# Deformation and mapping theorems *

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A class of theorems, (Q), may be characterized in a general way as asserting that in the deformation of a manifold, $M$, a point set, $P$, satisfying a certain property, contains an element for which a prescribed real valued continuous function $f$ takes on an assigned value. This would follow as a corollary if (T): $P$ contains a continuum joining points with $t = 0$ and those with $t = 1$. The demonstration of (T), in particular cases, depends on the fact that otherwise $P$ would admit a separation by a carrier, $C$, of a cycle homologous to the base cycle of the manifold though in the cases treated in this paper, contrary to its definition, $C$ must contain points of $P$.

This method, (T), would seem to have inherent interest, and is applied to some deformations of circles, etc. For instance, *if a circle of radius* 2 *is deformed in a plane into one of radius* $\frac{1}{2}$, *then a fixed circle of radius* 1 *intersects some intermediate curve of the deformation in a pair of points which are maps of antipodal points* or which (under weak regularity conditions) *bisect the length of the intermediate curve*, etc. When the $t$ segment is replaced by a circle, *applications to doubly periodic functions may be derived.* However, a theorem in the class (Q) may be valid without (T). For instance, if circles are replaced by $n$ spheres above, then without establishing (T), we show there are $n+1$ orthogonal points of intersection, *for some $t$, of the fixed unit sphere with the deformed sphere corresponding to this $t$ value.* In connection with the $n$ sphere, we demonstrate a *general criterion for the existence of a common image point for the map of orthogonal $k$ tuples of $S^{n-1}$ to $E^l$*, yielding the first general breakthrough on a problem of Knaster's and as a special case, for $n$ a prime, the generalized Kakutani theorem.

We use $S$, $S(\varrho)$, $E$ and $E(+)$ for the circle, the circle of radius $\varrho$ about the origin, the real axis and the positive real axis, respectively.

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$D$ is a disk, namely the circumference plus the interior of a circle. An open disk is one with the circumference discarded. A superscript is used to indicate dimension here, thus $E^l$ is the $l$ dimensional Euclidean space, etc. The notation $v = \infty$ means merely $v$ is sufficiently large and positive. The unit segment $\{t \mid 0 \leqslant t \leqslant 1\}$ is denoted by $I$. We sometimes write $S_u, I_t, E_u$ to indicate the parameter entering where this adds to clarity, and $u$ is a real number, or a real number mod 1, according as we have $E_u$ or $S_u$. *Maps* are understood to be continuous and the term *curve* $A$, is used both for the continuous correspondence $A(\lambda \mid I_\lambda)$ and for the graph $\Gamma(A)$. A curve is *in* a set $P$ if its graph is a subset of $P$. The interpretation intended for $S \times I$ is connoted by using terms like *cylinder* or *annulus*. The notions of separation of subsets used here are covered by two anterior results of the writer's ([1], Theorem 1A and 1C) which we refer to as BA and BC respectively. Unless otherwise stated the coefficient group for homologies is $I_2$ namely the integers mod 2.

LEMMA 1. *Let $f(x, t)$ be a real valued continuous function on $E \times I$ satisfying: $f(x, t)$ is periodic of period $2\pi$ in $x$ for each fixed $t$ and $f(x, t) = -f(x+\pi, t)$. Let $Z = \{(x, t) \mid f(x, t) = 0\}$. Then $Z$ contains a continuum connecting $E \times 0$ and $E \times 1$.*

Interpret $x$ as an angle coordinate and $r$ as radius. Let $S \times I$ be the annulus $\{(x, r) \mid 0 \leqslant x < 2\pi,\ r = 1+t\}$. Because of the periodicity requirement, we may consider $f(x, t)$ on $S \times I$ to the reals. Suppose the lemma's assertion false. We rephrase BA for our special situation. Thus, if $Z = C^0 \cup C'$ where $X_0 = C^0 \cup S \times 0$ and $X_1 = C' \cup S \times 1$ are disjunct compact sets, there is a set $Q$, the boundary of a finite number of open disks, covering $C^0 \cup S \times 0$, which is disjunct from $X_0 \cup X_1$ and carries a cycle homologous to that of $S \times 0$. Then either from the fact that $Q$ is a finite sum of arcs or, since $Q$ is locally connected, by closing $S \times I$ by two disks and applying a corollary of the Torhorst theorem ([10], Theorem 6.7, p. 114) henceforth referred to as CT, $Q$ contains a simple closed curve $K$ separating $S \times 0$ and $S \times 1$ by the Jordan Separation theorem ([6], p. 358). $S \times 0$ and $S \times 1$ are in the two disjoint domains into which $S \times I$ is separated by $K$. Moreover $K$ is homologous (= homotopic here) to $S \times 0$.

Let $K'$ be the reflection of $K$ in $r = \theta$ defined by $(x', r') = (x+\pi, r)$, $(x, r) \in K$. Let $r_1$ and $r_2$ be the maximum and minimum radii of $K$, taken on, in view of the compactness of $K$, in distinct points $(x_1, r_1)$ and $(x_2, r_2)$ respectively. To show $\emptyset \neq K \cap K'$ we may as well assume $(x_1, r_1) \cup (x_2, r_2) \notin K'$. Since $r_1$ and $r_2$ are likewise the maximum and minimum radii of $K'$, $(x_1, r_1)$ can be joined to $S \times 1$ by an arc not cutting $K'$ while $(x_2, r_2)$ can be joined to $S \times 0$ by an arc not cutting $K'$. Since $K'$ separates $S \times 0$ and $S \times 1$, the points $(x_1, r_1)$ and $(x_2, r_2)$ are separated by $K'$. Accordingly *each of the two sub continua of $K$ joining $(x_1, r_1)$ and $(x_2, r_2)$ cuts $K'$*. Let $(x_0, r_0) \in K \cap K'$ where $r_0 = 1+t_0$, then $(x_0+\pi, 1+t_0) \in K \cap K'$. Since $f(x_0, t_0) = -f(x_0+\pi, t_0)$ there is a point $(\bar{x}, 1+\bar{t})$ on a subcontinuum of $K$ joining $(x_0, 1+t_0)$ and $(x_0+\pi, 1+t_0)$ for which $f(\bar{x}, \bar{t}) = 0$. On the other hand, $(\bar{x}, 1+\bar{t}) \in K$ and hence cannot be in $Z$, a manifest contradiction.

For completeness we present three lemmas which seem intuitively evident.

LEMMA 2. *Let $a$ be the square $I_u \times I_v$ and let $A$ be a curve joining $0 \times I_v$ and $1 \times I_v$ in $\{(u, v) \mid 0 < v < 1\}$. Then $A$ separates $I_u \times 0$ from $I_u \times 1$.*

Let $u, v$, refer to Cartesian coordinates. We shall say $(u, v)$ is *over*, or *covers*, $u$, and write $p_v$ for projection on the hyperplane orthogonal to the $v$ axis. Thus $p_v(u, v) = u$. The curve $A = A(\lambda \mid I_\lambda)$ can be considered represented by a generally multi valued function of $u$ and the point set is indicated by the graph, where the independent variable is included in the symbol, when this lends clarity. Thus

$$\Gamma(A, u) = \{(u, v) \mid (u, v) = (u(\lambda), v(\lambda)) \in A(\lambda)\} \subset E_u \times E_v .$$

We interpret addition of graphs by

$$\Gamma(A_1, u) + \Gamma(A_2, u) = \{(u, v) \mid v = v_1 + v_2,\ (u, v_i) \in \Gamma(A_i, u)\} .$$

In general it is, of course, not true that $\Gamma(A_1, u) + \Gamma(A_2, u)$ is connected.

LEMMA 3. *Let $a$ be the strip $I_u \times E_v$. The two curves $A_i = A_i(\lambda \mid I_\lambda)$, $i = 1, 2$, are entirely in $I_u \times I_v$ and satisfy*

(a) *$A_i(0)$, $A_i(1)$ are over $u = 0$ and $u = 1$ respectively.*

*Furthermore*

(b) *$A_i$ covers $I_u$ finitely, i. e. $A_i$ contains a finite number, only, of points over each $u$ in $I_u$.*

*Then $\Gamma(A_1, u) + \Gamma(A_2, u)$ contains a curve $K$, where $p_v K = I_u$.*

If $A_1$ and $A_2$ represent single valued functions of $u$ the result is immediate. The argument given below assumes neither $A_1$ nor $A_2$ is single valued in $u$. Imbed $a$ in $a \times I_w$ where $w$ refers to the third Cartesian coordinate. Let $A_{2c}$ be the cylinder in $a \times I_w$ with generators parallel to the $w$ axis and generating curve $A_2$. Let $A_{1d}$ be the curve in $a \times I_w$ defined by

$$(u, v) \in A_2(\lambda), \qquad w = \lambda .$$

The cylinder $A_{1dc}$ has $A_{1d}$ as a generating curve, and the parallels to the $v$ axis as generators. Evidently each generator cuts the generating curve in a unique point. As easy consequence of hypothesis (b) is that $A_{1dc}$ cuts $A_{2c}$ in a *locally connected point set* $C$ (which projects by $p_w$

into $A_2$). Project $A_{1dc}$ into the $v, w$ plane, and in fact onto $\alpha' = E_v \times I_w$. Then write

$$p_u A_{1d} = A_2', \quad p_u C = B.$$

It is immediate from Lemma 2 that $A_{2c}$ separates points with $v = -\infty$ from those with $v = \infty$ in $I_u \times E_v \times I_w$ and so $C$ separates such points in $A_{1dc}$ whence $B$ separates such points in $E_v \times I_w$. Moreover $B$ *contains a continuum* $A_1'$ connecting $I_v \times O_w$ and $I_v \times 1_w$. (For instance using the device of taking two copies $\alpha'$, $\alpha''$ of $I_v \times I_w$ and identifying the horizontal sides $v = 0$ in $\alpha'$ and $\alpha''$ for one seam as well as the sides $v = 1$, for another, there results the circular cylinder $S \times I_w$. Since $B$ intersects every curve in $\alpha'$ joining $v = 0$ and $v = 1$ it follows that $B$ intersects every simple closed curve $K$ in $S \times I_w$ which is homologous to $S \times 0$. The italicized assertion now follows from BA.) If $A_1'$ is taken as the component of $B$ joining $I_v \times 0_w$ and $I_v \times 1_w$ then it is locally connected, since $C$ is locally connected and maps preserve local connectivity. By the Hahn-Mazurkiewicz theorem ([6], p. 185), $A_1'$ is a curve, and is a single valued function of $w$. One could reduce to the case both curves are single valued functions of the base variable by proceeding as above with $u, v, w$ going into $w, v, u'$ and $\alpha'$, $A_i'$ replacing $\alpha$, $A_i'$. However, this is unnecessary on noting $\Gamma(A_1', w) + \Gamma(A_2', w) = \Sigma$, is a homeomorphism of $\Gamma(A_2', w)$ with the points of $A_2'$ over $w = 0$ and $w = 1$ moving vertically. Hence $\Sigma$ contains a curve $K'$, where $K'$ is a curve covering $I_w$ since a subcontinuum of a curve is a curve. Let $P_w^{-1}$ be the projection of $\alpha'$ onto $A_{1dc}$ so $Q = p_w P_w^{-1}$ maps $\alpha'$ onto $\alpha$ preserving the $v$ coordinate. Write $Q(w, v) = (g(w), v)$ whence $Q(w, v_1 + v_2) = (g(w), v_1 + v_2)$. Then

$$K = QK' \subset Q\Gamma(A_1', w) + Q\Gamma(A_2', w) \subset \Gamma(A_1, u) + \Gamma(A_2, u).$$

The fact that $gI_w = I_u$ shows $p_v K = I_u$. The continuity of $Q$ insures $K$ is a curve.

LEMMA 4. *Let $\alpha$ be the circular cylinder $S_u \times E_v$ and let $A_1$ and $A_2$ be two curves each homotopic to $S_u \times 0$ and covering $S_u$ finitely. Then* (a) *$\Gamma(A_1, u) + \Gamma(A_2, u)$ contains a simple closed curve homotopic to $S_u \times 0$ and* (b) *no two distinct curves satisfy* (a).

There is no loss of generality if $A_1$ and $A_2$ are considered curves in $S_u \times \{v \mid 0 < v < \frac{1}{2}\}$ and we may even require that $A_1$ and $A_2$ be simple closed curves homotopic to the base circle. (Indeed $A_i$ considered on $S^2 = S \times I \cup D \times 0 \cup D \times 1$ separates the north and south poles whence by CT, $\Gamma(A_i, u)$ contains a simple closed curve with this property.)

Let $y$ be the radial coordinate measured orthogonally to the axis of the circular cylinder $\alpha$ so $u, v, y$ constitute the usual cylindrical coordinates. Let $A_{2c}$ be the cylinder determined by the radial rays starting



at $A_2$, i. e. the lines $(u, v)$ constant in $\alpha \times I_y$. Suppose $A_1$ given by $A_1(\lambda \mid I_\lambda)$, $A_1(0) = A_1(1)$. Then $A_{1d}$ is defined by

$$(u, v) \in A_1(\lambda), \quad y = \lambda.$$

This is a generating curve for the cylinder $A_{1dc}$ whose generators are the lines $(u, y) =$ constant. Denote by $C$ the intersection of $A_{1dc}$ and $A_{2c}$. Without our hypothesis that $A_i$ covers $S_u$ finitely we should not be assured, as here, that $C$ is locally connected. Let $A^0$ be the intersection of $A_{1dc}$ with $v = 0$. Note $C$ projects radially onto $A_2$.

In $A_{1dc}$ identify the generators $g_0 = \{0, v, 0\}$ and $g_1 = \{0, v, 1\}$. The result is a cylinder $\beta$ homeomorphic to a circular cylinder. Denote the effect of the passage from $A_{1dc}$ to $\beta$ by adding a dot. Thus $A^0$ becomes $A^{\cdot 0}$ and $C$ becomes $C^{\cdot}$, etc. Since $A_2$ separates $S \times 0$ from $S \times 1$ in $\alpha$, $C^{\cdot}$ separates $A^{\cdot 0}$ from points with $v = \infty$. We may replace $C^{\cdot}$ by one of its separating components and hence, applying CT, $C^{\cdot}$ contains a simple closed curve $B$ with the same separating property or, equivalently, $B$ is homologous or homotopic to $A^{\cdot 0}$ in $\beta$. $C^{\cdot}$ can contain no other simple closed curve, $D$, homologous to $A^{\cdot 0}$ and disjunct from $B$. Indeed under radial projection denoted by $p$, $pB \sim pA^{\cdot 0} \sim S \times 0$. Since the projection of $B$ is in $A_2$, it must therefore cover all of $A_2$. Similarly the radial projection of $D$ is onto $A_2$. Let $s = \sup\{v \mid (u, v) \in \Gamma(A_2)\}$ and let $l = \inf\{v \mid (u, v) \in \Gamma(A_2)\}$, where only the case $s \neq l$ is of interest. For suitable values of $u$ and $y$, $B$ must contain points $q_1$ and $q_2$ whose ordinates are $s$ and $l$ respectively. Since $D$ must lie between these ordinates, $q_1$ is above and $q_2$ is below $D$, whence the arcs of $B$ joining $q_1$ and $q_2$ must cut $D$ contrary to the postulated disjunction of $B$ and $D$.

Let the parameter $\{\varrho \mid I_\varrho\}$ increase monotonely in traversing $A^{\cdot 0}$ in one sense, with $A^{\cdot 0}(0) = A^{\cdot 0}(1)$. Then the situation in $\beta$ is that of one curve ($A_{1d}^{\cdot}$) a single valued function of $\varrho$ and the other ($\beta$) a simple closed curve. For this case the lemma is valid because

$$\Gamma(A_{1d}^{\cdot}, \varrho) + \Gamma(B, \varrho) = \Sigma$$

amounts merely to a homeomorphic displacement of $\Gamma(B, \varrho)$ and so $\Sigma$ contains a simple closed curve $K'$ homologous to $A^{\cdot 0}$. (Though the transformation $\nu$: $A_{1dc} \to B$ has no continuous inverse, the composition $\psi = \mu\nu$, $\mu$: $\alpha \to A_{1dc}$, admits the continuous inverse $\psi^{-1}$ which is simply the radial projection $p$ of $\beta$ (a folded cover of $\alpha$) onto $\alpha$.) Hence $K = pK'$ is a simple closed curve homologous to $S \times 0$ and

$$\Gamma(A_1, u) + \Gamma(A_2, u) \supset K.$$

The demonstration of (b) is immediate. $K$ and $L$ are two disjunct simple closed curves in $\Gamma(A_1, u) + \Gamma(A_2, u)$. Let $K_c$ and $L_c$ be the cylinders defined analogously to $A_{2c}$ with generating curves $K$ and $L$ respectively and generators the radii through $K$ and $L$. Then in $\beta$ we should have for the intersection of $A_{1dc}$ and $K_c$ and $L_c$ the sets $k_c$ and $l_c$ which in turn contain simple closed disjunct curves $E$ and $F$ respectively each homotopic to $A^{\cdot 0}$, and projecting radially onto $K$ and $L$. Since $A_{1d}$ has one point only on each vertical in $\beta$ there are unique disjunct simple closed curves $B$ and $D$ in $\beta$ homotopic to $A^{\cdot 0}$ and homeomorphic to $E$ and $F$ which satisfy

$$\Gamma(A_{1d}, \varrho)-\Gamma(E, \varrho)=\Gamma(B, \varrho), \qquad \Gamma(A_{1d}, \varrho)-\Gamma(F, \varrho)=\Gamma(D, \varrho).$$

These relations are preserved under radial projection onto $a$ and since the separate terms on the left hand side project onto $\Gamma(A_1, \varrho)$ and into $\Sigma$ respectively, it follows that the right hand side projects radially into $\Gamma(A_2, \varrho)$. Hence $B$ and $D$ are in $C^{\cdot}$. However, in the demonstration of part (a) the possibility of existence of two curves such as $B$ and $D$ was eliminated.

Write $E^2-\theta$ for the plane punctured by eliding the origin, $\theta$. Denote by $C(t)$ the closed curve $h$: $S_x \times t$ and by $C(t, x_1, x_2)$ the map of the arc $[x_1, x_2]$ of $S_x \times t$. Require that variation $C(t, x_1, x_2)$ for each fixed $t$ be continuous in $t$, $x_1$, and $x_2$. We then say that the family of closed curves $\{C(t) | I_t\}$ is of *equi-bounded variation or* EBV.

THEOREM 1. *Let $h$: $S_x \times I_t \to E^2-\theta$ be a homotopy and suppose that $C(0)$ is a circle of radius 2 about $\theta$ while $C(1)$ is contained in the interior of the disk of radius 1 about $\theta$. Then*

(a) *for some $x$ and $t$, say $x_0, t_0$, $h(x_0, t_0)$ and $h(x_0+\pi, t_0)$ are on $S(1)$;*

(b) *if the family $\{C(t)\}$ is* EBV *then for some $\bar{t}$, $C(\bar{t})$ intersects $S(1)$ in two points $w_1, w_2$ such that the two sensed arcs of $C(\bar{t})$ with these end points have equal lengths;*

(c) *the radial projection on the cylinder $S_w \times I_t$ of the antipodal point pairs of $C(t)$ for each $t \in I_t$, includes a continuum joining $S_w \times 0_t$ and $S_w \times I_t$.*

For (a): Let

$$f(x, t)=|h(x, t)|-|h(x+\pi, t)|$$

where $|h|$ is the norm or length of the vector to $h(x, t)$. The continuity of $h$ implies that of the norm, $|h(x, t)|$. (The immediate observation that for each $t$, $f(x, t)$ has a zero can be viewed, in the light of possible later generalization, as a trivial example of a theorem of Borsuk's ([6], p. 20) or of Dyson's [5].) Then $f$ satisfies the conditions of Lemma 1. Hence there is a continuum $K$ (contained in the set of zeros, $Z$, of $f(x, t)$) joining $S \times 0$ and $S \times 1$. For $t=0$, $|h(x, 0)|=2$ and for $t=1$, $|h(x, 1)|<1$ whence for some $x_0, t_0$ on $K$, $|h(x_0, t_0)|=1$. This is tantamount to the assertion (a) of the theorem.



For (b): Introduce the parameter

$$s=s(x, t)=2\pi l(x, t)/lC(t)$$

where $l(x, t)$ is the length of the curve $C(t, 0, x)$ and $lC(t)$ that of $C(t)$. Since the family $\{C(t)\}$ is EBV, $s(x, t)$ is continuous in $x$ and $t$ simultaneously.

Let $h(s, t)$ be the point on $C(t)$ with parameter $s$. That $h$ is continuous in $s$ and $t$ follows directly. (Thus suppose for some sequence $(s^n, t^n) \to (\bar{s}, \bar{t})$ yet $h(s^n, t^n)$ does not converge to $h(\bar{s}, \bar{t})$. Accordingly for some positive $\varepsilon$, there exists a subsequence $\{(s^m, t^m)\} \subset \{(s^n, t^n)\}$ such that $|h(\bar{s}, \bar{t})-h(s^m, t^m)|>\varepsilon$. Let $X^m=\{x \mid s(x, t^m)=s^m\}$. Choose $x^m$ arbitrarily in $X^m$. By compactness, $\{x^m\}$ contains a subsequence $\{x^l\}$ converging to a value $\bar{x}$, i. e. $(x^l, t^l) \to (\bar{x}, \bar{t})$. The continuity of $s$ in $x$ and $t$ implies that $s^l=s(x^l, t^l) \to s(\bar{x}, \bar{t})$ so that $\bar{s}=s(\bar{x}, \bar{t})$. Note that $h(s(x, t), t)=h(x, t)$. Since $h(x, t)$ is continuous, $h(x^l, t^l) \to h(\bar{x}, \bar{t})$, whence $h(s^l, t^l) \to h(\bar{s}, \bar{t})$, in contradiction with the definition of the subsequence $\{(s^m, t^m)\}$.) Writing $|h(s, t)|$ in place of $|h(x, t)|$ and $f(s, t)$ in place of $f(x, t)$ in the argument for (a) yields assertion (b) since $|h(s, t)|=1=|h(s+\pi, t)|$ is effectively the desired result.

For (c): We may, if necessary, deform $C(1)$ onto the circle $S(\frac{1}{2})$ by projecting radially from $\theta$. Since the truth of (c) is not affected by this, we therefore assume $C(1)=S(\frac{1}{2})$. We use polar coordinates $(w, y)$ in the punctured image plane, $E^2-\theta$. Thus $w$ can be interpreted as the arc length $\bmod 2\pi$ of $S(1)$ and $y$ is the norm $|Y|$ of the radius vector $Y$. If $Y$ is associated with $(w, y)$ then $-Y$ is associated with $(w+\pi, y)$. Let $p$ project $E^2-\theta$ radially on $S(1)$ and write $P(w, t)$ for the *closed* set $p^{-1}(w) \cap C(t)$. Let the possibly multiple valued representation of $C(t)$ in the polar coordinates be indicated by the closed set

$$R(w, t)=\{y \mid Y \in P(w, t)\}.$$

The elements of $R(w, t)$ are inferior to a fixed constant $\sup\{|h(x, t)| \mid S_x \times I_t\}$. Moreover $R(w, t)$ is upper semi continuous on $S_w \times I_t \to E_y(t)$. (Thus suppose $w^n, y^n, t^n \to \bar{w}, \bar{y}, \bar{t}$ where $y^n \in R(w^n, t^n)$. For some $x^n$, $(w^n, y^n)=h(x^n, t^n)$. For a suitable subsequence $\{x^{n'}\}$, $x^{n'} \to \bar{x}$ and $(w^{n'}, y^{n'}) \to (\bar{w}, \bar{y})=h(\bar{x}, \bar{t})$. Hence $\bar{w}, \bar{y} \in C(t)$ or $\bar{y} \in R(\bar{w}, \bar{t})$.) Define

$$f(w, t)=\inf_{\substack{y \in R(w,t) \\ y' \in R(w+\pi,t)}} |y-y'|.$$

The zero set

$$Z=\{(w, t) \mid f(w, t)=0\} \subset S_w \times I_t,$$

is closed. (Suppose for $\varepsilon=0, 1$, $Z$ contains $\{(w^n+\varepsilon\pi, t^n)\}$ which converges to $(\bar{w}+\varepsilon\pi, \bar{t})$. Thus some $y^n \in R(w^n, t^n) \cap R(w^n+\pi, t^n)$. For a suitable cofinal collection $\{n'\}$ of positive integers, $w^{n'}+\varepsilon\pi, y^{n'}, t^{n'} \to \overline{w}+ +\varepsilon\pi, \overline{y}, \overline{t}$. The upper semi continuity of $R$ guarantees $\overline{y} \in R(\overline{w}, \overline{t}) \cap \cap R(\overline{w}+\pi, \overline{t})$ so $(\overline{w}+\varepsilon\pi, \overline{t}) \in Z$.)

We assert: *$Z$ contains a continuum joining $S_w \times 0$ and $S_w \times 1$ in $S_w \times I$.*

Suppose the contrary. Then $Z = C^0 \cup C'$ where $C^0$ and $C'$ are compact and disjunct and $C^j \cap S_w \times j \neq \emptyset$, $j = 0, 1$. The fact that $S_w$ refers to the image space whereas in Lemma 1 $S_x$ was in the object space, is not significant. The argument in Lemma 1 applies verbatim, on replacing $x, t$ by $w, t$ and using $t$ in place of $r-1$, to show there is a simple closed curve $K$ (composed of a finite number of circular arcs) which separates $C^0 \cup S \times 0$ and $C' \cup S \times 1$ and contains a pair $(\overline{w}, \overline{t})$ and $(\overline{w}+\pi, \overline{t})$.

Let the reflection be indicated by $T$: $(w, t) \to (w+\pi, t)$. We assert that $K$ may be taken reflection invariant, i. e. $TK = K$. The type of argument used in Lemma 1 shows $K \cap TK \neq \emptyset$. Suppose that $(w_0, t_0)$ and $(w_0+\pi, t_0)$ are in $K \cap TK$. Let $K''$ be the closed curve obtained by piecing together one of the two arcs of $K$, say $K_1$, joining these two points and $TK_1$. Thus $TK'' = K''$. Since $Z$ is obviously reflection invariant, $Z \cap K'' = \emptyset$. In general $K''$ is not simple, but separates $S^2 \supset S \times I$ into the domains $D_0, D_1$ when $D_0 = TD_0 \supset S \times 0$ and $D_1 = TD_1 \supset S \times 1$ and $D_i$, $TD_i$ where $i = 3, \ldots, m$. Here $S^2$ is obtained by adding a lower and an upper unit disk to $S_w \times I_t$. Drop out the boundaries of the pairs $D_i, TD_i$. Since this amounts to dropping out cycles homologous to 0, the residual point set is homologous to $S \times 0$ and is a simple closed curve which we denote again by $K$ in the sequel.

Suppose the curve $K = K(\lambda | S_\lambda) = K\big((w, t) | (w, t) = (w(\lambda), z(\lambda))\big)$. The graph of $R(w, t)$ when $(w, t)$ traverses $K$ is indicated by

$$\Gamma(R; K) = \Big\{\big((w(\lambda), t(\lambda)), y\big) \Big| \; y \in R\big(w(\lambda), t(\lambda)\big)\Big\} \subset K \times E(+)\,.$$

Suppose that $\inf|h(x, t)| > \delta > 0$. Let $G$ be the cylinder (a homeomorph of a circular cylinder) with the generating curve $K$ and the rays, $(w, t) = \text{constant}$, $y \geqslant \delta$, as generators. Then $\Gamma(R, K)$ is the intersection of $\Gamma(h)$ and $G$, and, as a simple argument shows, separates $G$ since $h(S_x \times I_t)$ separates $S_w \times I_t \times E_y(t)$.

We proceed now on the assumption that $R(w, t)$ is a finite set for $(w, t) \in K$. Thus $\Gamma(R, k)$ is locally connected. Then, taking a suitable component of $\Gamma(R, K)$ and applying CT we infer $\Gamma(R; K)$ contains a simple, closed separating curve $A$ homotopic to $K$ (in $G$).

Write $A\big(w(\lambda), t(\lambda)\big)$ or $A(w, t)$ for the set of $y$ values corresponding to the intersection with $A$ of the ray through $K(\lambda) = K(w, t)$. Since $K$ is reflection invariant, $(w, t)$ in $K$ implies $(w+\pi, t) \in K$. Interpret $A\big(w(\lambda), t(\lambda)\big)$ as $A_1$ and $A\big(w(\lambda)+\pi, t(\lambda)\big)$ as $-A_2$ in Lemma 4. Thus



$\Gamma(A)+\Gamma(A_2)$ contains a closed curve $Q$ homotopic to $K$ in $G$. Let $\big(w(\lambda_0), t(\lambda_0)\big)$ be $(w_0, t_0)$ and $\big(w(\lambda_1), t(\lambda_1)\big) = (w_0+\pi, t_0)$. Then if $F(\lambda) = A\big(w(\lambda), t(\lambda)\big) - A\big(w(\lambda)+\pi, t(\lambda)\big)$,

$$F(\lambda_0) = -F(\lambda_1)\,. \tag{1}$$

Note $Q(\lambda)$ or $Q\big(w(\lambda), t(\lambda)\big)$ consists of a finite collection of real numbers constituting a subset of $F(\lambda)$. It is therefore conceivable, in the event $F(\lambda_0)$ contains both positive and negative reals, that $Q(\lambda_0)$ and $Q(\lambda_1)$ consist of real numbers of the same sign, and that in fact the numbers in $Q(\lambda)$, for all $\lambda$, are of the same sign and avoid 0. We assert *this last cannot happen*, for if it did, then if $-Q(\lambda)$ denotes $\{-y \mid y \in Q(\lambda)\}$, by (1) $P = \{P(\lambda) \mid P(\lambda) = -Q(\lambda)\}$ would constitute a simple closed curve included in $\Gamma(A_1)+\Gamma(A_2)$ homotopic to $K$, and disjunct from $Q$. The existence of such a pair $P$ and $Q$ contradicts Lemma 4 (b). Hence $0 \in Q(\overline{\lambda})$ for some $\lambda$, i. e. for $(\overline{w}, \overline{t}) = \big(w(\overline{\lambda}), t(\overline{\lambda})\big)$ on $K$, $f(\overline{w}, \overline{t}) = 0$ or $K \cap Z \neq \emptyset$ which is at variance with the defining property of $K$.

We show now that the restriction to the case that $R(w, t)$ is a finite set for $(w, t) \in K$, can be waived. Thus, for small enough positive $\eta$, the solid spheres of radius $\eta$ in $S_w \times I_t \times E_y(t)$ intersect $G$ in connected sets, termed *$\eta$ disks*. Let $\Gamma^\eta$ be the union of a finite number of $\eta$ disks, whose open part includes $\Gamma(R; K)$. Replace $\Gamma(R; K)$ by $\Gamma^\eta$ in our previous argument. Then $\Gamma^\eta$ is locally connected and hence contains a simple closed curve $A^\eta$ homotopic to $K$ in $G$. We can therefore conclude that for some point in $K$ denoted by $(w^\eta, t^\eta)$, $Q^\eta(w^\eta, t^\eta) \supset 0$.

Hence $f(w^\eta, t^\eta) \leqslant 2\eta$. Let $\eta \to 0$; for some sequence of values, $(w^{\eta_j}, t^{\eta_j}) = (w^j, t^j) \to (\overline{w}, \overline{t})$.

Since $R(w, t)$ is compact, for each $j$

$$|y_j - \mathfrak{z}_j| = \varepsilon_j < 2\eta_j\,,$$

with $y_j \in R(w^j, t^j)$, $\mathfrak{z}_j \in R(w^j+\pi, t^j)$. For an infinite subset $\{j'\}$

$$y_{j'} \to y\,, \quad \mathfrak{z}_{j'} \to \mathfrak{z}\,, \quad \varepsilon_{j'} \to 0\,,$$

whence $y - \mathfrak{z} = 0$. Since $R(w, t)$ is upper semi continuous,

$$\overline{y} \in R(\overline{w}, \overline{t})\,, \quad \overline{\mathfrak{z}} \in R(\overline{w}+\pi, \overline{t})\,.$$

Therefore

$$0 \leqslant f(\overline{w}, \overline{t}) \leqslant |y - \mathfrak{z}| = 0\,,$$

whence again the contradiction $(\overline{w}, \overline{t}) \in K \cap Z \neq \emptyset$.

Hence the italicized assertion about $Z$ is substantiated.

**Remark.** Two assertions are involved in Theorem 1. The first and most significant is that of the existence of a continuum in $Z$ on which

$h$ is symmetric. The second is derived here as the trivial consequence of the first, namely that this continuum joining two circles must intersect an intermediate circle, or for a more general conclusion for Theorem 1, the deformed circle must at some stageintersect a fixed symmetric closed curve (separating $S\times 1$ from $S\times 0$) in two points satisfying (a), (b). Moreover if only this second type of assertion is desired, it, and indeed much more general such conclusions, can be established without invoking the first assertion, which may indeed be invalid. (It is not difficult to give examples of a continuum in $Z$, joining $S_w\times 0_t$ and $S_w\times 1_t$ in Theorem 1 (c), which contains no antipodal pair $(w, 1, t)$, $(w+\pi, 1, t)$.) We illustrate this by the next theorem.

THEOREM 2. *Let $h$ represent a deformation of the $n$ sphere $S^n_x(\frac{3}{2})$ of radius $\frac{3}{2}$ onto a concentric sphere of radius $\frac{1}{2}$, in $R^{n+1}-\theta$. Then for some $t$, the deformed sphere meets a fixed unit sphere in $2n$ points which are*

(a) *the transforms of extremities of $n$ orthogonal diameters of $S^n_x$;*

(b) *the extremities of $n$ orthogonal diameters of the image unit sphere, $S^n_w(1)$.*

For (a): Let $W=\{(x,t)\mid \|h(x,t)\|=1\}$. Evidently $W$ separates $S^n\times 0$ and $S^n\times 1$. Let the closed set

$$W^+=\{(x,t)\mid \|h(x,t)|-\|h(-x,t)\|\geqslant 0\}\cap W\,.$$

Write $T$ for the fixed point free involution $T(x,t)=(-x,t)$. Then define $W^-$ by

$$W^-=\{(x,t)\mid (-x,t)\in W^+\}\,.$$

The set $W^+\cup W^-$ is $T$ symmetric and separates $S^n\times 0$ and $S^n\times 1$ in $S^n\times I$ conceived of as a spherical shell of radius varying from $\frac{1}{2}$ to $\frac{3}{2}$, and accordingly carries a $T$ symmetric $n$ cycle. Accordingly $W\supset W^+\cap W^-$ is a carrier for a symmetric $n-1$ cycle which bounds no symmetric chain on $W^+\cup W^-$ ([1], Theorem 2B) whence it follows that there are $n-1+1$ orthogonal diameters of some sphere $S^n\times t$ *(of radius $\frac{1}{2}+t$)* with end points on $W$ ([1], Theorem 3A) and this implies the assertion (a) of the theorem.

For (b): Let $w$ designate the generic point of the unit image sphere. Let

$$A=\{(w,t)\mid h(x,t)=(w,t)\ \text{for some}\ x\}$$
$$=h(S^n_x\times I_t)\cap S^n_w(1)\times I_t\subset S^n_w(1)\times I\,.$$

Let

$$TA=\{(w,t)\mid (-w,t)\in A\}\,.$$

Then $A\cup TA$ is symmetric and separates $S_w(1)\times 0$ from $S_w(1)\times 1$. Accordingly, using ([1], Theorems 2B and 3A) the argument follows that for (a), with $TA\cap A$ replacing $W^+\cap W^-$ for $S^n_w(1)\times t$ can be represented as a sphere of radius $1+t$.



Essentially as consequences of Lemma 1 and Theorem 1 we can make some assertions about doubly periodic functions. To make easier connection with earlier notation, we now interpret $I_s$ as $0\leqslant s\leqslant 2\pi$.

We require the Lemma 1 analogue for the torus $T=S\times S$ and we present a *method of proof which illustrates for our special case $M=S$ how certain results for $M\times S$ can be reduced to the case $M\times I$* for $M$ a closed simplicial manifold. Note again that $u$ is to be interpreted as a real number or as a real number mod $2\pi$ according as we write $E_u$, or $S_u$. Write $S_1, S_2$ for $S_x\times 0_t$, $0_x\times S_t$ respectively and $mI$ for $\{y\mid y=kt,\ k=1,2,\ldots,m,\ t\in I\}$. The cycles below are with integer coefficients.

LEMMA 5. *Suppose that $A$ is a closed subcomplex of the triangulated torus $T$, then*

(a) *if $A$ does not carry a non bounding cycle of $T$ the complement of $A$ carries cycles homologous respectively to the fundamental cycles $S_1$ and $S_2$ on $T$,*

(b) *if $A$ carries no non bounding cycle of $T$ homologous to $aS_1+bS_2$, $b\neq 0$, the complement of $A$ carries a cycle $\sim S_1$.*

Let $N$ be the number of vertices of $T$ and choose $m>N$. The cylinder $S_x\times E_t=\tilde{T}$ is a covering space for $T$ and contains the finite cylinder $Q=S_x\times mI_t$. Let $p$ project $\tilde{T}$ onto $T$ by $p(x,y)=(x, t\equiv y\,(\mathrm{mod}\,2\pi))$. Let $\tilde{A}=p^{-1}A\cap Q$. Evidently $\tilde{A}$ has no component containing both $(x,y)$ and $(x, y+2\pi j)$, $j\neq 0$ an integer, for this component would carry a curve, $\tilde{C}$, with $p\tilde{C}\sim aS_1+jS_2$ for some integer $a$. On the other hand if there were a component $\tilde{L}$ of $\tilde{A}$ joining $y=0$ and $y=2\pi m$, since the number of vertices on $\tilde{L}$ is at least $m$, there would needs be at least two distinct representatives of some vertex of $A$. Thus $\tilde{L}$ would contain a pair $(x,y)$ and $(x,y+2\pi j)$, $j\neq 0$. Accordingly there is no continuum in $\tilde{A}$ joining $y=0$ and $y=2\pi m$. Therefore by BA there is a separating (polygonal) curve $\tilde{K}\sim S_1$ on $Q$. Thus $K=p\tilde{K}$, considered on $T$ is disjoint from the original $A$. This establishes (b). For (a) we need merely consider $mI_x\times S_t$ also besides $Q$.

THEOREM 3. *Let $r$ and $w$ be continuous, real valued, doubly periodic functions on $E_x\times E_t$ of period $2\pi$ in each of $x$ and $t$. Suppose that $a$ is an arbitrary positive number. Then:*

(a) *There is a parallelogram, of base $\pi$ parallel to the $x$ axis, and height $a$, on the four vertices of which, $r$ assumes the same value.*

(b) *Suppose that $w$: $I_x\times t$ onto $I_w\times t$ and suppose that $r$ is positive. Write*

$$R(w,t)=\{r\mid r=r(x,t),\ (x,t)\in w^{-1}(w,t)\}\,.$$

*Then there is a parallelogram of base $\pi$ and height $a$ whose vertices satisfy*

$$R(w_0, t_0) \cap R(w_0+\pi, t_0) \neq \emptyset, \quad R(w_1, t_0+a) \cap R(w_1+\pi, t_0+a) \neq \emptyset,$$
$$R(w_0, t_0) \cap R(w_1, t_0+a) \neq \emptyset .$$

*If for each $t$, $w$ is a biunique map then $R(w,t)$ contains a single point and the same $r$ corresponds to the four vertices.*

(The result (a) is known to A. N. Milgram.) Triangulate $E'$ with intervals of length $\eta_n$ where $\eta_n$ approaches 0 when $n\to\infty$. For some positive $\varepsilon_n$ and a triangulation $T_n$ of $T$ of mesh $\varepsilon_n$ there is a simplicial approximation $r^{(n)}$ to $r$ so that $\|r-r^{(n)}\| < \varepsilon_n$. Let

$$f_1^{(n)}(x,t) = r^{(n)}(x,t) - r^{(n)}(x+\pi, t),$$
$$f_2^{(n)}(x,t) = r^{(n)}(x,t) - r^{(n)}(x, t+\pi),$$

and let $Z_i^{(n)}$ be the set of zeros of $f_i^{(n)}$, $i=1,2$. We drop the script $n$ till the end of the proof. We assert $Z_1$ for instance, carries a cycle, $C\sim aS_1 + bS_2$, $b\neq 0$. If not, there is a cycle $K\sim S_1$ in the complement of $Z_1$ by Lemma 5 (b). Moreover just as in the proof of Lemma 1, $K' = \{(x,t)|\,(x+\pi,t)\in K\}$ also is in the complement of $Z_1$ and $K\cap K'\neq\emptyset$. Then $(x_0,t_0)$, $(x_0+\pi, t_0)$ are on $K$ and so from $f_1(x_0,t_0)\,f_1(x_0+\pi,t_0) < 0$ we infer there is a point $(\bar{x},\bar{t})\in K$ for which $f_1(\bar{x},\bar{t}) = 0$ in contradiction with the disjointness of $K$ and $Z$.

It is convenient now to return to the covering space $E_x\times E_t$ so that $\tilde{C}$ is a curve joining points $(x_0, 0)$, $(x_0+2\pi a, 2\pi b)$. The values of $r(x,t)$ for $(x,t)\in\tilde{C}$ yield the graph $G = \{((x,t),r)|\,(x,t)\in\tilde{C},\ r = r(x,t)\}$ in $\tilde{C}\times E_r \subset E_x\times E_t\times E_r$. Let $\psi$ project $\tilde{C}$ orthogonally onto the $t$ axis so that $G$ goes into a graph $\Gamma$: $\{(t,r)\mid r = r((x,t)\mid\tilde{C})\}$ over the $t$ axis which constitutes a curve, generally with self intersections. Let $\Gamma_a = \{(t,r)\mid r = r(x,t+a),\ (x,t+a)\in\tilde{C}\}$. Let $M$ and $m$ be the maximum and minimum values of $r$ on $\Gamma$ and therefore on $\Gamma_a$ also. Since the point on $\Gamma$ with $r=M$ lies abowe $\Gamma_a$ and the point with $r=m$ does not lie above $\Gamma_a$, the continuum $\Gamma$ must intersect $\Gamma_a$. (This already familiar connectedness consequence will be referred to below as the $M,m$ argument.) Let $(t_0,r)\in\Gamma\cap\Gamma_a$. Then $r(x_0,t_0) = r(x_1,t_0+a)$ for $(x_0,t_0)\cup(x_1,t_0+a)\in Z_1$ and reference to the definition of $Z_1$ establishes the assertion (a) (for $r = r^{(n)}$). Since as we now recall $r, x_0, t_0, x_1$ bear the superscript $n$, the obvious compactness argument of taking limits of a suitable subsequence then establishes (a) for $r(x,t)$.

For (b) interpret $(w,r)$ as $h(x,t)$, proceed to the simplicial approximation, and define $Z^{(n)}$ by

$$Z^{(n)} = \{(w,t)\mid R^{(n)}(w,t)\cap R^{(n)}(w+\pi,t)\neq\emptyset\}.$$



We omit the superscript $n$ in the sequel. By the earlier arguments in this paper, $Z$ contains a simple curve $C$ in $E_w\times E_t$, joining $(w,0)$ and $(w_0+2\pi a, 2\pi b)$, $b\neq 0$. Then the graph $G_1 = \{((w,t),R)\mid (w,t)\in C\}$ contains a simple curve with graph $G$. (Indeed $G$ is the intersection of $L = h(E_x\times E_t)$ and the cylinder $A$ with generators $w=$ constant, $t=$ constant passing through $C$. Since $L$ separates $r=0$ from $r=\infty$ points, $G$ separates etc.) Let $\psi$ indicate the orthogonal projection onto $E_r\times E_t$. Hence $G$ becomes $\Gamma = \{(t,n)\mid ((t,w),r)\in G$ for some $w\}$. Let $\Gamma_a = \{(t,r)\mid (t-a, r)\in\Gamma\}$. The $M, m$ argument shows $\Gamma\cap\Gamma_a \subset t_0, r_0$. Thus for some $w_0$ and for some $w_1$ with $(w_0,t_0)\cup(w_1,t_0+a)\in C$, $r_0\in R(w_0,t_0)\cap R(w_1,t_0+a)$.

The mode of definition of $Z$ guarantees the validity of (b). The same sort of standard compactness argument indicated for (a) finishes up the proof.

COROLLARY. *Let $f$ be a map of $S^2(1)$ to $E^1$, then for assigned $a$, $0<a<\pi$, $f$ takes on the same value on four points $x_1, x_2$, and $y_1, y_2$ such that either*

(a) *$x_1, x_2$ are the extremities of a diameter $D$, $y_2$ is the reflection of $y_1$ in $D$ and the distance (on the sphere) from $y_1$ to $x_1$ is $a$ or, for some diameter $D$, $x_2$ and $y_2$ are the reflections in $D$ of $x_1$ and $y_1$ respectively and the distance between the latitude circles, orthogonal to $D$, through $x_1, x_2$ and $y_1, y_2$ respectively is*

(b) *$a$*

*or*

(c) *$\pi - a$.*

Evidently $f$ takes on the same value on some pair of antipodal points $\bar{x}, -\bar{x}$, and these determine the diameter $D$. So far as the map by $f$ is concerned, $\bar{x}$ and $-\bar{x}$ may be identified to give a pinched torus $T$ arising from pinching a meridian circle $C$ on a torus $T$ so that the deformed $C$ becomes the identified points $\bar{x}, -\bar{x}$. Accordingly $f$ may be viewed as a map of $T$ to $E$ with $f$ constant on $C$. We may consider $T$ represented by its covering surface consisting of translates of the fundamental rectangle (or square if vertical lengths are multiplied by 2) whose horizontal base, $B$, represents $C$. Theorem 3 (a) applies and the various cases, (a), (b) and (c) arise according as the parallelogram whose existence is asserted in that theorem has (a) a side coinciding with $B$, or (b) is contained in the fundamental rectangle, or (c) has its horizontal sides in different rectangles.

Let $f$ be a map of $S^{n-1}$ to $E^l$. Denote the generic point of $E^l$ by $z$. The $k$ tuple of points $z^{(1)}, z^{(2)}, \dots, z^{(k)}$ in $E^l$ may be considered a single point $\bar{z}$ in the $k$ fold product $(E^l)^k$. The diagonal $\{\bar{z}\mid z^{(1)} = z^{(2)} = \dots = z^{(k)}\}$ of $E^{lk}$ will be denoted by $\Delta$ and $s$ will be the cyclic permutation $s$: $(z^{(1)}, \dots, z^{(k)}) = (z^{(2)}, \dots, z^{(k)}, z^{(1)})$.

Thus $s^k$ is the identity permutation and the cyclic group $G = \{s^i | i = 0, \dots, k-1\}$ of order $k$ acts without fixed points on the complement of $\Delta$ in $E^{lk}$. We denote by $\pi$ the projection parallel to $\Delta$ of $E^{lk}$ onto the hyperplane $P$: $z^{(1)} + \dots + z^{(k)} = 0$.

Thus points of $\Delta$ alone project into the origin, $\theta$. Let, finally, $\varrho$ be the radial projection of $P - \theta$ onto the sphere $\Sigma$, in $P$,

$$\Sigma: \sum_{i=1}^{k} |z^i|^2 = 1 . \tag{4}$$

Observe that $G$ takes $P$ into $P$ and $\Sigma$ into $\Sigma$. Hence denote $G|P$, $G|\Sigma$ by $G'$ and $G''$ and $s|P$ and $s|\Sigma$ by $s'$ and $s''$ respectively. The orbit space of $\{\Sigma, G''\}$ is $Y$.

Let $e^1, \dots, e^n$ be the unit vectors along a fixed orthogonal frame in $E^n$. Suppose that $w = (w^1, \dots, w^k)$ is a $k$ tuple of orthogonal points of $S^{n-1}$. Let $M_k$ be the $k \times k$ matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and let $I_r$ be the $r$ dimensional identity matrix. Then

$$t = \left( \begin{array}{c:c} M_k & 0 \\ \hdashline 0 & I_{n-k} \end{array} \right) \tag{4.01}$$

induces a permutation of $e^1, \dots, e^k$ keeping $e^{k+1}, \dots, e^n$ fixed. Write $T_k$ for the cyclic group of order $k$ with generator $t$. Remark that the space $W$, of $k$ tuples of orthogonal points $w', \dots, w^k$ on $S^{n-1}$, is in 1-1 correspondence with the cosets of orthogonal transformation moving $e^1, \dots, e^k$ to $w^1, \dots, w^k$ where two orthogonal transformations are in the same coset if and only if they differ on $e^{k+1}, \dots, e^n$ alone. We shall suppose that $k$ is an odd prime so that the determinant of $M_k$ is 1 and $t \in \mathrm{SO}(n)$. Thus one shows easily $W$ is the Stiefel manifold $W = \mathrm{SO}(n)/\mathrm{SO}(n-k)$. The orbit space under the cyclic permutations $w^1, \dots, w^k \to w^2, \dots, w^k, w^1$ etc. is denoted by $X$ and is the homogeneous space

$$X = W/T_k = \mathrm{SO}(n)/(\mathrm{SO}(n-k) \times T_k) .$$

In terms of the boundary operator for the Smith homology groups one can define a homomorphism

$$\nu_*(n) : {}^{\varrho}H_n(W, T_k) \to I_k \tag{4.02}$$



where ${}^{\varrho}H_n(W, T_k)$ is the Smith homology group over $I_k$, and we may normalize by defining $\varrho$ as $\tau = 1 - t$ for $n$ even and $\tau = 1 + t + \dots + t^{k-1}$ for $n$ odd ([8]; [2], p. 183). A dual homomorphism can be defined on the cohomology groups. However, for convenience it is desirable to make use of the isomorphism between ${}^{\varrho}H^n(W, T_k)$ and $H^n(X, T_k)$, [7], and a homomorphism induced by $1 + 2t + \dots + (k-1)t^{k-2}$ on ${}^{\tau}H^n(W, T_k)$ to ${}^{\varrho}H_n(W, T_k)$. The dual homomorphism now indicated by $\nu^*(n)$ can be regarded as on $I_k$ to $H^n(X, T_k)$ where $H^m(X, T_k)$ is over $I_k$.

Suppose a space $V$ admits the cyclic group of fixed point free homeomorphisms $L = \{l^i \mid i = 0, 1, \dots, k-1\}$.

Let $\psi$ be a map of $W$ to $V$. Then $\psi$ is a *symmetry map* if $\psi t(\ ) = l\psi(\ )$.

The key result for our purpose is that if $\psi$ is a symmetry map, and if $\nu'_*(n)$ refers to the homomorphism (4.02) for $V, L$ then ([2], Lemma 2)

$$\nu'_*(n)\psi_* = \nu_*(n) . \tag{4.03}$$

Thus $\nu'_*(n)$ is non trivial if $\nu_*(n)$ is non trivial. The *index* of $(W, T_k)$ denoted by $\nu(W, T_k)$ or $\nu(X)$ is the maximum value of $n$ for which $\nu_*(n)$ is not trivial (the recursive definition of $\nu_*(n)$ in [2] guarantees that if $\nu_*(n)$ is not trivial then $\nu_*(m)$, $m < n$, is not trivial). Thus (4.03) includes the assertion (A): that *under a symmetric transformation the index does not decrease.*

(Wu [11], has also recently published (4.03) for a cohomology definition of an index (Wu's index may differ from ours since for one thing the coefficient group is taken alternately $I$ and $I_k$ starting with $H^0(X, T_k)$).

Let $a$ be a positive real number. Then $[a]$ will denote the integer part and $a' = [\frac{1}{2}a]$. We are now in a position to state the theorem [1] of this section:

**Theorem 4.** *If $f$ maps $S^{n-1}$ into $E^1$ then some $k$ tuple of orthogonal points on $S^{n-1}$, $k$ an odd prime, maps into the same point of $E^l$ provided* $2\left[\frac{n-1-\varepsilon(n)}{k-1}\right] > l$, *where* $\varepsilon(2m) = 1$, $\varepsilon(2m+1) = 0$, $m = 0, 1, \dots$

If the theorem were false, then for some $f$,

$$F(\overline{w} \mid \overline{w} = (w', \dots, w^k)) = f(w'), \dots, f(w^k) ,$$

is disjunct from $\Delta$ for all $\overline{w}$. Define $\psi$ on $W$ to $\Sigma$ by

$$\psi(\overline{w}) = \varrho\pi F(\overline{w}) .$$

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(1) Constituting a partial answer to a problem by B. Knaster, *Problem* 4, Colloquium Mathematicum 1 (1947), p. 30.

Observe
$$F(t\overline{w}) = f(w^2), \ldots, f(w^k), f(w') = sF(\overline{w}).$$
For $E^{lk}-\Delta$, $s$ commutes with $\varrho\pi$, i. e. $\varrho\pi s = s'\varrho\pi$, or $\psi$ is a symmetry map, indeed
$$\psi(t\overline{w}) = s''\psi(\overline{w}).$$
Accordingly

(4.04) $$\nu(X) = \nu(W, T_k) \leqslant \nu(\psi(W), G'').$$

Now $\psi(W)$ may be a proper subset of $\Sigma$, but since $s''$ takes $\psi(W)$ into itself and the inclusion map, $i$, of $\Sigma$ satisfies
$$is''\psi(w) = s'i\psi(w),$$
therefore

(4.05) $$\nu(\psi(W), G'') \leqslant \nu(Y).$$

We have need for the index of $Y$ and that of $X$. The index of $Y$ is essentially known for here our index coincides in value with Wu's calculated one

(4.06) $$\nu(Y) = (k-1)l-1.$$

The index of $X$ is a more recondite affair. I am indebted to A. Borel for information bearing on the calculation and exact value. Thus

(4.07) $$\nu(X) = 4c[\tfrac{1}{2}k]-1,$$

where $c$ is determined by

(4.08) $$(c-1)k' \leqslant (n-k)' < ck'.$$

Accordingly, since $k' = \frac{1}{2}(k-1)$, (4.08) yields
$$c = \left[\frac{n-1-\varepsilon(n)}{k-1}\right],$$
so
$$\nu(X) = 2(k-1)\left[\frac{n-1-\varepsilon(n)}{k-1}\right]-1.$$
*The contradiction*
$$\nu(X) > \nu(Y),$$
*is assured when*

(4.09) $$2\left[\frac{n-1-\varepsilon(n)}{k-1}\right] > l,$$

which implies the assertion of the theorem.

The *orthogonal*, or $\frac{1}{2}\pi$ $k$ *tuples, can be replaced by equispaced k tuples.* Distances are understood measured on the sphere. A $\varrho$ $k$ tuple has $\varrho$



for the common distance of all pairs. Let $a = \{a_i \mid i = 1, \ldots, k\}$ be a $\varrho$ $k$ tuple. Then the end point of
$$\frac{\sum_i a_i/k}{\left\|\sum_i a_i/k\right\|} = \overline{x}$$
is equidistant, and minimally so, from each of $a_1, \ldots, a_k$. Since the great circle, $C_i$, from $\overline{x}$ to $a_i$ is a geodesic, is it unique, and hence for some distance, $s$, the points on $C_1, \ldots, C_k$ at distance $s$ from $\overline{x}$ constitute an orthogonal $k$ tuple $b = \{b_i \mid i = 1, \ldots, k\}$. This correspondence $a \leftrightarrow b$ ensures the homeomorphism between the space of $\varrho$ $k$ tuples denoted by $X_\varrho$ and the $\frac{1}{2}\pi$ $k$ tuples which we have denoted above by $X = W/T_k$.

We remark that for $n$ even and $k-1 \mid n$, *the relation*

(4.10) $$2\left[\frac{n}{k-1}\right] > l+1$$

*guarantees for some* $\varrho_0$, $\varrho_0 \geqslant \frac{1}{2}\pi$, $f$ *maps a* $\varrho_0$ $k$ *tuple of* $S^{n-1}$ *into a common point of* $E^l$. (Most likely the stronger result $\varrho_0 = \frac{1}{2}\pi$ is true also.) Write $x = x_1, \ldots, x_n$ and consider $S^{n-1}$ the meridian sphere of $S^n$: $\sum_i^n x_i^2 + v^2 = 1$. Let $G(x, v)$ map $S^n$ into $E^l \times E_v$ by $G(x, v) = ((1-|v|)f(x/\|x\|), v)$.

Theorem (4) asserts (4.10) suffices for $G$ to map some orthogonal $k$ tuple

(4.11) $$\{(x(i), v(i)) \mid i = 1, \ldots, k\}$$

into a common point $(\tilde{z}, a)$ of $E^l \times E_v$. The definition of $G$ shows
$$v(1) = \ldots = v(k) = a \neq \pm 1.$$
Hence with $y = x/\|x\|$,
$$f(y(1)) = \ldots = f(y(k)).$$
If $a \neq 0$, $\{y(i)\}$ constitute a $\varrho_0$ $k$ tuple, $\varrho_0 > \frac{1}{2}\pi$. (The inticated generalization of Theorem 4 to $\varrho$ $k$ tuples so that (4.11) refers to $\varrho$ $k$ tuples with $\varrho \to 0$ does not ensure $\{y(i)\}$ need be a $\frac{1}{2}\pi$ $k$ tuple.)

Remark. The only result heretofore known on the equispaced case of the Knaster problem with $l \neq 1$, $n$ odd, is included in Theorem 4 as the special case $k = 3$, $l = n-2$. Theorem 4 gives et once (for $n$ prime) as the case $k = n$, $l = 1$, the generalized Kakutani theorem.

Evidently, it is more than enough for the validation of (4.07) and (4.08) to present the cohomology ring of $X$. This latter can be determined by the method of Borel in his thesis [4]. All details of notation used below are found in [3] and [4]. We merely mention that $\Lambda$ and $Z_k$ refer to exterior algebra and polynomial ring respectively, that $f_1$ and $f_2$ are generators of dimension 1 and 2 respectively, that $E_r$ is the spectral sequence term with

$$E_2 = H^* G_{SO(n-k)\times T_k} \otimes H^* SO(n) \qquad ([4], \text{ Theorem } 22).$$

Then, using Theorem 22 and Proposition 22.1 of [4] and the boundedness of the spectral sequence $\{E_r\}$, the following representation of the cohomology ring is obtained where to establish $E_{4ck'} = E_{4(n-k)'+1}$ (crucial for $c$), one shows $d_r = 0$, $4(n-k)'+1 \leqslant r < 4ck'$, by computation using the simple number theoretic fact (the only one not covered in [3] and [4]) that the $i$th elementary symmetric function in $1, 2^2, \ldots, [\frac{1}{2}n]^2$ is $0 \bmod n$ for $i < \frac{1}{2}n$.

$$(4.12) \qquad H^*(X) = E_\infty = E_{4ck'+1} = E_{4(n-k)'+1}$$
$$= \Lambda(f_1) \otimes Z_k(f_2)/(f_2)^{2ck'} \otimes \Lambda(z_{4ck'+3}, \ldots z_{4n'-5}, q) \otimes D(n)$$

where $q$ is $z_{4n'-1}$ for $n$ odd and is $z_{n'-1}$ for $n$ even,

$$D(2m+1) = 1, \qquad D(2m) = Z_k(w_{2(n-k)'})/(w_{2(n-k)'})^2,$$

and the subscripts of $f$, $z$, $w$ indicate dimension. The significant part to us is that there are no elements of base degree $2ck'$. Let $u(1) = \nu^*(1) f_0$, $u(2) = \nu^*(2) f_0$. It may be shown that

$$(4.13) \qquad \nu^*(2m) f_0 = u(2)^m, \qquad \nu^*(2m+1) f_0 = u(1) u(2)^m,$$

where $f_0$ is the 0 dimensional cohomology class of the cocycle identically 1 on $X$ and the products (and powers) are cup products. It is at once verified that $u(1)$ and $u(2)$ can be interpreted as the images of $f_1$ and $f_2$ in $H(X)$ in (4.12). Accordingly the absence of elements of base degree $2ck'$ in (4.12) indicates the lowest value of $m$ for which the right side of (4.13) vanishes is $4ck'$ as asserted in (4.07) and (4.08). More complete details on (4.03) and (4.07) and (4.08) will be appear elsewhere.

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