On the topological structure of 0-dimensional topological groups

by

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The following theorem is well known:

(i) *Each compact 0-dimensional, perfect, separable space is homeomorphic with the Cantor set, i. e. with the Cartesian product of denumerably many spaces each consisting of two elements.*

The Cartesian product of arbitrarily many spaces each consisting of two elements will be called the generalized Cantor set.

The family of closed-open sets of a 0-dimensional compact space constitutes a Boolean algebra. The Boolean algebra of all closed-open sets of a generalized Cantor set is a free Boolean algebra [1].

Theorem (i) formulated in the language of Boolean algebras has by Stone's representation theorem the following form:

(ii) *Each denumerable atom-free Boolean algebra is free.*

A question arises what kinds of assumptions must be added in order to obtain theorems like (ii) omitting the assumption of countability. In the topological language that means: which 0-dimensional compact spaces are homeomorphic with generalized Cantor sets?

It is easy to find perfect 0-dimensional compact spaces which are not homeomorphic with any generalized Cantor set even when they have the same character at each of their points. Namely: the space $\beta(N)\setminus N$, where $\beta(N)$ is the Čech compactification of the set of integers N, is compact, 0-dimensional, has at each of its points the character of the continuum, and is not homeomorphic with a generalized Cantor set [2].

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(1) A Boolean algebra is free whenever there exists a system $\{a_\iota\}$ of (free) generators such that $a_{\iota_1}^{(\varepsilon_1)} \cap \ldots \cap a_{\iota_k}^{(\varepsilon_k)} \neq \emptyset$ where $\varepsilon_i = 0$ or 1 and $a_{\iota_j}^{(1)} = a'_{\iota_j}$, $a_{\iota_j}^{(0)} = a_{\iota_j}$. Note that in the Boolean algebra of closed-open sets of a generalized Cantor set elements $\{(i_t)_{t\in T}\colon i_{t_0} = 0,\ i_t = 0 \text{ or } 1 \text{ if } t \neq t_0\}$ constitute a system of free generators.

(2) Indeed: by Novàk's theorem 2 in [3] for each closed infinite set $C \subset \beta(N)\setminus N$ we have $\overline{\overline{C}} = 2^{2^{\aleph_0}}$. One can find easily in each generalized Cantor set a closed subset of cardinal $\aleph_0$.

The aim of this note is to prove the following

THEOREM. *Each compact (infinite) 0-dimensional topological group is homeomorphic with the generalized Cantor set* [3].

In the language of Boolean algebras that means:

The Boolean algebra of all closed-open sets of a compact (infinite) 0-dimensional group is free.

LEMMA 1. *An infinite Boolean algebra B is free if and only if there exists a transfinite sequence of subalgebras*

$$B_1 \subset \ldots \subset B_\beta \subset \ldots \subset B_\gamma = B$$

such that (1) B_1 *consists of two elements,* $\emptyset$ *and* $\emptyset'$, (2) *if* β *is a limit-number,* $B_\beta = \bigcup_{\alpha<\beta} B_\alpha$, (3) *if* $\beta' = \beta+1$, *there exists a finite number of disjoint elements* $\mathfrak{a}_1^\beta, \ldots, \mathfrak{a}_{k_\beta}^\beta \in B$, *with* $\bigcup_{i=1}^{k_\beta} \mathfrak{a}_i^\beta = \emptyset'$, *such that* $\mathfrak{a}_i^\beta$ $(i = 1, \ldots, k_\beta)$ *is independent of* B_β *and* $B_{\beta'} = [B_\beta, \mathfrak{a}_1^\beta, \ldots, \mathfrak{a}_{k_\beta}^\beta]$.

Proof. Take the product $Y = \Pr_{\beta<\gamma} Y_\beta$ where $Y_\beta = (s_\beta^1, \ldots, s_\beta^k)$. Let B_Y be the algebra of all closed-open sets of the space Y. If $\overline{\gamma} \geqslant \aleph_0$, then B_Y is free. We define an application of the algebra B onto B_Y by

$$\varphi(\mathfrak{a}_i^\beta) = s_\beta^i \times \Pr_{\substack{\alpha\neq\beta \\ \alpha<\gamma}} Y_\alpha .$$

Since $\mathfrak{a}_1^\beta, \ldots, \mathfrak{a}_{k_\beta}^\beta$ are disjoint and $\mathfrak{a}_i^\beta$, $\mathfrak{a}_j^\alpha$ (with $\alpha \neq \beta$) are independent (just as $s_\beta^i \times \Pr_{\substack{\alpha\neq\beta \\ \alpha<\gamma}} Y_\alpha$ and $s_\alpha^j \times \Pr_{\substack{\xi\neq\alpha \\ \xi<\gamma}} Y_\xi$ are), φ can be extended to an isomorphism of B and B_Y.

Since each free Boolean algebra is representable as a B_Y, the conditions (1)-(3) are satisfied with $\mathfrak{a}_i^\beta = s_\beta^i \times \Pr_{\substack{\alpha\neq\beta \\ \alpha<\gamma}} Y_\alpha$.

Now let G be a 0-dimensional compact group and let $\{V_\alpha\}_{\alpha<\gamma}$ constitute a family of all neighbourhoods of the unity of G given by invariant subgroups [4]. We assume that $G = V_1 \in \{V_\alpha\}_{\alpha<\gamma}$. For each V_α take all its different translations $a_\alpha^1 V_\alpha, \ldots, a_\alpha^{n_\alpha} V_\alpha$. Since V_α are open subgroups and G is compact, n_α are finite for all α. The family of open sets $\{a_\alpha^i V_\alpha\}_{\alpha<\gamma}$ $(0 < i < n_\alpha)$ constitutes a base of closed-open sets in G. The Boolean

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[3] In the case where the group is assumed to be a Galois group with Krull's topology the theorem was proved by A. Białynicki-Birula in [1].

[4] The existence of a full set of neighbourhoods of the unity which are invariant subgroups of a compact 0-dimensional group is proved, e. g. in [2], p. 56.

algebra generated by the family $\{a_\alpha^i V_\alpha\}_{\alpha<\gamma}$ $(0 < i \leqslant n_\alpha)$ will be denoted in the following by B. Take a subalgebra $B_\beta \subset B$ generated by the family $\{a_\alpha^i V_\alpha\}_{\alpha<\beta}$ $(0 < i \leqslant n_\alpha)$.

LEMMA 2. $B_\beta = \{N_\beta V\}_{V\in B}$, *where* $N_\beta = \bigcap_{\alpha<\beta} V_\alpha$.

Proof. It is evident that each set $\mathfrak{a} \in B_\beta$ is of the form $N_\beta V$: it suffices to take $V = \mathfrak{a}$. In order to prove the converse, note first that for each $a \in G$ if $U \in B_\beta$ then $aU \in B_\beta$, and that for each $V \in B$, $V = a_1 V_{\alpha_1} \cup \ldots \cup a_n V_{\alpha_n}$ $(a_i \in G)$. Since N_β is an invariant subgroup (because V_α are invariant), it suffices to prove that $N_\beta V \in B_\beta$, where $V \in \{V_\alpha\}_{\alpha<\gamma}$. Assuming this take all finite intersections of the sets V_α $(\alpha < \beta)$ and well-order them in a transfinite sequence $\widetilde{V}_1, \widetilde{V}_2, \ldots, \widetilde{V}_\alpha, \ldots$ $(\alpha < \beta)$. We have

$$N_\beta V = \bigcap_{\alpha<\beta} \widetilde{V}_\alpha V . \tag{1}$$

Indeed, $N_\beta = \bigcap_{\alpha<\beta} V_\alpha = \bigcap_{\alpha<\beta} \widetilde{V}_\alpha$ and hence $N_\beta V = (\bigcap_{\alpha<\beta} \widetilde{V}_\alpha) V$. In order to prove (1) it suffices to show the following inclusion: $\bigcap_{\alpha<\beta} \widetilde{V}_\alpha V \subset (\bigcap_{\alpha<\beta} \widetilde{V}_\alpha) V$ (the converse is trivial). Then let $x \in \bigcap_{\alpha<\beta} \widetilde{V}_\alpha V$. It means that for each α $x = x_\alpha y_\alpha$, where $x_\alpha \in \widetilde{V}_\alpha$ and $y_\alpha \in V$. Hence $x_\alpha^{-1} x = y_\alpha$, which implies $V \cap \widetilde{V}_\alpha^{-1} x \neq 0$. Since V is compact and each finite intersection of sets belonging to $\{\widetilde{V}_\alpha^{-1} x\}_{\alpha<\beta}$ belongs to $\{\widetilde{V}_\alpha^{-1} x\}_{\alpha<\beta}$ again, $\emptyset \neq \bigcap_{\alpha<\beta} (V \cap \widetilde{V}_\alpha^{-1} x) = V \cap \bigcap_{\alpha<\beta} \widetilde{V}_\alpha^{-1} x$. Let $z \in V \cap \bigcap_{\alpha<\beta} \widetilde{V}_\alpha^{-1} x$, that is $z \in V$ and $z = h^{-1} x$, $h \in \bigcap_{\alpha<\beta} \widetilde{V}_\alpha$. Then $x = hz$, which proves $x \in (\bigcap_{\alpha<\beta} \widetilde{V}_\alpha) V$. Since $N_\beta V$ is open and $V_\alpha V$ are closed, equality (1) implies $N_\beta V = \widetilde{V}_{\alpha_1} V \cap \ldots \cap \widetilde{V}_{\alpha_k} V$ $(\alpha_i < \beta)$. Hence, since the unity of the group G belongs to V, $N_\beta V \supset \widetilde{V}_{\alpha_1} \cap \ldots \cap \widetilde{V}_{\alpha_k}$. But, $\widetilde{V}_{\alpha_1} \cap \ldots \cap \widetilde{V}_{\alpha_k} = \widetilde{V}$ being an open subgroup of $N_\beta \cdot V$, we have $N_\beta V = a_1 \widetilde{V} \cap \ldots \cap a_s \widetilde{V}$ $(a_i \in G)$. Then by $\widetilde{V} \in B_\beta$ we obtain $N_\beta V \in B_\beta$.

Proof of the Theorem. Take the algebra B and a transfinite sequence of its subalgebras $B_1 \subset B_2 \subset \ldots \subset B_\beta \subset \ldots \subset B_\gamma = B$, such that B_β is generated by the family $\{a_\alpha^i V_\alpha\}_{\alpha<\beta}$ $(0 < i \leqslant n_\alpha)$. We are going to prove that this sequence fulfils conditions (1)-(3) of lemma 1. It is evident that the sequence fulfils conditions (1) and (2). It remains to prove condition (3). Take a subalgebra $B_\beta \subset B$ and the sets

$$a_\beta^1 V_\beta, \ldots, a_\beta^{n_\beta} V_\beta . \tag{2}$$

We say that $a_\beta^i V_\beta$ is equivalent to $a_\beta^r V_\beta$ if $a_\beta^i N_\beta V_\beta = a_\beta^r N_\beta V_\beta$. It is evident that in each class of equivalent sets in the sequence (2) there is the same number k_β of elements. Let $m_\beta = n_\beta / k_\beta$. If $k_\beta = 1$, then in each class of equivalent elements there is exactly one element, that is $a_\beta^i N_\beta V_\beta \cap a_\beta^j N_\beta V_\beta = \emptyset$ if only $j \neq i$. Since $G = \bigcup_i^{n_\beta} a_\beta^i V_\beta$ and $a_\beta^i V_\beta \subset a_\beta^i N_\beta V_\beta$ we have $a_\beta^i V_\beta = a_\beta^i N_\beta V_\beta$. This proves that, in the case $k_\beta = 1$, $V_\beta \in B_\beta$ and hence $B_\beta = B_\beta + 1$. Consider the case $k_\beta \geqslant 2$. Denote by $a_\beta^{ij} V_\beta$, $1 \leqslant i \leqslant k_\beta$, elements of jth class of equivalent sets of the sequence (2). Put

$$\mathfrak{A}_i^\beta = \bigcup_{j=1}^{m_\beta} a_\beta^{ij} V_\beta \qquad (i = 1, 2, \ldots, k_\beta).$$

We are going to show that the elements $\mathfrak{a}_i^\beta$ have all the properties mentioned in (3) of Lemma 1. Indeed:

1° $\mathfrak{a}_i^\beta \cap \mathfrak{a}_r^\beta = \emptyset$ if only $r \neq i$,

2° all $\mathfrak{a}_i^\beta$ are independent of B_β,

3° $[B_\beta, a_\beta^1 V_\beta, \ldots, a_\beta^{n_\beta} V_\beta] = [B_\beta, \mathfrak{a}_1^\beta, \ldots, \mathfrak{a}_{k_\beta}^\beta]$.

1° is evident. To prove 2° take an arbitrary element $\mathfrak{b} \in B$. We have to prove $\mathfrak{a}_i^\beta \cap b \neq \emptyset$ and $\mathfrak{b} \cap (\mathfrak{a}_i^\beta)' \neq \emptyset$. By Lemma 2, $\mathfrak{b} = N_\beta V$. Since $\bigcup_{j=1}^{m_\beta} a_\beta^{ij} N_\beta V_\beta = G$, we have $\mathfrak{b} \cap a_\beta^{ij_0} N_\beta V_\beta \neq \emptyset$ or, which means the same, $N_\beta V \cap a_\beta^{ij_0} N_\beta V_\beta \neq \emptyset$, and hence we obtain $N_\beta V \cap a_\beta^{ij_0} V_\beta \neq \emptyset$. In order to obtain $\mathfrak{b} \cap (\mathfrak{a}_i^\beta)' \neq \emptyset$ note that $(\mathfrak{a}_i^\beta)' = \bigcup_{j=1}^{m_\beta} \bigcup_{r \neq i}^{k_\beta} a_\beta^{rj} V_\beta = \bigcup_{r \neq i}^{k_\beta} \mathfrak{a}_r^\beta$. Hence, since $k_\beta \geqslant 2$, $\mathfrak{b} \cap (\mathfrak{a}_i^\beta)' \neq \emptyset$.

3° It is evident that $[B_\beta, a_\beta^1 V_\beta, \ldots, a_\beta^{n_\beta} V_\beta] \supset [B_\beta, \mathfrak{a}_1^\beta, \ldots, \mathfrak{a}_{k_\beta}^\beta]$. In order to obtain the converse inclusion note that $a_\beta^{ij} V_\beta = a_\beta^{ij} N_\beta V_\beta \cap \mathfrak{a}_i^\beta$.

References

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Concerning the classification of topological spaces from the stand-point of the theory of retracts

by

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We consider in topology two spaces as different if they are not homeomorphic and we identify all homeomorphic spaces. It is clear, from the intuitive point of view, that difference between two not homeomorphic spaces may be more or less essential. But the notions which allow us to state precisely how far one space differs topologically from another are not numerous. To such notions belong the notion of *homoia (type of dimension)* due to Fréchet [2], the notion of *type of homotopy* due to Hurewicz [3], the notion of *domination* due to J. H. C. Whitehead [8] and also the notion of $\mathfrak{R}$-*type* [1].

In this note I introduce some notions intimately related to the notion of the $\mathfrak{R}$-type. Those notions allow us in some cases to formulate precisely the sense of the statement that one space is topologically more complicated than another, and also to formulate precisely in some cases the degree of diversity between two topological spaces. Moreover, I give some examples, determine the number of all $\mathfrak{R}$-types among the AR-sets and pose some problems.

1. Basic definitions. By an $\mathfrak{R}$-*mapping* of a space X onto a space Y we understand here a function φ satisfying the following three conditions:

1° X is the set of arguments and Y the set of values for φ.

2° φ is continuous.

3° There exists a continuous mapping ψ for which Y is the set of arguments and the values belong to X and for which we have $\varphi\psi(y) = y$ for every $y \in Y$.

In particular every *retraction*, i. e. every continuous mapping of X onto a subset Y of X, identical on Y, is an $\mathfrak{R}$-mapping. It is easy to show [1] that the $\mathfrak{R}$-mappings are the same as the mappings of the form hr, where r is a retraction and h a homeomorphism.

Two spaces X and Y are said to be $\mathfrak{R}$-*equal*, symbolically

(1) $$X \underset{\mathfrak{R}}{=} Y,$$