A note on Kosiński's r-spaces *

by

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Following Kosiński [1] we call a point x in a space X an *r-point* if x has arbitrarily-small neighborhoods U such that for each $y \in U$ there is a deformation retraction of $\overline{U}-y$ onto $\overline{U}-U$. A space X is an *r-space* if it is finite dimensional, compact metric and each point is an r-point. Problem 7 of [1] asks if (a, b) being an r-point of $A \times B$ implies that a and b are r-points of A and B respectively. We answer this question in the negative by giving a 4-dimensional finite polyhedron P^4 which is not an r-space but is such that its Cartesian product $P^4 \times S^1$ with a 1-sphere S^1 is an r-space. This example also furnishes a negative answer to Problem 6 of [1]. The polyhedron P^4 is the suspension of a Poincaré space M^3; i. e. M^3 is a polyhedral orientable closed 3-manifold such that $H_1(M^3, Z) = 0$ but $\pi_1(M^3) \neq 0$. It is not known if $P^4 \times S^1$ is a topological manifold.

One can readily show that P^4 has the homotopy type of the 4-sphere S^4. This fact also follows from Lemma 9 of [1]. Let $P^4 = M^3 \vee (a \cup b)$ where a and b are points and $\vee$ denotes the join. Clearly, P^4-a and P^4-b are contractible. Since P^4 is locally Euclidean at all other points and has the homotopy type of S^4, it follows that P^4-x is contractible for any $x \in P^4$. It follows from Theorem 6 of [1] that P^4 is not an r-space. We will show that $P^4 \times S^1$ is an r-space. We note that $P^4 \times S^1$ is an r-space if and only if the double suspension M^5 of M^3 is an r-space. (Indeed, for any space X we may represent the suspension X' of X as $X \vee (a \cup b)$ and the double suspension X'' as $X' \vee (c \cup d)$ where a, b, c, d are points. Then any point y in $P = (a \cup b) \times S^1$ in $X' \times S^1$ and any point z in $Q = (a \cup b) \vee (c \cup d)$ in X'' have homeomorphic neighborhoods. Similarly any point y in $(X' \times S^1) - P$ and any point z in $X''-Q$ have homeomorphic neighborhoods.)

Let $\widetilde{X} = X \vee p$ be the cone over X. Each point of $\widetilde{X}$ can be represented as (x, r) with $x \in X$ and $r \in I$. The representation is unique except for p which can be written as $(x, 1)$ for any $x \in X$.

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LEMMA 1 (1). *If X is contractible, then X is a deformation retract of $\widetilde{X}$.*

Proof. Let $\varphi(x,t)$ be a contraction of X to a point x_0. Let

$$\psi\big((x,r),t\big)=\begin{cases}\big(\varphi(x,t),r\big) & \text{for} \quad \tfrac{1}{2}\leqslant r\leqslant 1,\\ \big(\varphi(x,2tr),r\big) & \text{for} \quad 0\leqslant r\leqslant \tfrac{1}{2}.\end{cases}$$

Then ψ is constant on X and $\psi(\widetilde{X},1)$ is composed of a set A: $\big(\varphi(x,2r),r\big)$, $0\leqslant r\leqslant\frac{1}{2}$, and of a segment B: (x_0,r), $\frac{1}{2}\leqslant r\leqslant 1$. Thus if we follow ψ by the deformation which is the identity on A and deforms B to a point $(x_0,\frac{1}{2})$, the resulting deformation is still constant on X and deforms $\widetilde{X}$ to A. Since A does not contain p it may be deformed to X by a deformation which is constant on X. This proves the lemma.

LEMMA 2. *If X is completely regular at a point x_0 in X and $J=\{(x_0,t)\,|\,0<t<1\}\subset\widetilde{X}$, then for any t_0 such that $0<t_0<1$, there exists a deformation of $\widetilde{X}-(x_0,t_0)$ onto $\widetilde{X}-J$ which is constant on X.*

Proof. Choose a neighborhood U of x_0 and a map f: $\overline{U}\to I$ such that $f(x_0)=0$ and $f(\overline{U}-U)=1$. For each $x\in U$ we let

$$t_x=t_0+f(x)(1-t_0)\,,\qquad t'_x=t_0-f(x)t_0\,.$$

We deform each line segment $\langle(x,0),p\rangle$ onto itself by deforming the segment $\langle(x,0),(x,t'_x)\rangle$ to $(x,0)$, deforming the segment $\langle(x,t_x),p\rangle$ to p and deforming the open segment $\big((x,t'_x),(x,t_x)\big)$ linearly onto $\big((x,0),p\big)$. It is clear that this yields the desired deformation.

THEOREM 1. *If X is completely regular and has the property that for every $x\in X$, the space $X-x$ is contractible in itself to a point, then the point p in the cone $\widetilde{X}=X\vee p$ over X is an r-point of $\widetilde{X}$.*

Proof. We shall show that for any point (x_0,t_0) with $0<t_0<1$, there is a deformation retraction of $\widetilde{X}-(x_0,t_0)$ onto X. Since p has arbitrarily-small neighborhoods with closures homeomorphic with $\widetilde{X}$, it will follow that p is an r-point.

By Lemma 2 there is a weak deformation retraction of $\widetilde{X}-(x_0,t_0)$ to $\widetilde{X}-J$. By Lemma 1 there exists a deformation retraction of $(\widetilde{X}-x_0)\vee p$ onto $X-x_0$. The first of these deformations followed by the second gives a deformation retraction of $\widetilde{X}-(x_0,t_0)$ onto X if we simply let the deformations be constant on $(x_0,0)$. This proves the theorem.

(1) The present form of this lemma was suggested by the referee.

Now P^4 fulfils the hypotheses on X in Theorem I, so we get the following result:

THEOREM 2. *If P^4 is the suspension of a Poincaré space M^3 and M^5 is the suspension of P^4, then both M^5 and $P^4\times S^1$ are r-spaces.*

Reference

[1] A. Kosiński, *On manifolds and r-spaces*, Fund. Math. 43 (1955), p. 111-124.

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