warszawa)

We consider in topology two spaces as different if they are not homeomorphic and we identify all homeomorphic spaces. It is clear, from the intuitive point of view, that difference between two not homeomorphic spaces may be more or less essential. But the notions which allow us to state precisely how far one space differs topologically from another are not numerous. To such notions belong the notion of *homoia (type of dimension)* due to Fréchet [2], the notion of *type of homotopy* due to Hurewicz [3], the notion of *domination* due to J. H. C. Whitehead [8] and also the notion of $\mathfrak{R}$-*type* [1].

In this note I introduce some notions intimately related to the notion of the $\mathfrak{R}$-type. Those notions allow us in some cases to formulate precisely the sense of the statement that one space is topologically more complicated than another, and also to formulate precisely in some cases the degree of diversity between two topological spaces. Moreover, I give some examples, determine the number of all $\mathfrak{R}$-types among the AR-sets and pose some problems.

1. Basic definitions. By an $\mathfrak{R}$-*mapping* of a space X onto a space Y we understand here a function φ satisfying the following three conditions:

1° X is the set of arguments and Y the set of values for φ.

2° φ is continuous.

3° There exists a continuous mapping ψ for which Y is the set of arguments and the values belong to X and for which we have $\varphi\psi(y) = y$ for every $y \in Y$.

In particular every *retraction*, i. e. every continuous mapping of X onto a subset Y of X, identical on Y, is an $\mathfrak{R}$-mapping. It is easy to show [1] that the $\mathfrak{R}$-mappings are the same as the mappings of the form hr, where r is a retraction and h a homeomorphism.

Two spaces X and Y are said to be $\mathfrak{R}$-*equal*, symbolically

(1) $$X \underset{\mathfrak{R}}{=} Y,$$

provided that there exist an $\mathfrak{R}$-mapping of X onto Y and also an $\mathfrak{R}$-mapping of Y onto X. Evidently the relation "$\underset{\mathfrak{R}}{=}$" is an equivalence relation. Hence every class $\mathcal{X}$ of spaces can be split up into a family of subclasses, called $\mathfrak{R}$-types, two spaces $X, Y \in \mathcal{X}$ belonging to the same $\mathfrak{R}$-type if and only if $X \underset{\mathfrak{R}}{=} Y$.

If there exists an $\mathfrak{R}$-mapping of X onto Y, but an $\mathfrak{R}$-mapping of Y onto X does not exist, then X will be said to be $\mathfrak{R}$-*larger* than Y, or Y to be $\mathfrak{R}$-*smaller* than X, symbolically

(2) $$X \underset{\mathfrak{R}}{>} Y \quad \text{or} \quad Y \underset{\mathfrak{R}}{<} X.$$

Manifestly the relation $\underset{\mathfrak{R}}{<}$ (hence also the inverse relation $\underset{\mathfrak{R}}{>}$) is transitive, not reflexive and not symmetric. Every class $\mathcal{X}$ of spaces is partially ordered by this relation.

$X \underset{\mathfrak{R}}{\leqslant} Y$ means that $X \underset{\mathfrak{R}}{<} Y$ or $X \underset{\mathfrak{R}}{=} Y$, and $X \underset{\mathfrak{R}}{\geqslant} Y$ means that $X \underset{\mathfrak{R}}{>} Y$ or $X \underset{\mathfrak{R}}{=} Y$.

If between two spaces X and Y none of the relations $X \underset{\mathfrak{R}}{\leqslant} Y$ or $X \underset{\mathfrak{R}}{\geqslant} Y$ holds, then the spaces X and Y are said to be $\mathfrak{R}$-*uncomparable*. In particular the empty space and every non-empty space are always $\mathfrak{R}$-uncomparable.

2. Elementary properties and examples. Recalling the known properties of retraction (see, for instance [1]) we have:

(3) *If* $X \underset{\mathfrak{R}}{\geqslant} Y$ *then* Y *is homeomorphic with a closed subset of* X.

It follows in particular that:

(4) *If* $X \underset{\mathfrak{R}}{\geqslant} Y$ *then* $\dim X \geqslant \dim Y$.

(5) *If* $X \underset{\mathfrak{R}}{\geqslant} Y$ *then the number of points of order* $\geqslant n$ (in the sense of Menger and Urysohn [5], [7]) *is for the space* X *not smaller than the corresponding number for the space* Y.

It follows, for instance, that if X and Y are two dendrites and if there exist two integers m and n such that the number of points of order $\geqslant m$ is for X larger than the corresponding number for Y and the number of points of order $\geqslant n$ is for X smaller than the corresponding number for Y, then X and Y are $\mathfrak{R}$-uncomparable. This remark allows us to construct a family of $\mathfrak{R}$-uncomparable dendrites (even polytopes) containing an arbitrarily given finite number of elements. The problem of constructing an infinite family of $\mathfrak{R}$-uncomparable dendrites is notably more difficult (comp. the following paper by K. Sieklucki [6]).

(6) *If* $X \underset{\mathfrak{R}}{\geqslant} Y$ *then every group of homology* $H_n(Y, \mathfrak{A})$ (where n denotes the dimension and $\mathfrak{A}$ the group of coefficients) *is isomorphic with some direct summand of the group* $H_n(X, \mathfrak{A})$.

It follows in particular that:

(7) *If* $X \underset{\mathfrak{R}}{\geqslant} Y$ *then every Betti-number of* X *is not smaller than the corresponding Betti-number of* Y.

The properties (4), (5), and (7) constitute singular cases of the following statement:

(8) *If* $X \underset{\mathfrak{R}}{\geqslant} Y$ *then every property of the space* X *invariant under all* $\mathfrak{R}$-*mappings holds also for the space* Y.

To the invariants of all $\mathfrak{R}$-mappings belong in particular the following properties: normality, separability, metrizability, connectedness, compactness, local compactness, local connectedness, arcwise connectedness, local connectedness in the dimension n, contractibility, local contractibility, property ANR, property AR, the existence of a fixed point for continuous mappings into itself.

3. $\mathfrak{R}$-minorant and $\mathfrak{R}$-majorant. The space X_0 is said to be an $\mathfrak{R}$-*minorant* (respectively, an $\mathfrak{R}$-*majorant*) for a class $\mathcal{X}$ of spaces provided $X_0 \underset{\mathfrak{R}}{\leqslant} X$ for every $X \in \mathcal{X}$ (respectively, $X_0 \underset{\mathfrak{R}}{\geqslant} X$ for every $X \in \mathcal{X}$).

Let us observe that

(9) *For every class* $\mathcal{X}$ *of non-empty spaces there exist an* $\mathfrak{R}$-*minorant and an* $\mathfrak{R}$-*majorant.*

For, every space containing only one point is an $\mathfrak{R}$-minorant for $\mathcal{X}$. On the other hand, the Cartesian product of all spaces belonging to the class $\mathcal{X}$ is evidently an $\mathfrak{R}$-majorant for $\mathcal{X}$.

If the class $\mathcal{X}$ is countable or finite and if its elements are metrizable and separable spaces, then the Cartesian product of all those spaces is also metrizable and separable. Hence

(10) *For a countable or finite class of non-empty, metrizable and separable spaces there exist a metrizable and separable* $\mathfrak{R}$-*minorant and a metrizable and separable* $\mathfrak{R}$-*majorant.*

If all spaces of the family $\mathcal{X}$ are compact, then their Cartesian product is also compact. Hence

(11) *For a class of not empty, compact spaces there exist a compact* $\mathfrak{R}$-*minorant and a compact* $\mathfrak{R}$-*majorant.*

Let us observe the following:

(12) *Let $\mathfrak{X}$ denote the class of all subsets of a metric, separable and complete space M of power $\mathfrak{c}$. Then a metrizable and separable $\mathfrak{R}$-majorant for $\mathfrak{X}$ does not exist.*

In fact, in M there exists a family of $2^{\mathfrak{c}}$ subsets of which no two are homeomorphic. On the other hand in every metric separable space there exist only $\mathfrak{c}$ closed subsets and, by (3) every space $Y \underset{\mathfrak{R}}{\leqslant} X$ is homeomorphic with a closed subset of X.

4. $\mathfrak{R}$-closed classes of spaces. A class $\mathfrak{X}$ of spaces is said to be *$\mathfrak{R}$-closed on the left* (respectively *on the right*) provided it contains an $\mathfrak{R}$-minorant (respectively an $\mathfrak{R}$-majorant) of $\mathfrak{X}$.

Henceforth we shall always understand by a space a separable, metric space. Since every such space is homeomorphic to a subset of the Hilbert space E_ω, we can always assume that the considered spaces are subsets of E_ω. Thus we can speak about the class of all separable metric spaces and about various subclasses of it. For instance, we can speak about the class of all (compact) AR-sets or about the class of all (compact) ANR-sets. Let us observe that

(13) *The class of all compact 0-dimensional spaces is $\mathfrak{R}$-closed on the left and on the right.*

For the set containing only one point constitutes an $\mathfrak{R}$-minorant of this class and the Cantor discontinuum constitutes its $\mathfrak{R}$-majorant.

However

(14) *For $n>0$ the class of all compact n-dimensional spaces is not $\mathfrak{R}$-closed on the left.*

For an n-dimensional compactum X which is an $\mathfrak{R}$-minorant in the class of all compact n-dimensional spaces would be homeomorphic to a subset of the n-dimensional Euclidean cube Q_n. It follows, by $\dim X = n$, that X contains Q_n topologically. Hence every n-dimensional compactum would contain the n-dimensional cube Q_n, which is manifestly not true. By the same reasoning we find also that:

(15) *For $n>0$ the class of all n-dimensional* AR*-sets (respectively* ANR*-sets) is not $\mathfrak{R}$-closed on the left.*

However

(16) *The class of all* AR*-sets is $\mathfrak{R}$-closed on the left and on the right.*

For the set containing only a single point constitutes its $\mathfrak{R}$-minorant and the Hilbert cube Q_ω constitutes its $\mathfrak{R}$-majorant.

Moreover

(17) *The class of all non-empty* ANR*-sets is $\mathfrak{R}$-closed on the left, but not $\mathfrak{R}$-closed on the right.*

For the set containing only a single point constitutes its $\mathfrak{R}$-minorant. However if a set X is an $\mathfrak{R}$-majorant for the class of all non-empty ANR-sets, then, by (7), the 0-dimensional Betti number of X is infinite. Hence X is not an ANR-set. By the same reasoning we find also that:

(18) *For every $n = 0, 1, 2, \ldots$ the class of all* ANR*-sets of dimension n is not $\mathfrak{R}$-closed on the right.*

The following problems remain open:

1. *Is the class of all non-empty metric compact spaces $\mathfrak{R}$-closed on the right?*

2. *Let n be an integer $\geqslant 0$. Is the class of all n-dimensional metric compact spaces $\mathfrak{R}$-closed on the right?*

5. An $\mathfrak{R}$-decreasing sequence of polytopes. Let us consider in the 3-dimensional Euclidean space E_3 an orthogonal system of coordinates (x, y, z). Let us set

$$P_0 = \underset{(x,y,z)}{E}[x=0,\ y^2+z^2 \leqslant 1], \qquad P_1 = \underset{(x,y,z)}{E}[x=1,\ y^2+z^2 \leqslant 1],$$

$$Q_{n,k} = \underset{(x,y,z)}{E}\left[z=0, \left(x-\frac{2k-1}{2n}\right)^2 + y^2 \leqslant \frac{1}{4n^2}\right]$$

for $n = 1, 2, \ldots$ and $k = 1, 2, \ldots, n$. It is clear that the set

$$A_n = \bigcup_{k=1}^{n} Q_{n,k} \cup P_0 \cup P_1$$

has the following properties:

1° A_n is a 2-dimensional curvilinear polytope which is an AR-set.

2° For every point of A_n, different from the points $(0, 0, 0)$ and $(1, 0, 0)$, there exists a neighbourhood in A_n homeomorphic with a plane set.

3° None of the neighbourhoods of the points $(0, 0, 0)$ and $(1, 0, 0)$ in A_n is homeomorphic with a plane set.

4° The set A_n contains exactly $n-1$ points decomposing it between the points $(0, 0, 0)$ and $(1, 0, 0)$, namely the points $(1/n, 0, 0)$, $(2/n, 0, 0)$, ... $\big((n-1)/n, 0, 0\big)$.

5° For $m<n$ the set A_n is an image by an $\mathfrak{R}$-mapping of the set A_m.

6° For $m<n$ the set A_m is not homeomorphic with any subset of A_n.

For suppose that h is a homeomorphism mapping A_m onto a set $h(A_m) \subset A_n$. By 2° and 3°, h maps the points $(0, 0, 0)$ and $(1, 0, 0)$ onto the same points. It follows, by 4°, that the set $h(A_m)$ contains $n-1$ points decomposing it between the points $h(0, 0, 0)$ and $h(1, 0, 0)$. Hence the set A_m contains $n-1 > m-1$ points decomposing A_m between $(0, 0, 0)$ and $(1, 0, 0)$, which contradicts statement 4°.

It follows by 6° and (3) that

7° If $m < n$ then A_m is not $\underset{\Re}{\leqslant} A_n$.

But by 5° we have for $m < n$ the relation $A_n \underset{\Re}{\leqslant} A_m$. Consequently

(19) *If* $m < n$ *then* $A_n \underset{\Re}{<} A_m$.

Hence we can say that the sequence $\{A_n\}$ is $\Re$-*decreasing*.

6. A sequence of $\Re$-uncomparable polytopes. Now let us construct a sequence of 2-dimensional polytopes, which are $\Re$-uncomparable AR-sets. Let E_{n+3} denote the $(n+3)$-dimensional Euclidean space with the points $(x_1, x_2, \ldots, x_{n+1}, y, z)$. Setting for $k = 1, 2, \ldots, n+1$

$$\varphi_k(x, y, z) = (x_1, x_2, \ldots, x_{n+1}, y, z) \quad \text{with} \quad x_i = 0 \quad \text{for} \quad i \neq k \text{ and } x_k = x,$$

we obtain a homeomorphism φ_k mapping E_3 into E_{n+3}. It is readily seen that the set

$$B_n = \bigcup_{k=1}^{n+1} \varphi_k(A_n)$$

is a 2-dimensional polytope which is an AR-set. It may also be defined as the space obtained from $n+1$ copies of the polytope A_n by the identification of the corresponding points belonging to the discs P_0. For the sake of arithmetic simplicity, we have constructed the polytope B_n in the space E_{n+3}. But it is readily seen that B_n is homeomorphic with a subset of the space E_3.

Let us observe that

(20) *For* $n \neq m$ *neither of the sets* B_m, B_n *is homeomorphic with a subset of the other.*

To prove it, let us assume that there exists a homeomorphism h mapping B_m into B_n. It is easily seen that then $h(0, 0, \ldots, 0) = (0, 0, \ldots, 0)$ and that for every index $i = 1, 2, \ldots, m+1$ there exists an index $j(i) \leqslant n+1$ such that

$$h[\varphi_i(A_m)] \subset \varphi_{j(i)}(A_n). \tag{21}$$

Moreover it is clear that for different indices i the indices $j(i)$ are different. It follows $m+1 \leqslant n+1$, whence $m \leqslant n$. On the other hand, we infer by (21) and (19) that $m \geqslant n$. Consequently $m = n$, i. e. statement (20) is proved.

7. The number of $\Re$-types. It is known (see for instance [4], p. 333) that in every separable and complete space Z of the power $\mathfrak{c}$ there exists a family of $2^{\mathfrak{c}}$ subsets none of which is a continuous image of another. It follows that under subsets of Z there exists $2^{\mathfrak{c}}$ of different (even uncomparable) $\Re$-types. On the other hand, it is evident that the number of $\Re$-types under compact (metric) spaces cannot be larger than $\mathfrak{c}$, because the number of all compact subsets of the Hilbert space is equal to $\mathfrak{c}$. Let us prove that already the number of different $\Re$-types under 2-dimensional compacta is $\mathfrak{c}$. More exactly, let us prove the following statement:

(22) *There exists in the space* E_3 *a family of* $\mathfrak{c}$ *two-dimensional* $\Re$-*uncomparable* AR-*sets.*

Let us denote by W_n the 3-dimensional ball defined in E_3 by the inequality

$$\left(x - \frac{1}{n}\right)^2 + y^2 + z^2 \leqslant \frac{1}{25n^4}.$$

Let L denote the segment with endpoints $(0, 0, 0)$ and $(1, 0, 0)$. It is clear that there exists a homeomorphism h_n mapping the set B_{2n-1} (defined in Nr. 6) onto a subset of W_{2n-1}, satisfying the condition

$$L \cap h_n(B_{2n-1}) = \left(\frac{1}{2n-1}, 0, 0\right).$$

Let $\varkappa = \{k_n\}$ be an arbitrary sequence of natural numbers. For every $n = 1, 2, \ldots$ let us denote by g_n a homeomorphic mapping of the set B_{2k_n} onto a subset of W_{2n} satisfying the condition

$$L \cap g_n(B_{2k_n}) = \left(\frac{1}{2n}, 0, 0\right).$$

It is easily seen that the set

$$C_\varkappa = \bigcup_{n=1}^{\infty} [h_n(B_{2n-1}) \cup g_n(B_{2k_n})] \cup L \tag{23}$$

is an AR-set. Since there exist $\mathfrak{c}$ different sequences of natural numbers, it remains to prove that for every two sequences $\varkappa = \{k_n\}$ and $\varkappa' = \{k'_n\}$ the relation $C_\varkappa \underset{\Re}{\leqslant} C_{\varkappa'}$ implies $\varkappa = \varkappa'$. By (3) it suffices to show that

(24) *If there exists a homeomorphism* h *mapping* $C_\varkappa$ *into* $C_{\varkappa'}$ *then* $\varkappa = \varkappa'$.

Let the set $C_{\varkappa'}$ be given by the formula

$$C_{\varkappa'} = \bigcup_{n=1}^{\infty} [h'_n(B_{2n-1}) \cup g'_n(B_{2k'_n})] \cup L, \tag{23'}$$

where h'_n, g'_n are homeomorphisms of B_{2n-1} into W_{2n-1} and of $B_{2k'_n}$ into W_{2n} such that

$$L \cap h'_n(B_{2n-1}) = \left(\frac{1}{2n-1}, 0, 0\right), \qquad L \cap g'_n(B_{2k'_n}) = \left(\frac{1}{2n}, 0, 0\right).$$

It is clear that each of the sets $h_n(B_{2n-1})$, $g_n(B_{2k_n})$ is maximal with respect to the property of being a subcontinuum of $C_\varkappa$ which is 2-dimensional at each of its points. Analogously each of the sets $h'_n(B_{2n-1})$, $g'_n(B_{2k'_n})$ is maximal with respect to the property of being a subcontinuum of $C_{\varkappa'}$ which is 2-dimensional at each of its points. It follows that the homeomorphism h maps each of the sets $h_n(B_{2n-1})$ and $g_n(B_{2k_n})$ onto a subset of one of the sets $h'_m(B_{2m-1})$ and $g'_{m'}(B_{2k'_{m'}})$. By (20) we infer that

$$h[h_n(B_{2n-1})]\subset h'_n(B_{2n-1}).$$

It follows that h sends the points of the segment L lying between the points $(1/(2n-1),0,0)$ and $(1/(2n+1),0,0)$ onto points lying also between the same points. We infer that h sends the set $g_n(B_{2k_n})$ into the set $g'_n(B_{2k'_n})$. Again applying (20), we conclude that $k_n=k'_n$ for every $n=1,2,\ldots$, i. e. the sequences $\varkappa$ and $\varkappa'$ are identical. Thus (24), and consequently also (22) is proved.

Let us observe that

(25) *The number of different $\mathfrak{R}$-types between all polytopes is countable.*

In fact, every n-dimensional polytope is homeomorphic with a polytope lying in the Euclidean space E_{2n+1}, and every polytope lying in E_{2n+1} is evidently homeomorphic with a polytope for which there exists a triangulation all vertices of which have all coordinates rational.

It follows that among the $\mathfrak{R}$-types of ANR-sets the $\mathfrak{R}$-types containing polytopes are rather exceptional. It seems interesting to study the topological properties of ANR-sets belonging to those $\mathfrak{R}$-types, i. e. the ANR-sets $\mathfrak{R}$-equal to the polytopes.

8. $\mathfrak{R}$-neighbours. Suppose that X, Y are two spaces such that

$$X \underset{\mathfrak{R}}{<} Y,$$

but a space Z satisfying the condition

$$X \underset{\mathfrak{R}}{<} Z \underset{\mathfrak{R}}{<} Y$$

does not exist. Then the sets X and Y are said to be *$\mathfrak{R}$-neighbours*. More exactly, X is said to be an *$\mathfrak{R}$-neighbour of Y on the left* and Y an *$\mathfrak{R}$-neighbour of X on the right.*

Examples: 1. It is easy to see that for the segment there exists only one $\mathfrak{R}$-neighbour on the left, namely the set containing only one point, but many $\mathfrak{R}$-neighbours on the right, for instance the circle, the dendrite which is the sum of three segments with one common endpoint, the set composed of the segment and of one isolated point, the ray, the closure of the diagram of the function $y=\sin(1/x)$, with $0<x\leqslant 1$ and so on.

2. The disc is the $\mathfrak{R}$-neighbour on the left for the sphere and also for the projective plane.

3. The set which is the sum of a circle and one of its rays has two $\mathfrak{R}$-neighbours on the left: the circle and the sum of three segments having one endpoint in common.

The notion of the $\mathfrak{R}$-neighbour leads in a natural way to the notion of the *index of $\mathfrak{R}$-proximity* $I(X,Y)$ of two spaces X and Y, which we define as follows:

A finite sequence $X=X_0,X_1,\ldots,X_n,X_{n+1}=Y$ of spaces is said to be a *transition* from X to Y of index n, if for every $i=0,1,\ldots,n$ the spaces X_i and X_{i+1} are $\mathfrak{R}$-neighbours. If there exists a transition from X to Y, then we define $I(X,Y)$ as the minimum of indices of all transitions from X to Y. If, however, a transition from X to Y does not exist, then we set $I(X,Y)=\infty$. For instance, the index of $\mathfrak{R}$-proximity for two $\mathfrak{R}$-neighbours is equal to 0. If X is the 2-dimensional sphere and Y is the projective plane, then, by 2, we have $I(X,Y)=1$. But we are able to calculate the index of $\mathfrak{R}$-proximity only in very special cases. Already in the case where X is the 2-dimensional sphere and Y is the surface of the torus, we do not know whether $I(X,Y)$ is finite or infinite.

9. Problems concerning families of spaces ordered by the relation $\mathfrak{R}$. In Nr. 5 we have constructed a sequence of 2-dimensional polytopes $\{A_n\}$ such that

$$A_\infty \underset{\mathfrak{R}}{<} A_{n+1} \underset{\mathfrak{R}}{<} A_n \underset{\mathfrak{R}}{<} A_1 \quad \text{for} \quad n=2,3,\ldots,$$

where $A_\infty=P_0\cup P_1\cup L$ (we use here the notation of Nr. 5 and 7). It is easily seen that A_{n+1} is an $\mathfrak{R}$-neighbour (on the left) of A_n, i. e. there exists no space between A_n and A_{n+1} (in the sense of the relation $\underset{\mathfrak{R}}{<}$). On the other hand, it is easy to define a space lying between A_∞ and all sets $A_1,A_2,\ldots$ To obtain such a space, let us denote by Q_n^* the disc defined in the space E_3 by the conditions

$$\left(x-\frac{2n+1}{2n(n+1)}\right)^2+y^2\leqslant\frac{1}{4n^2(n+1)^2}; \qquad z=0.$$

It suffices to set

$$A_\omega=\bigcup_{n=1}^{\infty}Q_n^*\cup A_\infty,$$

to obtain a 2-dimensional AR-set satisfying the relation

$$A_\infty \underset{\mathfrak{R}}{<} A_\omega \underset{\mathfrak{R}}{<} A_n \quad \text{for every} \quad n = 1, 2, \ldots$$

It is easy to formulate numerous problems concerning the families of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$. Let us mention some of them:

1° Does there exist an uncountable family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$?

2° Does there exist a family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$ in a dense manner (i. e. such that for every two spaces $X \underset{\mathfrak{R}}{<} Y$ of this family there exists in it a space Z satisfying the relation $X \underset{\mathfrak{R}}{<} Z \underset{\mathfrak{R}}{<} Y$)?

3° Does there exist a family of spaces ordered by the relation $\underset{\mathfrak{R}}{<}$ similarly to the set of all real numbers?

All those problems concern quite arbitrary spaces, but each of them may also be formulated for a special class of spaces, for instance for AR-sets or ANR-sets. For the class of polytopes, problems 1° and 3° have manifestly negative answers, but problem 2° remains open.

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On a family of power $\mathfrak{c}$ consisting of $\mathfrak{R}$-uncomparable dendrites

by

K. Sieklucki (Warszawa)

K. Borsuk has introduced the notion of $\mathfrak{R}$-type ([1], p. 322) and proved that the set of different $\mathfrak{R}$-types among 2-dimensional compacta has power $\mathfrak{c}$. It is a consequence of the theorem, proved in that paper ([1], p. 327), on the existence of the family of power $\mathfrak{c}$ consisting of $\mathfrak{R}$-uncomparable 2-dimensional AR-sets. The purpose of the present paper is to prove that even among 1-dimensional AR-sets (i. e. among dendrites (1)) there exist $\mathfrak{c}$ different $\mathfrak{R}$-types. More strictly we shall prove that

On the plane E^2 there exists a family of power $\mathfrak{c}$ consisting of $\mathfrak{R}$-uncomparable dendrites.

It is easy to construct such a family having power n, where n is a natural number. (See [1], p. 322.) Beginning the construction of the family of power $\mathfrak{c}$ we shall prove first that there exists an

1. $\mathfrak{R}$-decreasing sequence of dendrites. Let d_0 be a closed segment on the plane E^2 with the endpoints $(0,0)$, $(0,1)$ and let $T(d_0)$ denote the dendrite (Fig. 1) consisting of the points of d_0 and those points $(x, y) \in E^2$ for which

$$x = \frac{2i-1}{2^j}, \quad 0 < y \leqslant \frac{1}{2^{j+1}} \quad \text{where} \quad i = 1, 2, \ldots, 2^{j-1};\ j = 1, 2, \ldots$$

while $K(d_0)$ denotes the dendrite (Fig. 2) consisting of the points of d_0 and those points $(x, y) \in E^2$ for which

$$x = \frac{2i-1}{2^j}, \quad 0 < |y| \leqslant \frac{1}{2^{j+1}} \quad \text{where} \quad i = 1, 2, \ldots, 2^{j-1};\ j = 1, 2, \ldots$$

If $d \subset E^2$ is a closed segment parallel either to the axis of abscissae or to the axis of ordinates and φ denotes the affine mapping preserving

(1) A dendrite is a locally conected continuum containing no simple closed curve. See for example [2], p. 224.