On the isomorphism of Haar measures

by

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I. Preliminary definitions. Let G be a group (not necessarily commutative). A Hausdorff topology $\mathcal{T}$ for G will be called *compact topology for* G if G is a compact topological group with respect to the topology $\mathcal{T}$. If $\mathcal{T}$ is a compact topology for G, then by $\mathcal{B}_{\mathcal{T}}$ we shall denote the σ-field of all Baire subsets of G, i. e. the smallest σ-field that includes all compact G_δ sets (with respect to the topology $\mathcal{T}$). Further, by $\mu_{\mathcal{T}}$ we shall denote the Haar measure defined on $\mathcal{B}_{\mathcal{T}}$ and normalized by supposing $\mu_{\mathcal{T}}(G)=1$.

Let $\mathcal{I}_{\mathcal{T}}$ be the σ-ideal of all sets $E \in \mathcal{B}_{\mathcal{T}}$ with $\mu_{\mathcal{T}}(E)=0$. Consider the Boolean σ-algebra $\mathbf{B}_{\mathcal{T}}=\mathcal{B}_{\mathcal{T}}/\mathcal{I}_{\mathcal{T}}$. The element (the coset) of $\mathbf{B}_{\mathcal{T}}$ determined by a set $E \in \mathcal{B}_{\mathcal{T}}$ will be denoted by $[E]$. The Haar measure $\mu_{\mathcal{T}}$ determines the measure $\tilde{\mu}_{\mathcal{T}}$ on $\mathbf{B}_{\mathcal{T}}$:

$$\tilde{\mu}_{\mathcal{T}}([E]) = \mu_{\mathcal{T}}(E) \qquad (E \in \mathcal{B}_{\mathcal{T}}).$$

Let $\mathbf{A}$ and $\mathbf{B}$ be two Boolean σ-algebras with measures μ and ν respectively. An isomorphism h of $\mathbf{A}$ into $\mathbf{B}$ is said to be *measure-preserving* if $\nu(h(A)) = \mu(A)$ for each $A \in \mathbf{A}$. If there is a measure-preserving isomorphism of $\mathbf{A}$ into $\mathbf{B}$, the measures μ and ν are said to be *isomorphic.*

The Haar measures $\mu_{\mathcal{T}_1}$ and $\mu_{\mathcal{T}_2}$ are said to be *almost isomorphic* if the measures $\tilde{\mu}_{\mathcal{T}_1}$ and $\tilde{\mu}_{\mathcal{T}_2}$ determined by $\mu_{\mathcal{T}_1}$ and $\mu_{\mathcal{T}_2}$ respectively are isomorphic.

Let $\mathcal{B}$ be a σ-field with a measure μ and let $\mathcal{B}_0$ be a σ-subfield of $\mathcal{B}$. The measure μ restricted to sets $A \in \mathcal{B}_0$ will be denoted by $\mu|\mathcal{B}_0$. Further two σ-subfields $\mathcal{B}_1$ and $\mathcal{B}_2$ ($\mathcal{B}_1 \subset \mathcal{B}$, $\mathcal{B}_2 \subset \mathcal{B}$) are said to be independent if $\mu(A_1 \cap A_2) = \mu(A_1) \cap \mu(A_2)$ for each $A_1 \in \mathcal{B}_1$ and $A_2 \in \mathcal{B}_2$.

For every element A_0 of a Boolean σ-algebra $\mathbf{A}$, the symbol $A_0 \cap \mathbf{A}$ will denote the relativized Boolean σ-algebra formed of all $A \subset A_0$, $A \in \mathbf{A}$.

For any Boolean σ-algebra $\mathbf{A}$, let $\tau(\mathbf{A})$ denote the least cardinal that is the power of a set $\mathbf{S} \subset \mathbf{A}$ such that the least σ-subalgebra containing $\mathbf{S}$ is $\mathbf{A}$ itself.

A Boolean σ-algebra $\mathbf{A}$ is said to be of the type $\alpha \geqslant 0$ if $\tau(A \cap \mathbf{A}) = \aleph_\alpha$ for every $A \in \mathbf{A}$, $A \neq 0$.

For a fixed ordinal $\alpha \geqslant 0$, let

$$K_\alpha = \mathop{\mathcal{P}}_{\xi \in \Xi} I_\xi \qquad (\overline{\overline{\Xi}} = \aleph_\alpha)$$

be the Cartesian product of $\aleph_\alpha$ sets I_ξ each of which contains only two elements 0 and 1. Let K_α^ξ be the set of all points $p \in K_\alpha$ whose ξth co-ordinate is 0, and let $\mathcal{X}_\alpha$ be the least σ-field generated by the sets K_α^ξ ($\xi \in \Xi$). In each I_ξ we define the measure λ putting $\lambda(0) = \lambda(1) = \frac{1}{2}$. These measures on I_ξ induce the product measure m_α on $\mathcal{X}_\alpha$. Let $\mathcal{I}_\alpha$ be the σ-ideal of all sets $A \in \mathcal{X}_\alpha$ with $m_\alpha(A) = 0$. The measure m_α determine the measure $\widetilde{m}_\alpha$ on the Boolean σ-algebra $\mathcal{X}_\alpha/\mathcal{I}_\alpha = \mathbf{K}_\alpha$.

A measure μ on a Boolean σ-algebra $\mathbf{A}$ is said to be *strictly positive normed* if $\mu(A) > 0$ for $A \in \mathbf{A}$, $A \neq 0$ and $\mu(0') = 1$. For example, $\widetilde{\mu}_{\mathcal{J}}$ and $\widetilde{m}_\alpha$ are strictly positive normed measures on $\mathbf{B}_{\mathcal{J}}$ and $\mathbf{K}_\alpha$ respectively.

D. Maharam ([4]) has proved the following fundamental theorem:

(M) *If a Boolean σ-algebra with a strictly positive normed measure μ is of the type $\alpha \geqslant 0$, then μ is isomorphic to $\widetilde{m}_\alpha$.*

II. Problems. In connection with the study of the foundations of the theory of probability J. Łoś has raised the following problem:

Let G be an infinite group and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be compact topologies for G. Does there exist a common invariant extension of $\mu_{\mathcal{J}_1}$ and $\mu_{\mathcal{J}_1}$? In other words, does there exist an invariant measure μ defined on the least σ-field containing the σ-fields $\mathcal{B}_{\mathcal{J}_1}$ and $\mathcal{B}_{\mathcal{J}_2}$ such that

$$\mu|\mathcal{B}_{\mathcal{J}_1} = \mu_{\mathcal{J}_1},$$

$$\mu|\mathcal{B}_{\mathcal{J}_1} = \mu_{\mathcal{J}_2}?$$

S. Hartman has proved that for $\mathcal{J}_1 \neq \mathcal{J}_2$ there exists no extension of $\mu_{\mathcal{J}_1}$ and $\mu_{\mathcal{J}_2}$ defined on a Baire σ-field induced by a compact topology for G.

S. Hartman has formulated the following problem, which is a modification of Łoś's problem:

Let G be an infinite group and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be compact topologies for G. Does there exist a compact topology $\mathcal{J}$ for G such that $\mathcal{B}_{\mathcal{J}}$ contains two independent σ-subfields $\mathcal{B}_1$ and $\mathcal{B}_2$ for which $\mu_{\mathcal{J}}|\mathcal{B}_j$ and $\mu_{\mathcal{J}_j}$ ($j = 1, 2$) are almost isomorphic?

At the same time S. Hartman has proved that the answer to this problem is affirmative for divisible Abelian groups. In the present note we shall give, using Maharam's theorem, the general solution of Hartman's problem. Moreover, we shall examine isomorphism types of Boolean σ-algebras $\mathbf{B}_{\mathcal{J}}$.

III. Theorems. In the present note we use Zermelo's axiom of choice. Consequently, powers of sets are alephs.

THEOREM 1. *Let G be an infinite group and let $\mathcal{J}_1$ and $\mathcal{J}_2$ be compact topologies for G. There exists then a compact topology $\mathcal{J}$ for G such that $\mathcal{B}_{\mathcal{J}}$ contains two independent σ-subfields $\mathcal{B}_1$ and $\mathcal{B}_2$ for which $\mu_{\mathcal{J}}|\mathcal{B}_j$ and $\mu_{\mathcal{J}_j}$ ($j = 1, 2$) are almost isomorphic.*

By (*) we shall denote the following implication:

(*) *If $2^{\aleph_\alpha} = 2^{\aleph_\beta}$, then $\aleph_\alpha = \aleph_\beta$.*

It is well known that the generalized continuum hypothesis ($\aleph_{\alpha+1} = 2^{\aleph_\alpha}$) implies (*).

THEOREM 2. *The implication (*) is equivalent to the following one: Let G be a group and let $\overline{\overline{G}} = 2^{\aleph_\alpha}$. Then for every compact topology $\mathcal{J}$ for G the Boolean σ-algebra $\mathbf{B}_{\mathcal{J}}$ is of the type α.*

A. Hulanicki has shown that the power of every infinite compact topological group is of the form $2^{\aleph_\alpha}$ ([2]). Hence, according to Maharam's theorem, we immediately obtain the following

COROLLARY. *Let G be an infinite group. Under the assumption (*) for all compact topologies $\mathcal{J}$ for G the Haar measures $\mu_{\mathcal{J}}$ are almost isomorphic.*

IV. Proofs. In the sequel we shall denote by G an infinite group.

LEMMA 1. *Let $\mathcal{J}$ be a compact topology for G. Then for every $A \in \mathbf{B}_{\mathcal{J}}$, $A \neq 0$, the equality*

$$\tau(\mathbf{B}_{\mathcal{J}}) = \tau(A \cap \mathbf{B}_{\mathcal{J}})$$

holds.

Proof. Let $E \in \mathcal{B}_{\mathcal{J}}$ and $[E] \neq 0$. It is well known that there exists a sequence $x_1, x_2, \ldots$ ($x_j \in G$) satisfying the equality

$$\mu_{\mathcal{J}}(G \setminus \bigcup_{j=1}^{\infty} x_j E) = 0$$

(cf. [1], § 60, exercise 4). Consequently,

$$[G] = \bigcup_{j=1}^{\infty} [x_j E].$$

Hence we obtain the following inequality:

$$\tau([x_j E] \cap \mathbf{B}_{\mathcal{J}}) \leqslant \tau(\mathbf{B}_{\mathcal{J}}) \leqslant \sum_{j=1}^{\infty} \tau([x_j E] \cap \mathbf{B}_{\mathcal{J}}).$$

Since $\tau([E] \cap \mathbf{B}_{\mathcal{J}}) = \tau([xE] \cap \mathbf{B}_{\mathcal{J}})$ for each $x \in G$, we have, according to the last inequality,

$$(1) \qquad \tau([E] \cap \mathbf{B}_{\mathcal{J}}) \leqslant \tau(\mathbf{B}_{\mathcal{J}}) \leqslant \aleph_0 \tau([E] \cap \mathbf{B}_{\mathcal{J}}).$$

The Haar measure on an infinite compact group is non-atomic. Therefore $\tau([E]\cap \mathbf{B}_{\mathcal{G}})\geqslant \aleph_0$. Consequently, in view of (1), $\tau([E]\cap \mathbf{B}_{\mathcal{G}})=\tau(\mathbf{B}_{\mathcal{G}})$. The Lemma is thus proved.

Proof of Theorem 1. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be compact topologies for G and $\tau(\mathbf{B}_{\mathcal{G}_1})=\aleph_{a_1}$, $\tau(\mathbf{B}_{\mathcal{G}_1})=\aleph_{a_2}$. We may suppose that the inequality $a_1\geqslant a_2$ holds. In virtue of Lemma 1 and Maharam's theorem (M) the measures $\widetilde{\mu}_{\mathcal{G}_j}$, $\widetilde{m}_{a_j}$ $(j=1,2)$ are isomorphic. Let $\Xi=\Xi_1\cup\Xi_2$, $\Xi_1\cap\Xi_2=0$, $\overline{\overline{\Xi}}=\aleph_{a_1}$, $\overline{\overline{\Xi}}_1=\aleph_{a_1}$, and $\overline{\overline{\Xi}}_2=\aleph_{a_2}$. Then

$$K_{a_1}=\mathop{\mathcal{P}}_{\xi\in\Xi_1} I_\xi\times \mathop{\mathcal{P}}_{\xi\in\Xi_2} I_\xi .$$

Let $\mathbf{K}_{a_1}^{(1)}$ be the least σ-subalgebra of $\mathbf{K}_{a_1}$ containing all elements of the form $[E]$, where $E\in\mathcal{K}_{a_1}$, $E=A\times \mathop{\mathcal{P}}_{\xi\in\Xi_2} I_\xi$, $A\subset \mathop{\mathcal{P}}_{\xi\in\Xi_1} I_\xi$, and let $\mathbf{K}_{a_1}^{(2)}$ be the least σ-subalgebra of $\mathbf{K}_{a_1}$ containing all elements of the form $[E]$, where $E\in\mathcal{K}_{a_1}$, $E=\mathop{\mathcal{P}}_{\xi\in\Xi_1} I_\xi\times A$, $A\subset \mathop{\mathcal{P}}_{\xi\in\Xi_2} I_\xi$. It is easy to verify that the measures $\widetilde{m}_{a_1}|\mathbf{K}_{a_1}^{(j)}$ and $\widetilde{m}_{a_j}$ $(j=1,2)$ are isomorphic. Moreover, for any $A_1\in\mathbf{K}_{a_1}^{(1)}$, $A_2\in\mathbf{K}_{a_1}^{(2)}$ the equality $\widetilde{m}_{a_1}(A_1\cap A_2)=\widetilde{m}_{a_1}(A_1)\widetilde{m}_{a_1}(A_2)$ is true. We denote by h a measure-preserving isomorphism of $\mathbf{B}_{\mathcal{G}_1}$ into $\mathbf{K}_{a_1}$. Let $\mathcal{B}_j$ be a σ-field of all sets E $(E\in\mathbf{B}_{\mathcal{G}_1})$ for which $h([E])\in\mathbf{K}_{a_1}^{(j)}$ $(j=1,2)$. Obviously, for any $E_1\in\mathcal{B}_1$ and $E_2\in\mathcal{B}_2$

$$\begin{aligned}\mu_{\mathcal{G}_1}(E_1\cap E_2)&=\widetilde{\mu}_{\mathcal{G}_1}([E_1]\cap[E_2])=\widetilde{m}_{a_1}\big(h([E_1])\cap h([E_2])\big)\\&=\widetilde{m}_{a_1}\big(h([E_1])\big)\widetilde{m}_{a_1}\big(h([E_2])\big)=\widetilde{\mu}_{\mathcal{G}_1}([E_1])\widetilde{\mu}_{\mathcal{G}_1}([E_2])=\mu_{\mathcal{G}_1}(E_1)\mu_{\mathcal{G}_1}(E_2).\end{aligned}$$

Thus $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent σ-subfields of $\mathcal{B}_{\mathcal{G}_1}$. Moreover, from the equality $\mathcal{B}_j/\mathcal{G}_{\mathcal{G}_1}=h(\mathbf{K}_{a_1}^{(j)})$ it follows that the measures $\widetilde{\mu_{\mathcal{G}_1}|\mathcal{B}_j}$ and $\widetilde{m}_{a_1}|\mathbf{K}_{a_1}^{(j)}$ $(j=1,2)$ are isomorphic. Since $\widetilde{m}_{a_1}|\mathbf{K}_{a_1}^{(j)}$, $\widetilde{m}_{a_j}$, $\widetilde{\mu}_{\mathcal{G}_j}$ are isomorphic, the measures $\widetilde{\mu_{\mathcal{G}_1}|\mathcal{B}_j}$ and $\widetilde{\mu}_{\mathcal{G}_j}$ $(j=1,2)$ are also isomorphic. In other words the measures $\mu_{\mathcal{G}_1}|\mathcal{B}_j$ and $\mu_{\mathcal{G}_j}$ $(j=1,2)$ are almost isomorphic. Putting $\mathcal{G}=\mathcal{G}_1$ we obtain the assertion of Theorem 1.

LEMMA 2. *For every compact topology $\mathcal{G}$ for G the inequality*

$$\overline{\overline{G}}\leqslant 2^{\tau(\mathbf{B}_{\mathcal{G}})}$$

is true.

Proof. The Boolean σ-algebra $\mathbf{B}_{\mathcal{G}}$ will be considered as a metric space with the distance

$$\varrho(A_1,A_2)=\widetilde{\mu}_{\mathcal{G}}(A_1\setminus A_2)+\widetilde{\mu}_{\mathcal{G}}(A_2\setminus A_1)\qquad (A_1,A_2\in\mathbf{B}_{\mathcal{G}}).$$

Moreover, every element $x\in G$ will be considered as a distance-preserving transformation of $\mathbf{B}_{\mathcal{G}}$ into itself: $\varphi_x([E])=[xE]$ $(E\in\mathcal{B}_{\mathcal{G}})$. It is easy to verify that for every pair $x,y\in G$, $x\neq y$ there exists a neighbourhood V such that $xV\cap yV=0$. Hence, taking into account the inequality $\mu_{\mathcal{G}}(V)>0$ we obtain $[xV]\neq[yV]$, which implies $\varphi_x\neq\varphi_y$. Consequently,

$$\overline{\overline{G}}=\overline{\overline{\{\varphi_x\}}}_{x\in G}. \tag{2}$$

According to Lemma 1, $\mathbf{B}_{\mathcal{G}}$ is of the type β, where $\tau(\mathbf{B}_{\mathcal{G}})=\aleph_\beta$. Consequently, the least cardinal of a set dense in $\mathbf{B}_{\mathcal{G}}$ is $\tau(\mathbf{B}_{\mathcal{G}})$ (cf. [5], Lemma (iii)). It is clear that the transformation φ_x is determined by its values on a dense subset of $\mathbf{B}_{\mathcal{G}}$. Hence

$$\overline{\overline{\{\varphi_x\}}}_{x\in G}\leqslant \overline{\overline{\mathbf{B}}}_{\mathcal{G}}^{\tau(\mathbf{B}_{\mathcal{G}})},$$

which implies, according to (2), the inequality

$$\overline{\overline{G}}\leqslant \overline{\overline{\mathbf{B}}}_{\mathcal{G}}^{\tau(\mathbf{B}_{\mathcal{G}})}.$$

Further, from the definition of the cardinal $\tau(\mathbf{B}_{\mathcal{G}})$ it follows that $\overline{\overline{\mathbf{B}}}_{\mathcal{G}}\leqslant\tau(\mathbf{B}_{\mathcal{G}})^{\aleph_0}$. Consequently,

$$\overline{\overline{G}}\leqslant \tau(\mathbf{B}_{\mathcal{G}})^{\aleph_0\tau(\mathbf{B}_{\mathcal{G}})}=\tau(\mathbf{B}_{\mathcal{G}})^{\tau(\mathbf{B}_{\mathcal{G}})}=2^{\tau(\mathbf{B}_{\mathcal{G}})}$$

q. e. d.

LEMMA 3. *Let $\mathcal{G}$ be a compact topology for G. If $\Theta(\mathcal{G})$ is the least cardinal that is the power of a basis for all open subsets of G, then the inequality*

$$\tau(\mathbf{B}_{\mathcal{G}})\leqslant\Theta(\mathcal{G})$$

holds.

Proof. Let $\mathfrak{B}$ be a basis for all open subsets of G and $\overline{\overline{\mathfrak{B}}}=\Theta(\mathcal{G})$. It is well known that a compact Hausdorff space is normal. Therefore, by Urysohn's Lemma, for every pair of neighbourhoods $U,V\in\mathfrak{B}$, $\overline{U}\cap\overline{V}=0$, there exists a continuous real-valued function $f_{U,V}$ on G such that $f_{U,V}=1$ on $\overline{U}$ and $f_{U,V}=0$ on $\overline{V}$. Let $\mathcal{F}$ be the smallest class of continuous real valued functions on G, containing all linear combinations $pf_{U,V}+qf_{V,U}$ with rational coefficients, that is closed under the operations $\max(f,g)$ and $\min(f,g)$. Obviously,

$$\overline{\overline{\mathcal{F}}}=\Theta(\mathcal{G}). \tag{3}$$

Let f be a continuous real-valued function on G. Then for every pair $x,y\in G$ and for every $\varepsilon>0$ there exist rational numbers p,q such that

$$|f(x)-p|<\varepsilon,\qquad |f(y)-q|<\varepsilon. \tag{4}$$

If $x\neq y$, there are neighbourhoods $U,V\in\mathfrak{B}$ such that

$$x\in U,\quad y\in V,\quad \overline{U}\cap\overline{V}=0.$$

Hence, according to (4), we obtain the inequalities

$$|f(x)-pf_{U,V}(x)-qf_{V,U}(x)|<\varepsilon\,, \qquad |f(y)-pf_{U,V}(y)-qf_{V,U}(y)|<\varepsilon\,.$$

Consequently, every continuous real-valued function on G can be approximated at every pair of points by a function of $\mathcal{F}$. Therefore, by a version of Stone's theorem, the uniform closure of $\mathcal{F}$ contains every continuous real-valued function on G (cf. [3], § 4).

Let us denote by $\mathcal{L}(\mu_{\mathcal{J}})$ the space of all real-valued functions $\mu_{\mathcal{J}}$-integrable on G, with the norm

$$\|f\|=\int_G |f(x)|\,\mu_{\mathcal{J}}(dx)\,.$$

It is well known that the set of all continuous functions is dense in $\mathcal{L}(\mu_{\mathcal{J}})$ (cf. [1], § 55, exercise 2). Consequently, the set $\mathcal{F}$ is also dense in $\mathcal{L}(\mu_{\mathcal{J}})$. Put

$$E_f=\{x\colon x\in G,\ f(x)>\tfrac{1}{2}\}\qquad (f\in\mathcal{F})\,. \tag{5}$$

Since f are continuous functions, we infer that E_f are Baire sets (cf. [1], § 50, theorem 3). Now we shall prove that the family of cosets $\{[E_f]\}_{f\in\mathcal{F}}$ is dense in the space $\mathbf{B}_{\mathcal{J}}$.

Let $E\in\mathcal{B}_{\mathcal{J}}$ and let χ_E denote the characteristic function of E, i. e. $\chi_E=1$ on E and $\chi_E=0$ on $G\setminus E$. Since $\chi_E\in\mathcal{L}(\mu_{\mathcal{J}})$, there exists a sequence $f_1, f_2, \ldots$ of functions belonging to $\mathcal{F}$ such that

$$\lim_{n\to\infty}\|\chi_E-f_n\|=0\,. \tag{6}$$

In virtue of definition (5), we obtain the following inequalities:

$$\int_{E\setminus E_{f_n}} |\chi_E(x)-f_n(x)|\,\mu_{\mathcal{J}}(dx)=\int_{E\setminus E_{f_n}} |1-f_n(x)|\,\mu_{\mathcal{J}}(dx)\geqslant \tfrac{1}{2}\mu_{\mathcal{J}}(E\setminus E_{f_n})\,,$$

$$\int_{E_{f_n}\setminus E} |\chi_E(x)-f_n(x)|\,\mu_{\mathcal{J}}(dx)=\int_{E_{f_n}\setminus E} |f_n(x)|\,\mu_{\mathcal{J}}(dx)\geqslant \tfrac{1}{2}\mu_{\mathcal{J}}(E_{f_n}\setminus E)\,.$$

Hence we have the inequality

$$\varrho([E],[E_{f_n}])=\mu_{\mathcal{J}}(E\setminus E_{f_n})+\mu_{\mathcal{J}}(E_{f_n}\setminus E)\leqslant 2\|\chi_E-f_n\|\,,$$

which, in view of (6), implies the convergence

$$\lim_{n\to\infty}\varrho([E],[E_{f_n}])=0\,.$$

Thus $\{[E_f]\}_{f\in\mathcal{F}}$ is dense in $\mathbf{B}_{\mathcal{J}}$.

Since $\tau(\mathbf{B}_{\mathcal{J}})$ is the least cardinal of a set dense in $\mathbf{B}_{\mathcal{J}}$, we have, according to (5), the inequality

$$\tau(\mathbf{B}_{\mathcal{J}})\leqslant\overline{\overline{\{[E_f]\}_{f\in\mathcal{F}}}}\leqslant\overline{\overline{\mathcal{F}}}=\Theta(\mathcal{J})\,.$$

The lemma is thus proved.

Proof of theorem 2. Sufficiency of (*). Let us suppose that (*) is true. By a theorem of Hulanicki ([2])

$$\overline{\overline{G}}=2^{\Theta(\mathcal{J})}\,, \tag{7}$$

where $\Theta(\mathcal{J})$ denotes the least cardinal that is the power of a basis for all open subsets of G. Hence, in virtue of (*), $\Theta(\mathcal{J})$ does not depend on a compact topology $\mathcal{J}$: $\Theta(\mathcal{J})=\aleph_\alpha$. Hence and from (7), in virtue of Lemmas 2 and 3, we obtain the inequalities

$$2^{\aleph_\alpha}\leqslant 2^{\tau(\mathbf{B}_{\mathcal{J}})}\,,\qquad \tau(\mathbf{B}_{\mathcal{J}})\leqslant\aleph_\alpha\,. \tag{8}$$

From the last inequality it follows that

$$2^{\tau(\mathbf{B}_{\mathcal{J}})}\leqslant 2^{\aleph_\alpha}\,.$$

Thus, in virtue of (8), the equality

$$2^{\tau(\mathbf{B}_{\mathcal{J}})}=2^{\aleph_\alpha}$$

holds. Hence, applying (*), we have the equality $\tau(\mathbf{B}_{\mathcal{J}})=\aleph_\alpha$ for every compact topology $\mathcal{J}$. Consequently, by Lemma 1, for every compact topology $\mathcal{J}$ for G, the Boolean σ-algebra $\mathbf{B}_{\mathcal{J}}$ is of the type α. The sufficiency of (*) is thus proved.

Necessity of (*). Let us suppose that if $\overline{\overline{G}}=2^{\aleph_\alpha}$, then for every compact topology $\mathcal{J}$ for G the Boolean σ-algebra $\mathbf{B}_{\mathcal{J}}$ is of the type α.

Let us assume that the equality

$$2^{\aleph_{\alpha_1}}=2^{\aleph_{\alpha_2}} \tag{9}$$

holds. Let $G=\mathop{P}\limits_{\xi\in\Xi} G_\xi$ $(\overline{\overline{\Xi}}=\aleph_{\alpha_1})$ be the product, with the Tychonov topology $\mathcal{J}_0$, of $\aleph_{\alpha_1}$ groups G_ξ each of which contains only two elements 0 and $\frac{1}{2}$ and the group-addition is mod 1. Obviously, G is a compact topological group and $\overline{\overline{G}}=2^{\aleph_{\alpha_1}}$. Consequently, taking into account equality (9), the Boolean σ-algebra $\mathbf{B}_{\mathcal{J}_0}$ is of the type α_j $(j=1,2)$. This implies the equality $\alpha_1=\alpha_2$. Hence $\aleph_{\alpha_1}=\aleph_{\alpha_2}$. The implication (*) is thus proved.

References

[1] P. R. Halmos, *Measure theory*, New York, 1950.

[2] A. Hulanicki, *On cardinal numbers related with locally compact groups*, Bull. Acad. Pol. Sci., Cl. III, 6 (1958), p. 67-70.

[3] L. H. Loomis, *An introduction to abstract harmonic analysis*, Toronto-New York-London 1953.

[4] D. Maharam, *On homogeneous measure algebras*, Proceedings of the National Academy of Sciences 28 (1942), p. 108-111.

[5] E. Marczewski and R. Sikorski, *On isomorphism types of measure algebras*, Fund. Math. 38 (1951), p. 92-98.

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Deformation and mapping theorems *

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A class of theorems, (Q), may be characterized in a general way as asserting that in the deformation of a manifold, M, a point set, P, satisfying a certain property, contains an element for which a prescribed real valued continuous function f takes on an assigned value. This would follow as a corollary if (T): P contains a continuum joining points with $t=0$ and those with $t=1$. The demonstration of (T), in particular cases, depends on the fact that otherwise P would admit a separation by a carrier, C, of a cycle homologous to the base cycle of the manifold though in the cases treated in this paper, contrary to its definition, C must contain points of P.

This method, (T), would seem to have inherent interest, and is applied to some deformations of circles, etc. For instance, *if a circle of radius* 2 *is deformed in a plane into one of radius* $\frac{1}{2}$, *then a fixed circle of radius* 1 *intersects some intermediate curve of the deformation in a pair of points which are maps of antipodal points* or which (under weak regularity conditions) *bisect the length of the intermediate curve*, etc. When the t segment is replaced by a circle, *applications to doubly periodic functions may be derived.* However, a theorem in the class (Q) may be valid without (T). For instance, if circles are replaced by n spheres above, then without establishing (T), we show there are $n+1$ orthogonal points of intersection, *for some* t, *of the fixed unit sphere with the deformed sphere corresponding to this* t *value.* In connection with the n sphere, we demonstrate a *general criterion for the existence of a common image point for the map of orthogonal* k *tuples of* S^{n-1} *to* E^l, yielding the first general breakthrough on a problem of Knaster's and as a special case, for n a prime, the generalized Kakutani theorem.

We use S, $S(\varrho)$, E and $E(+)$ for the circle, the circle of radius ϱ about the origin, the real axis and the positive real axis, respectively.

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