On strengthening the Lebesgue Density Theorem

by

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Suppose that E is any Lebesgue measurable set on the line, then the density theorem due to Lebesgue states that

(1) $$\lim_{\substack{x \in I \\ |I| \to 0}} \frac{|I \cap \mathrm{C}E|}{|I|} = 0$$

for all points x in E except for a possible subset $E' \subset E$ of zero linear measure. Here $\mathrm{C}E$ denotes the complement of E, I denotes a closed interval, and $|A|$ denotes the linear measure of the measurable set A. The problem arises as to whether or not this result can be improved either (a) for each particular set E or (b) uniformly for all measurable sets E. We can look for sharper results in two possible directions, leading to four problems in all.

(i.a) Suppose that E is a fixed set, and E' is the exceptional subset where (1) is not satisfied. Can one always find a measure function of class 1 (¹) $\varphi(x)$ such that (²) $\varphi - m^*(E') = 0$? Certain special cases of this problem were considered by Besicovitch [2], [3], who actually determined the function $\varphi(x)$ for the sets he considered. We will see that the answer to (i.a) is affirmative.

(i.b) Does there exist a measure function of class 1, $\varphi(x)$, such that for any set E the subset E' where (1) is not satisfied has zero φ-measure? The answer is negative. Problems (i.a) and (i.b) are dealt with in § 1.

(ii.a) Suppose that E is a fixed set, can we strengthen (1)? I. e. Does there exist a real function $\psi(x)$, depending on E, monotonic increasing, defined for positive x with $\lim_{x \to 0+} \psi(x) = 0$ such that

(2) $$\lim_{\substack{x \in I \\ |I| \to 0}} \frac{|I \cap \mathrm{C}E|}{|I|\psi(|I|)} = 0$$

(¹) $\varphi(x)$ is a measure function of class 1 if $\varphi(x)$ is defined, continuous, monotonic increasing for positive x, $x/\varphi(x)$ is increasing, and $\lim_{x \to 0+} \varphi(x) = 0$, $\lim_{x \to 0+} \varphi(x)/x = +\infty$.

(²) $\varphi - m^*(A)$, for any set A, denotes the outer Hausdorff measure of A with respect to $\varphi(x)$, first defined in [4].

for all points x in E except for a subset $E'' \subset E$ of zero linear measure? Problem (ii.a) was posed by S. Ulam in 1937 in the *Scottish Book*, and was recently brought to my attention by P. Erdös. The answer to this question is again affirmative.

(ii.b) Does there exist a function $\psi(x)$ of the above type, independent of E, and such that (2) is satisfied at all points of any measurable set E, except for a subset of zero linear measure? We prove that no such function exists. Problems (ii.a) and (ii.b) are dealt with in § 2.

Thus we see that the Lebesgue density theorem is "best possible" in the sense that it cannot be improved uniformly for all measurable sets. However a stronger result (in two directions) is true for each particular measurable set on the line.

In § 3 we consider briefly the analogous results for measurable sets in Euclidean space of n-dimensions.

For notation and a summary of the definition of Hausdorff measures the reader is referred to [7].

1. Hausdorff measure of a linear set of zero Lebesgue measure. In this section we show that there is no linear set A such that [3] $\Lambda(A) = 0$ but $\varphi - m^*(A) > 0$ for every measure function $\varphi(x)$ of class 1. Since the measure functions of class 1 classify linear sets of zero Lebesgue measure, in the sense that $\varphi - m^*(A) < +\infty$ implies $\Lambda(A) = 0$ when $\varphi(x)$ is of class 1, this result is of some interest in itself. It provides an immediate solution to problem (i.a). We need several results proved previously.

If E is any linear set $E^{(x)}$ denotes a set on the x-axis congruent to E, $E^{(y)}$ denotes a set on the y-axis congruent to E. The Cartesian product $E^{(x)} \times F^{(y)}$ denotes the plane set of points (x, y) with x in $E^{(x)}$, y in $F^{(y)}$.

LEMMA 1. *Given any linear set E such that $\Lambda(E) = 0$, there exists a perfect linear set P such that*

$$\Lambda(E^{(x)} \times P^{(y)}) = 0 .$$

This is a restatement of theorem 1 of [7].

LEMMA 2. *For any perfect set P, there exists a measure function of class 1, with respect to which the measure of P is positive.*

[3] $\Lambda(A)$ denotes the Hausdorff measure with respect to x. When A is linear $\Lambda(A) = |A|$.

This is lemma 7 of [7].

LEMMA 3. *Suppose that $\varphi_1(x)$, $\varphi_2(x)$ are two measure functions each of class 1, and $\varphi(x) = \varphi_1(x)\varphi_2(x)$. Then there exists a constant $k > 0$ such that, if E is a φ_1-measurable linear set, F is any linear set, then*

$$\varphi - m^*(E^{(x)} \times E^{(y)}) \geqslant k\varphi_1 - m(E^{(x)})\varphi_2 - m^*(F^{(y)}) .$$

In the case where both factors on the right-hand side of the above inequality are positive, and at least one is infinite, the lemma should be interpreted as stating that the left-hand side is also infinite. A special case of lemma 3 was proved as theorem 3 of [7]. The lemma as stated can be proved using the methods of Marstrand [5]. As no new idea is involved the proof is omitted.

THEOREM 1. *Given any linear set A of zero Lebesgue measure, there exists a measure function $\varphi_1(x)$, of class 1, such that $\varphi_1 - m(A) = 0$* [4].

Proof. Apply lemma 1 to the set A, to obtain a perfect linear set P such that $\Lambda(A^{(x)} \times P^{(y)}) = 0$. $P^{(y)}$ being a closed set, it is measurable with respect to any measure function. Further by lemma 2, there exists a function $\varphi_2(x)$ of class 1 such that $\varphi_2 - m(P) > 0$.

Now if $\varphi_1(x) = x/\varphi_2(x)$, $\varphi_1(x)$ is also a measure function of class 1, and $\varphi_1(x)\varphi_2(x) = x$. Applying lemma 3 we see that if $\varphi_1 - m^*(A) > 0$, then $\Lambda(A^{(x)} \times P^{(y)}) > 0$ which is not true. Hence $\varphi_1 - m^*(A) = 0 = \varphi_1 - m(A)$.

COROLLARY. *For any Lebesgue measurable set E, there exists a measure function $\varphi(x)$ of class 1 such that the subset E' of those points x in E where*

$$\limsup_{\substack{x \in I \\ |I| \to 0}} \frac{|I \cap CE|}{|I|} \neq 0$$

satisfies $\varphi - m(E') = 0$.

This solves problem (i.a). To deal with problem (i.b) we need

LEMMA 4. *Given any measure function $\varphi(x)$ of class 1, there exists a linear set A measurable with respect to $\varphi(x)$, such that $0 < \varphi - m(A) < \infty$.*

This is proved in [4].

Suppose that $\varphi(x)$ is any measure function of class 1. Let $A \subset [2, 3]$ be such that $0 < \varphi - m(A) < \infty$. Let $E \subset [0, 1]$ be a Lebesgue measurable set with $|E| > 0$. Put $E_1 = E \cup A$. Then $|E_1| = |E|$, since $|A| = 0$. However if $x \in A$, then

$$\lim_{\substack{x \in I \\ |I| \to 0}} \frac{|I \cap CE_1|}{|I|} = 1$$

[4] (Added in proof.) It has been pointed out to me that a direct proof of Theorem 1 was given by Besicovitch; see Proc. Camb. Phil. Sov. 52 (1956), p. 28.

so that the exceptional subset of E_1 where (1) is not satisfied has positive φ-measure. That is, there is no "universal" function $\varphi(x)$ of class 1 which solves (i.a).

2. Strengthening the limit in the definition of density. We consider problem (ii.a) first in the case where E is a closed subset of the open interval $(0,1)$. It will be easy to deduce the general case from this one. Let G be the complement of E in $(0,1)$. Then G is open, and consists of an enumerable sequence of disjoint open intervals K_i, $i=1,2,\ldots$ Let $|K_i|=l_i$, and suppose that the K_i are enumerated so that

$$l_{i+1}\leqslant l_i\,,\qquad i=1,2,\ldots \tag{3}$$

Then $\sum_{i=1}^{\infty} l_i\leqslant 1$. The essential idea behind our proof is to use the fact that a convergent series of positive terms always converges with a definite rapidity: that is, there are series which converge more slowly that $\sum l_i$.

Choose a sequence $\{a_i\}$ $(i=1,2,\ldots)$ of positive numbers such that

$$a_i\geqslant a_{i+1}>0,\qquad a_i\to 0\quad \text{as}\quad i\to\infty, \tag{4}$$

$$\frac{l_i}{a_i}\geqslant\frac{l_{i+1}}{a_{i+1}}, \tag{5}$$

and

$$\sum_{i=1}^{\infty}\frac{l_i}{a_i} \tag{6}$$

converges. Note that (3) and (5) together imply that, if $l_i=l_{i+1}$, then $a_i=a_{i+1}$. We define a function $\psi(x)$ in terms of the sequences $\{l_i\}$ and $\{a_i\}$ as follows.

When $x=l_i$ put $\psi(x)=a_i$ $(i=1,2,\ldots)$. When $l_{i+1}\neq l_i$ and $l_{i+1}\leqslant x\leqslant l_i$, define $\psi(x)$ by linear interpolation. Put $\psi(0)=0$.

From (3) and (4) we deduce that

(7) $\psi(x)$ is continuous and monotonic increasing for $0\leqslant x\leqslant l_1$ with $\psi(0)=0$.

From (5) and (6) we deduce that

(8) $x/\psi(x)$ is monotonic increasing with $\lim_{x\to 0+} x/\psi(x)=0$.

We will see that $\psi(x)$ is a function such that (2) is satisfied. In the proof we will need

LEMMA 5. *Suppose that I_a is a closed interval on the line (not a single point), for each index a, and $J=\bigcup_{a\in K} I_a$, where the index set is not necessarily enumerable, is a bounded set. Then J consists of an enumerable set of disjoint intervals, and for any $\delta>0$, there is a finite set of some of the intervals I_a, $J'=\bigcup_{i=1}^{q} I_{a_i}$ such that $I_{a_i}\cap I_{a_j}$ is void $(i\neq j)$ and $|J'|>(\frac12-\delta)|J|$.*

The proof of this lemma is not difficult, and is left to the reader.

THEOREM 2. *Suppose that E is a closed linear set contained in the open interval $(0,1)$: then there exists a function $\psi(x)$ which is defined for positive x, is continuous, and decreases to zero as x decreases to zero, such that*

$$\lim_{\substack{x\in I\\ |I|\to 0}}\frac{|I\cap CE|}{|I|\psi(|I|)}=0$$

for all x in E except for a subset of zero Lebesgue measure.

Proof. Points of E which are end-points of intervals of the complementary set are clearly in the exceptional subset where (2) is not satisfied. Let E_1 be the set of points of E which are not end-points of complementary intervals. Then $|E_1|=|E|$. From the set G construct the function $\psi(x)$ satisfying the conditions (6), (7) and (8). Then

$$\sum_{i=1}^{\infty} l_i/\psi(l_i)\ \text{converges}. \tag{9}$$

For each positive rational k, let $E(k)$ be the subset of E_1 such that, for $\xi\in E(k)$,

$$\lim_{h\to 0+}\sup\frac{|[\xi,\xi+h]\cap G|}{h\psi(h)}>k\ (^5). \tag{10}$$

For any $\varepsilon>0$, we prove that $|E(k)|<\varepsilon$, and therefore $|E(k)|=0$. By (9) we can choose an integer N such that

$$\sum_{i=N+1}^{\infty} l_i/\psi(l_i)<\tfrac13\varepsilon k. \tag{11}$$

For every point ξ of $E(k)$, there is a $\delta=\delta(\xi)>0$ such that no point of $\bigcup_{i=1}^{N} K_i$ is within δ of ξ. For each point ξ of $E(k)$, choose an interval $I_\xi=[\xi,\xi+h]$ such that (10) is satisfied and $0<h<\delta(\xi)$. Then, for each ξ, $I_\xi\cap\bigcup_{i=1}^{N} K_i$ is void, and so

$$|I_\xi\cap G|=|I_\xi\cap\bigcup_{i=N+1}^{\infty} K_i|. \tag{12}$$

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(5) $[x,x+h]$ denotes the closed interval $\xi\leqslant x\leqslant\xi+h$.

Let $J = \bigcup_{\xi\in E(k)} I_\xi$. Then

(13) $$J \supset E(k).$$

Apply lemma 5 with $\delta = \frac{1}{6}$ to obtain a finite set of points $\xi_1, \xi_2, \ldots, \xi_q$ such that $I_{\xi_i} \cap I_{\xi_j}$ is void when $i \neq j$, and, if $J' = \bigcup_{i=1}^{q} I_{\xi_i}$, then

(14) $$|J'| > \tfrac{1}{3}|J|.$$

Put $h_j = |I_{\xi_j}|$, $j = 1, 2, \ldots, q$. Then by (10) and (12),

$$h_j < \frac{1}{k}\cdot\frac{|[\xi_j, \xi_j+h_j] \cap \bigcup_{i=N+1}^{\infty} K_i|}{\psi(h_j)}.$$

Now $[\xi_j, \xi_j+h_j] \cap \bigcup_{i=N+1}^{\infty} K_i$ consists of a finite or infinite number of whole intervals $K_{i_1}, K_{i_2}, \ldots$ and possibly one partial interval $K'_{i_1} \subset K_{i_2}$. Put $l'_{i_0} = |K'_{i_0}|$. Then, by (7) and (8),

$$\frac{|[\xi_j, \xi_j+h_j] \cap \bigcup_{i=N+1}^{\infty} K_i|}{\psi(h_j)} \leqslant \frac{l'_{i_0}}{\psi(l'_{i_0})} + \sum_{K_i \subset I_{\xi_j}} \frac{l_i}{\psi(l_i)} \leqslant \sum_{K_i \cap I_{\xi_j} \neq \emptyset} \frac{l_i}{\psi(l_i)}.$$

Since the I_{ξ_j} are disjoint and all the points ξ_j are in E, no interval K_i can have a non-void intersection with more than one I_{ξ_j}. Hence

$$\sum_{j=1}^{q} h_j < \frac{1}{k} \sum_{K_i \cap J' \neq \emptyset} \frac{l_i}{\psi(l_i)} \leqslant \frac{1}{k} \sum_{i=N+1}^{\infty} \frac{l_i}{\psi(l_i)}$$

since $J \cap \bigcup_{i=1}^{N} K_i$ is void by construction. By (11), we have

$$|J'| = \sum_{j=1}^{q} h_j < \tfrac{1}{3}\varepsilon;$$

and therefore, by (13) and (14) we have, as required,

$$|E(k)| < \varepsilon.$$

Let $E(+)$ be the subset of E_1 for which

$$\limsup_{h\to 0+} \frac{|[\xi, \xi+h] \cap G|}{h\psi(h)} > 0$$

when ξ is in $E(+)$. Since

$$E(+) = \bigcup_{\substack{k \text{ rational} \\ k>0}} E(k),$$

it follows that $E(+)$ has zero Lebesgue measure. Similarly, if $E(-)$ is the subset of those ξ of E_1 for which

$$\limsup_{h\to 0+} \frac{|[\xi-h, \xi] \cap G|}{h\psi(h)}$$

we also have $|E(-)| = 0$.

Now, if

$$\frac{|[\xi-h', \xi+h''] \cap CE|}{(h'+h'')\psi(h'+h'')} > a > 0,$$

it follows from (7) and (8) that at least one of the inequalities

$$\frac{|[\xi-h', \xi] \cap G|}{h'\psi(h')} > a; \quad \frac{|[\xi, \xi+h''] \cap G|}{h''\psi(h'')} > a$$

must be satisfied.

Hence if (2) is not satisfied at a point ξ of E, ξ must be in $E(+) \cup E(-)$, or is an end point of a complementary interval. Thus the subset $E'' \subset E$ where (2) is not satisfied has zero Lebesgue measure.

I state a special case of theorem 2 which is of some interest.

THEOREM 2A. *Suppose that E is a closed subset of $(0,1)$ whose complement consists of open intervals of lengths $\{l_i\}$ $(i = 1, 2, \ldots)$ and a is such that $1 \geqslant a > 0$ and $\sum l_i^{1-a}$ converges. Then at almost all points x of E*

$$\lim_{\substack{x\in I \\ |I|\to 0}} \frac{|I \cap CE|}{|I|^{1+a}} = 0.$$

This result can be proved by the methods used in the proof of the last theorem using the function $\psi(x) = x^a$. It is analogous to the main result of [2], which states that under the conditions of theorem 2A, the exceptional set where (1) is not satisfied has zero measure with respect to $\varphi(x) = x^{1-a}$.

The main theorem of the paper now follows easily.

THEOREM 3. *Given any Lebesgue measurable linear set E, there exists a function $\psi(x)$ which is defined for positive x, is continuous, and decreases to zero as x decreases to zero, such that*

$$\lim_{\substack{\xi\in I \\ |I|\to 0}} \frac{|I \cap CE|}{|I|\psi(|I|)} = 0$$

for all ξ in E except for a subset of zero Lebesgue measure.

Proof. There exists a sequence of bounded closed sets $F_1, F_2, \ldots, F_n, \ldots$ such that $F = \bigcup_{i=1}^{\infty} F_i \subset E$, and $|E - F| = 0$. Hence it is sufficient to prove the theorem for sets F which are enumerable unions of bounded closed sets. For each F_i, define a function $\psi_i(x)$ which satisfies theorem 2. That is, there is a subset $F_i' \subset F_i$ such that

$$\lim_{\substack{\xi \in I \\ |I| \to 0}} \frac{|I \cap CF_i|}{|I| \psi_i(|I|)} = 0 \tag{15}$$

whenever ξ is in $F_i - F_i'$ and $|F_i'| = 0$. Let $F' = \bigcup_{i=1}^{\infty} F_i'$, and choose a function $\psi(x)$ which tends to zero more slowly than any of the $\psi_i(x)$ as $x \to 0$. Thus $\psi(x)$ is monotonic,

$$\lim_{x \to 0+} \psi(x) = 0 \quad \text{and} \quad \lim_{x \to 0+} \frac{\psi(x)}{\psi_i(x)} = +\infty, \qquad i = 1, 2, \ldots \tag{16}$$

If $\xi \in F - F'$, then ξ is in some F_i but not in F_i' and therefore (15) is true, and, a fortiori, by (16),

$$\lim_{\substack{\xi \in I \\ |I| \to 0}} \frac{|I \cap CF|}{|I| \psi(|I|)} = 0 .$$

Since $|F'| = 0$, the theorem is proved.

This completes the solution of problem (ii.a). Problem (ii.b) is solved by

THEOREM 4. *Given any function $\psi(x)$, defined for $0 < x \leqslant 1$, which decreases to zero as x decreases to zero, and a real number a, $0 < a < 1$, there exists a perfect linear set $E \subset [0, 1]$ such that $|E| = a$, and for all ξ in E,*

$$\limsup_{\substack{\xi \in I \\ |I| \to 0}} \frac{|I \cap CE|}{|I| \psi(|I|)} = +\infty .$$

Proof. The theorem is proved by actually constructing a Cantor type set with the required properties. We use a method similar to one of the constructions of [1]. Let $\psi(a/2^k) = a_k$, $k = 0, 1, 2, \ldots$ Let $r_0 = 1 - a$, and define a decreasing sequence $r_0 > r_1 > r_2 > \ldots > 0$ such that $r_k / a_{k-1} \to +\infty$ and $r_k \to 0$ as $k \to \infty$. Let $F_0 = [0, 1]$. Obtain F_1 from F_0 by removing from the centre of F_0 an open interval of length $r_0 - r_1$. Then F_1 consists of two equal closed intervals. Suppose that F_i has been defined, and consists of 2^i equal disjoint closed intervals ($i = 1, 2, \ldots$). Obtain F_{i+1} from F_i by removing from the centre of each F_i an open interval of length $2^{-i}(r_{i-1} - r_i)$. Then F_{i-1} will again consist of 2^{i+1} equal disjoint closed intervals. Finally put $P = \bigcap_{i=1}^{\infty} F_i$. Clearly $|P| = a$, and P is a perfect linear set. Let ξ be any point of P. Then there is a sequence of closed intervals

$$I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_k \supset \ldots \supset \xi ,$$

such that I_k is one of the closed intervals making up F_k, and therefore has length $2^{-k}(a + r_k) = l_k$. Further $|CP - CF_k| = r_k$, and the set $CP - CF_k$ is equally divided among the intervals of F_k. Hence

$$|I_k \cap CP| = 2^{-k} r_k , \quad \text{and} \quad \frac{|I_k \cap CP|}{|I_k| \psi(|I_k|)} = \frac{r_k 2^{-k}}{2^{-k}(a + r_k)\psi(l_k)} .$$

When $k \geqslant k_0$, $r_k < a$ and therefore $\psi(l_k) \geqslant \psi(a/2^{k-1}) = a_{k-1}$. Hence, for $k \geqslant k_0$,

$$\frac{|I_k \cap CP|}{|I_k| \psi(|I_k|)} \geqslant \frac{1}{2a} \cdot \frac{r_k}{a_{k-1}} \to \infty \quad \text{as} \quad k \to \infty .$$

This completes the proof of the theorem.

3. Analogous results for measurable sets in Euclidean n-space.

The results in n dimensions analogous to theorem 1 and lemma 4 are true, and can be proved by essentially the same methods. Hence, whatever definition of density is being used, if it is known that the density theorem is valid, then the answer to the problem analogous to (i.a) is affirmative, and to that analogous to (i.b) is negative. Of course in n dimensions one needs to consider Hausdorff measure functions of class n, i. e. functions $\varphi(x)$ which are monotonic increasing, defined for positive x and such that $x^n/\varphi(x)$ is also monotonic with

$$\lim_{x \to 0+} \varphi(x) = 0, \quad \lim_{x \to 0+} \frac{x^n}{\varphi(x)} = 0 .$$

Before we can pose the n-dimensional analogues of (ii.a) and (ii.b) we must decide how zero density (analogous to (1)), will be defined in n dimensions. There are two possible definitions which are relevant: we use the terminology of Saks [6], p. 106 and p. 129.

For any set E in n dimensions, $d(E)$ denotes the diameter of E, and $|E|$ denotes the outer Lebesgue measure of E. R denotes a closed rectangle with sides parallel to the axes. A decreasing sequence of closed rectangles

$$R_1 \supset R_2 \supset \ldots \supset R_k \supset \ldots$$

is said to be *regular* if the ratio of the lengths of the longest to the shortest side of R_k is bounded, and $d(R_k) \to 0$ as $k \to \infty$. The Lebesgue density theorem can then be stated in two ways.

A. If E is any measurable set in n-space, then there is a subset $E' \subset E$ with $|E - E'| = 0$ such that when $x \in E'$,

$$\lim_{k \to \infty} \frac{|R_k \cap CE|}{|R_k|} = 0 ,$$

for every regular sequence of closed rectangles $\{R_k\}$ tending to x.

B. If E is any measurable set in n-space, then there is a subset $E' \subset E$ with $|E-E'|=0$ such that, when $x \in E'$,

$$\lim_{k\to\infty} \frac{|R_k \cap CE|}{|R_k|} = 0,$$

for every decreasing sequence of closed rectangles $\{R_k\}$ containing x such that $d(R_k)\to 0$.

A and B are equivalent in 1 dimension: however in n-space ($n \geqslant 2$), B is stronger than A. Both A and B are true in n-space (see [6], p. 129 for a proof of B). I can show that A may be strengthened in the same way that (1) was strengthened to give (2) for linear sets.

THEOREM 5. *Given any Lebesgue measurable set E in Euclidean n-space, there exists a function $\psi(x)$ which is defined and continuous for positive x, decreases to zero as x decreases to zero, and satisfies*

$$\lim_{k\to\infty} \frac{|R_k \cap CE|}{|R_k|\,\psi(|R_k|)} = 0$$

when $\{R_k\}$ is any regular sequence of rectangles decreasing to ξ and ξ is any point of a subset $E' \subset E$ with $|E-E'|=0$.

It is sufficient to prove the analogous result for density with respect to a sequence of cubes tending to ξ. This can be done by making a few modifications to the proof of theorems 2 and 3 given in § 2. Since no essentially new idea is involved, I omit the details.

The theorem in n-dimensions analogous to theorem 4 is true and can be proved by defining the n-dimensional analogue of the Cantor set in the obvious way.

There remains the following unsolved problem arising from the definition B of density.

PROBLEM. Given an n-dimensional Lebesgue measurable set E, does there exist a function $\psi(x)$ which is monotonic increasing and continuous, with $\lim_{x\to 0+}\psi(x)=0$, and a subset $E' \subset E$ with $|E-E'|=0$, such that for any $\xi \in E'$,

$$\lim_{\substack{\xi\in R\\ d(R)\to 0}} \frac{|R \cap CE|}{|R|\,\psi(|R|)} = 0\ ?$$

I have been unable to decide the answer to this question. The methods of § 2 break down in this case essentially because the analogue of lemma 5 is not valid without a restriction on the ratio of the lengths of the longest and shortest sides of closed rectangles making up J.

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