A theory of extensions of map-systems I

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Introduction. The concept of a map-system makes it possible to find out the common structure of a number of theories similar to Schwartz's theory of distributions [7], Mikusiński's operational calculus [5], and Gelfand-Silov generalized functions [2].

This idea was prompted by Sikorski's approach [9] to the theory of distributions. Therefore the reader who is already acquainted with Schwartz distributions and still finds it difficult to follow our theory closely or pick up its intuitive background is advised to look into Sikorski's brief paper as an example. It should be noted here that Sikorski's idea is in fact very similar to one due to Bochner [1].

It is the purpose of this paper to characterize some classes of extensions of linear map-systems or topological map-systems which are similar to those of Schwartz, Mikusiński, and Gelfand-Silov respectively, as well as many others important classes. The principal results of this paper were announced in [10] and [11]

A map-system is a pair $\mathfrak{A} = (S, X)$ which consists of an abelian group X and a semi-group S of homomorphisms $A \in S$ of some subgroups $G_A \subset X$ into X. We assume that, for each $A, B \in S$, $(AB)x = A(Bx)$ whenever the left side exists.

If X is a linear space, we say that $\mathfrak{A} = (S, X)$ is a linear map-system provided all G_A are linear spaces and $A \in S$ are linear mappings.

The chief problem concerning map-systems is to find all the possible extensions of an arbitrary map-system to an algebraically closed one, i. e. such a map-system that the domains of its operators coincide with the whole underlying space.

In general there are many different ways of extension of a given map-system. If the linear space of a given linear map-system $\mathfrak{A}$ is a topological one, we can impose on the extension some additional topological conditions, for instance we can admit only algebraically closed extensions of $\mathfrak{A}$ whose spaces are topologized so that all the extended operators are continuous and so is the imbedding of $\mathfrak{A}$ into its extension.

Therefore the theory of extensions of map-systems can be divided into algebraic and topological parts (cf. [12]). Consequently the paper consists of several parts each of which gives a systematic review of possible extensions that satisfy some additional conditions.

The notion of ideal is the principal one and is exploited throughout the paper. In a sense it is a generalization of that used in the theory of rings. If $\mathfrak{A} = (S, X)$ is a map-system then an ideal $\mathfrak{J}$ of $\mathfrak{A}$ is a certain function $\mathfrak{J}(A)$ that maps S into the family of subgroups of X (or linear subspaces if $\mathfrak{A}$ is linear). It is very natural to introduce some algebraically closed map-systems $\mathfrak{A}/\mathfrak{J} = (S, X')$ named "quotient map-systems", where each value $\mathfrak{J}(A)$ of an ideal for a fixed A can be thought of as the vanishing-set of composition of A with elements of X. The notion of ideal of an operator-system goes back to S. L. Sobolev [16]-[18] who had considered it in a particular case in functional analysis.

In particular if $\mathfrak{J}$ is a so called extensor [1], the group (or the linear space) X can be considered as a subgroup (or a linear subspace) of X', and we may write $\mathfrak{A} \subset \mathfrak{A}/\mathfrak{J}$. This is the subject matter of the Fundamental Theorem stated below. Although it has many applications to distribution-like theories in general functional analysis, the Fundamental Theorem itself belongs, as a matter of fact, to the abstract group theory. It describes some extensions of an abstract commutative group by a given semigroup. The reader who is interested in the purely algebraic theory would look rather into [14] and [15].

The first part of this paper deals only with the algebraic theory of some special types of map-systems named operator-systems, i. e. map-systems $\mathfrak{A} = (S, X)$ with the additional condition that S is commutative and each $A \in S$ maps its domain G_A onto the whole group X. It is very convenient to do so, since a lot of important examples of map-systems satisfy this special condition and a great deal of tiresome considerations can be avoided or made more clear in this particular case.

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1. The notion of map-system [13]. DEFINITION 1.1. A *map-system* is an ordered pair $\mathfrak{A} = (S, X)$ where:

a. S is a semigroup written multiplicatively with the unit element I (the cancellation law is not assumed);

b. X is a commutative group written additively (in what follows S will often be denoted by $l\mathfrak{A}$ — left member of $\mathfrak{A}$ and similarly X will be denoted by $r\mathfrak{A}$ — right member of $\mathfrak{A}$);

[1] In [10] extensors were named *zero-ideals*.

c. with each element $A \in S$ there is associated a subgroup G_A of X and with each pair $A \in S$ and $x \in X$ there is associated an element $y = Ax$ (the composition of A and x) in such a way that

1. the mapping $y = Ax$ is a homomorphism of G_A into the whole group X,
2. for every $A, B \in S$

$$G_{AB} \subset G_B\,; \qquad B(G_{AB}) \subset G_A$$

and $(AB)x = A(Bx)$ for $x \in X$.

If $G_A = X$ for each $A \in S$ then $\mathfrak{A}$ is said to be *algebraically closed*; $\mathfrak{A}$ is said to be a *linear map-system* if and only if X is a linear space and the condition $x \in G_A$ implies $\lambda x \in G_A$ and $A(\lambda x) = \lambda(Ax)$ for every scalar λ; $\mathfrak{A}$ is said to be an *operator-system* [10] if and only if S is commutative and all the homomorphisms $A \in S$ map their domains G_A onto X.

Let $\mathfrak{A} = (S, X)$ and $\mathfrak{A} = (S', X')$ be two map-systems (linear map-systems). A pair $\mathfrak{H} = (H, h)$ is said to be a *homomorphism* of $\mathfrak{A}$ into $\mathfrak{A}'$ if and only if H is a homomorphism of S into S', h is a homomorphism (linear mapping) of X into X', $h(G_A) \subset G'_{HA}$ and

$$h(Ax) = (HA)(hx) \quad \text{for} \quad A \in S \text{ and } x \in G_A\,; \tag{1.1}$$

$\mathfrak{H}$ is called an *isomorphism* if and only if H and h are one-to-one.

We say that $\mathfrak{H}$ *maps* $\mathfrak{A}$ *almost onto* $\mathfrak{A}'$ or that $\mathfrak{A}'$ is an *almost image* of $\mathfrak{A}$ if and only if:

(1.2) α. $H(S) = S'$,

β. for each $x' \in X'$ there are $A_{ij} \in S$, and $x_j \in X$, $i = 1, 2, \ldots, k_j$, $j = 1, 2, \ldots, n$, such that

$$x' = \sum_{j=1}^{n} HA_{1j}\big(HA_{2j} \ldots \big(HA_{k_j j}(hx_j)\big) \ldots\big)\,.$$

If $\mathfrak{A}$ is an operator-system and $\mathfrak{A}'$ is algebraically closed, condition β can be replaced by a more simple one:

β'. for each $x' \in X'$ there are $A \in S$ and $x \in X$ such that

$$x' = (HA)(hx)\,.$$

Clearly β' implies β.

If S is a subsemigroup of S' and $HA = A$ for each $A \in S$, then the homomorphism $\mathfrak{H}$ will be called a *simple homomorphism* and will be denoted by (h).

If homomorphisms $\mathfrak{H} = (H, h)$, $\mathfrak{H}_1 = (H_1, h_1)$, and $\mathfrak{H}_2 = (H_2, h_2)$ map $\mathfrak{A}_2$ into $\mathfrak{A}$, $\mathfrak{A}_1$ into $\mathfrak{A}$, and $\mathfrak{A}_2$ into $\mathfrak{A}_1$ respectively then we write $\mathfrak{H} = \mathfrak{H}_1\mathfrak{H}_2$ if and only if $H = H_1H_2$ and $h = h_1h_2$.

Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two map-systems (linear map-systems) where $l\mathfrak{A}$ is a subsemigroup of $l\mathfrak{A}'$ and $r\mathfrak{A}$ is a subgroup (linear subspace) of $r\mathfrak{A}'$. $\mathfrak{A}$ is said to be *contained* in $\mathfrak{A}'$: $\mathfrak{A} \subset \mathfrak{A}'$ if and only if the identical imbedding $\boldsymbol{h}$ of $r\mathfrak{A}$ into $r\mathfrak{A}'$ induces a simple homomorphism $(\boldsymbol{h})$ of $\mathfrak{A}$ into $\mathfrak{A}'$; $\mathfrak{A}$ is said to be *algebraically dense* in $\mathfrak{A}'$ if and only if $(\boldsymbol{h})$ maps $\mathfrak{A}$ almost onto $\mathfrak{A}'$.

Let $\mathfrak{A} = (\boldsymbol{S}, \boldsymbol{X})$ be an arbitrary map-system (a linear map-system) and let $\boldsymbol{S}'$ be a subsemigroup of $\boldsymbol{S}$. We set

$$S'\mathfrak{A} = (\boldsymbol{S}', \boldsymbol{X}) \subset \mathfrak{A} \tag{1.3}$$

where the composition in $\boldsymbol{S}'\mathfrak{A}$ coincides with that in $\mathfrak{A}$. Similarly, let $\boldsymbol{X}'$ be a subgroup (a linear subspace) of $\boldsymbol{X}$. We set

$$\mathfrak{A}\boldsymbol{X}' = (\boldsymbol{S}, \boldsymbol{X}') \tag{1.3'}$$

where the composition in $\mathfrak{A}\boldsymbol{X}'$ coincides with that in $\mathfrak{A}$ and the domain of $A \in \boldsymbol{S}$ in $\mathfrak{A}\boldsymbol{X}'$ consists of all $x \in \boldsymbol{X}'$ that belong to the domain of A in $\mathfrak{A}$ and $Bx \in \boldsymbol{X}'$ for all $B \in \boldsymbol{S}$ such that $A = CB$, for some $C \in \boldsymbol{S}$.

Now we are able to present a certain abstract example of a linear operator-system (cf. [8], Théorème 1). As its particular cases several examples can be obtained which are considered later on.

Let Δ be an abstract set where $\overline{\overline{\Delta}} = \aleph_\xi$ and let (A_ν) denote the totality of linear operators A_ν, $\nu \in \Delta$, that map linear subsets $\boldsymbol{G}_\nu$ of a given linear space $\boldsymbol{Z}$ into $\boldsymbol{Z}$. Further let (L_ν) be a family of commutative endomorphisms of $\boldsymbol{Z}$ satisfying the following conditions:

(1.4) $L_\nu x \in G_\nu$ and $A_\nu(L_\nu x) = x$ for each $x \in \boldsymbol{Z}$ and $\nu \in \Delta$.
If $\nu' \neq \nu''$ and $x \in \boldsymbol{G}_{\nu'}$ then $L_{\nu''}x \in \boldsymbol{G}_{\nu'}$ and

$$A_{\nu'}(L_{\nu''}x) = L_{\nu''}(A_{\nu'}x).$$

The first condition asserts that the operator L_ν plays the special role of forcing an arbitrary element of $\boldsymbol{X}$ into a domain G_ν of A_ν.

The second asserts that on the domain $\boldsymbol{G}_\nu, L_\nu$ is a sort of rightside inverse operator for A_ν.

The third ensures that no $L_{\nu''}$ stirs too much the domains of $A_{\nu'}$ with ν' different from ν''.

If we regard A_ν as differential operators, then L_ν correspond to associated integral operators.

We denote by $\boldsymbol{N}^{\aleph_\xi}$ the weak product of $\aleph_\xi$ copies of the semigroup $\boldsymbol{N}$ of non-negative integers with usual addition as the semigroup operation. Thus $\boldsymbol{N}^{\aleph_\xi}$ can be identified with the semigroup of all $\boldsymbol{N}$-valued functions $p = (p_\nu)$ defined on Δ, each of which vanishes outside a finite subset of Δ, with the ordinary pointwise group operation. We write $p = (p_{\nu_1}, p_{\nu_1}, \dots, p_{\nu_k})$ if and only if p vanishes out of $(\nu_1, \nu_2, \dots, \nu_k) \subset \Delta$.

We denote by $\boldsymbol{G}_p$, $p \in \boldsymbol{N}^{\aleph_\xi}$, the set of all $x \in \boldsymbol{X}$ such that

$$A_{\nu_{i_1}}(A_{\nu_{i_2}} \dots (A_{\nu_{i_m}} x) \dots)$$

exist and are equal for each permutation (i_j) of numbers

$$\underbrace{1, 1, \dots, 1}_{p_{\nu_1}},\ \underbrace{2, 2, \dots, 2}_{p_{\nu_2}},\ \dots,\ \underbrace{k, k, \dots, k}_{p_{\nu_k}},$$

where $p = (p_{\nu_1}, p_{\nu_2}, \dots, p_{\nu_k})$.

For each $x \in \boldsymbol{G}_p$ we set

$$A^p x \overset{\text{def}}{=} A_{\nu_1}^{p_{\nu_1}}(A_{\nu_2}^{p_{\nu_2}} \dots (A_{\nu_k}^{p_{\nu_k}} x) \dots). \tag{1.5}$$

Similarly let

$$L^p x \overset{\text{def}}{=} L_{\nu_1}^{p_{\nu_1}}(L_{\nu_2}^{p_{\nu_2}} \dots (L_{\nu_k}^{p_{\nu_k}} x) \dots) \quad \text{for} \quad x \in \boldsymbol{Z}. \tag{1.5'}$$

By virtue of (1.4), $L^p x \in \boldsymbol{G}_p$ and $A^p(L^p x) = x$. Therefore the pair

$$\mathfrak{A} = (\{A_\nu\}, \boldsymbol{Z}), \tag{1.6}$$

where $\{A_\nu\}$ denotes the semigroup of all formal operators A^p, $p \in \boldsymbol{N}^{\aleph_\xi}$, with multiplication $A^p A^q = A^{p+q}$, is an operator-system.

Basing on this example we obtain some more special examples. Let $\boldsymbol{E}^k$ denote the k-dimensional Euclidean space with the Cartesian coordinate system.

A point $t^0 \in \boldsymbol{E}^k$ is said to be a *z-symmetric point of the set* Ω if and only if $(t_1, t_2, \dots, t_k) \in \Omega$ implies $(t_1, \dots, t_{i-1},\ \lambda t_i^0 + (1-\lambda)t_i,\ t_{i+1}, \dots, t_k) \in \Omega$ for each $0 \leqslant \lambda \leqslant 1$ and $i = 1, 2, \dots, k$.

Let $\boldsymbol{X}$ be an arbitrary sequentially complete linear locally convex space and let Ω be a subset of $\boldsymbol{E}^k$, where $\Omega \subset \overline{\operatorname{Int}\Omega}$, with at least one z-symmetric point t^0. We denote by $\boldsymbol{X}^\Omega$ the linear space of all continuous $\boldsymbol{X}$-valued functions defined on Ω. Let $a_i(t_i)$ be continuous real positive valued functions such that

$$a(t) \overset{\text{def}}{=} (a_1(t_1), a_2(t_2), \dots, a_k(t_k))$$

maps Ω into $\boldsymbol{E}^k$. Then the linear operators

$$\left(\frac{l}{a}\right)_i x \overset{\text{def}}{=} \left\{ \int_{t_i^0}^{t_i} \frac{1}{a_i(\tau)} x(t_1, \dots, t_{i-1}, \tau, t_{i+1}, \dots, t_k)\, d\tau \right\} \tag{1.7}$$

exist and map $\boldsymbol{X}^\Omega$ into $\boldsymbol{X}^\Omega$ (²). Clearly all the operators $(l/a)_i$ are commutative.

(²) As regards integrals in linear spaces see [3], p. 40-45.

We introduce operators aD_i setting

$$(1.7')\qquad aD_ix \overset{\text{def}}{=} \Big\{a_i(t_i) \lim_{\substack{h\to 0\\(t_1,\dots,t_{i-1},t_i+h,t_{i+1},\dots,t_k)\in\Omega}} \Big(\frac{x(t_1,\dots,t_{i-1},t_i+h,t_{i+1},\dots,t_k)}{h} - \frac{x(t_1,\dots,t_{i-1},t_i,t_{i+1},\dots,t_k)}{h}\Big)\Big\}$$

whenever the right-hand limit exists and belongs to X^Ω. The domain of aD_i is denoted by G_i.

It is easy to check that

$$(1.8)\qquad (l/a)_ix\in G_i \text{ and } aD_i\big((l/a)_ix\big) = x \text{ for } x\in X^\Omega,\ i=1,2,\dots,k;$$

if $i\neq j$ and $x\in G_j$ then $(l/a)_ix\in G_j$ and

$$aD_j\big((l/a)_ix\big) = (l/a)_i(aD_jx).$$

Therefore conditions (1.4) are satisfied and setting $\Delta = \{1,2,\dots,k\}$, $Z = X^\Omega$, and $A_i = aD_i$ we obtain a linear operator-system

$$(1.9)\qquad \mathfrak{A}_{\Omega,a} \overset{\text{def}}{=} (\{aD_i\}, X^\Omega)$$

as a particular case of example (1.6).

Let Ω_1 and Ω_2 be two fixed subsets of E^k, $\Omega_2\supset\Omega_1$ and let each Ω_1,Ω_2 contain at least one z-symmetric point. Then we can introduce a simple homomorphism $(h^{(\Omega_1,\Omega_2)})$ of $\mathfrak{A}_{\Omega_2,a}$ onto $\mathfrak{A}_{\Omega_1,a}$

$$(1.9')\qquad h^{(\Omega_1,\Omega_2)}x = \text{restriction of } x \text{ to } \Omega_1, \quad \text{where} \quad x\in X^{\Omega_2}.$$

Such homomorphisms will be needed later. Let

$$g = \Big\{\int_{t_1^0}^{t_1}\frac{d\tau}{a_1(\tau)},\dots,\int_{t_k^0}^{t_k}\frac{d\tau}{a_k(\tau)}\Big\}.$$

We notice here that the operator-systems $\mathfrak{A}_{\Omega,a}$ and $\mathfrak{A}_{g(\Omega),\{1\}}$ are isomorphic. In fact, setting

$$(1.9'')\qquad \begin{gathered} H_aD^p \overset{\text{def}}{=} aD^p,\\ h_ax \overset{\text{def}}{=} \{x(g(t))\} \quad \text{for} \quad x\in X^{g(\Omega)},\end{gathered}$$

we obtain

$$(H_aD^p)(h_ax) = h_a(D^px)$$

clearly if g is one-to-one [3], and so is h_a.

[3] Example (1.9'') with more general substitutions than g will be considered in the next section.

In what follows let $X^{\Omega,0}$ denote a linear subspace of X^Ω which consists of all $x\in X^\Omega$ such that

$$x(t_1,\dots,t_{i-1},t_i^0,t_{i+1},\dots,t_k) = 0$$

for each $i=1,2,\dots,k$, and each $(t_1,\dots,t_{i-1},t_i^0,t_{i+1},\dots,t_k)\in\Omega$. The linear map-system

$$(1.10)\qquad \mathfrak{A}^0_{\Omega,a} \overset{\text{def}}{=} \mathfrak{A}_{\Omega,a}X^{\Omega,0}$$

is an operator-system, for $(l/a)_i$ maps $X^{\Omega,0}$ into $X^{\Omega,0}$. Since $(l/a)^{(1,1,\dots,1)}x \in X^{\Omega,0}$ for each $x\in X^\Omega$, we see that $\mathfrak{A}^0_{\Omega,a}$ is algebraically dense in $\mathfrak{A}_{\Omega,a}$. Clearly the operator system (1.10) is also a particular case of (1.6).

We now introduce another linear subspace of X^Ω. Let $X^{\Omega,1}$ consist of all $x\in X^\Omega$ satisfying the condition

$$x(t_1^0,t_2,\dots,t_k) = 0 \quad \text{for each} \quad (t_1^0,t_2,\dots t_k)\in\Omega,$$

where t_1^0 is the first coordinate of the point t^0. The linear map-system

$$(1.11)\qquad \mathfrak{A}^1_{\Omega,a} \overset{\text{def}}{=} \mathfrak{A}_{\Omega,a}X^{\Omega,1}$$

is an operator system, for $(l/a)_i$ maps $X^{\Omega,1}$ into $X^{\Omega,1}$ and since $X^{\Omega,0}$ is a linear subspace of $X^{\Omega,1}$, we have the inclusion

$$\mathfrak{A}_{\Omega,a}\supset\mathfrak{A}^1_{\Omega,a}\supset\mathfrak{A}^0_{\Omega,a}.$$

The last example can also be treated as a particular case of (1.6).

Let $X^{E^k,\infty}$ denote the greatest linear subspace $Y\subset X^{E^k}$ with the following properties:

1. If $x\in Y$, then for every continuous pseudonorm $|\cdot|$ on X, and arbitrary real numbers t_i, $i=1,2,\dots,k$,

$$\int_{-\infty}^{t_i}\frac{|x(\tau)|}{a_i(\tau_i)}\,d\tau_i < \infty.$$

2. If $x\in Y$, then $(l/a)_{\infty,i}x\in Y$, where

$$(l/a)_{\infty,i}y \overset{\text{def}}{=} \Big\{\int_{-\infty}^{t_i}\frac{y(\tau)}{a_i(\tau_i)}\,d\tau_i\Big\} \quad \text{for} \quad y\in X^{E^k}.$$

Setting in (1.6) $Z = X^{E^k,\infty}$ and $A_i = aD_i$, where aD_i are defined by (1.7), we obtain as a particular case of (1.6)

$$(1.12)\qquad \mathfrak{A}^\infty_{E^k,a} \overset{\text{def}}{=} (\{aD_i\}, X^{E^k,\infty}).$$

It is obvious that

$$\mathfrak{A}^{\infty}_{E^k,a} \subset \mathfrak{A}_{E^k,a}.$$

Clearly $\mathfrak{A}^{\infty}_{E^k,a}$ is not algebraically dense in $\mathfrak{A}_{E^k,a}$, but all the operators ${}_aD^p$ are one-to-one in $\mathfrak{A}^{\infty}_{E^k,a}$, which does not hold in $\mathfrak{A}_{E^k,a}$.

Similarly, we denote by $X_1^{E^k,\infty}$ the greatest linear subspace $Y \subset X^{E^k}$ such that

1'. If $x \in Y$, then for every continuous pseudonorm $|\cdot|$ on X and arbitrary real numbers t_1,

$$\int_{-\infty}^{t_1} \frac{|x(\tau)|}{a_1(\tau_1)}\, d\tau_1 < +\infty .$$

2'. If $x \in Y$, then $(l/a)_{\infty,1} x \in Y$,

and again, as a particular case of (1.6) we obtain

(1.12') $$\mathfrak{A}^{\infty,1}_{E^k,a} \overset{\text{def}}{=} (\{{}_aD_i\}, X_1^{E^k,\infty}).$$

Clearly

$$\mathfrak{A}^{\infty}_{E^k,a} \subset \mathfrak{A}^{\infty,1}_{E^k,a} \subset \mathfrak{A}_{E^k,a}.$$

Here is an example of another kind. Let T^k denote the space obtained by identification of points of E^k congruent modulo 1 [4]. This identification leads to a continuous image of E^k which is topologically identical with the k-dimensional thorus. We denote by X^{T^k} a linear space of all X-valued continuous functions on T^k. It is clear that X^{T^k} is a linear subspace of X^{E^k}, where each member of X^{T^k} is considered as a function from X^{E^k} with the same values at congruent points.

We introduce a linear map-system setting

(1.13) $$\mathfrak{A}_{T^k,a} \overset{\text{def}}{=} \mathfrak{A}_{E^k,a} X^{T^k}.$$

Clearly it is not an operator-system.

Let Q^k denote the closed cube $\langle 0,1;\dots;0,1\rangle \subset E^k$. The space T^k can also be treated as a continuous image of Q^k, where the points of Q^k which are congruent modulo 1 are identified. In the same way as previously we can imbed X^{T^k} into X^{Q^k} and then X^{T^k} is also a linear subspace of X^{Q^k}. It can easily be verified that

$$\mathfrak{A}_{T^k,a} = \mathfrak{A}_{Q^k,a} X^{T^k}.$$

[4] $(t_1,\dots,t_k) \equiv (\tau_1,\dots,\tau_k) \pmod 1$ if and only if $t_i \equiv \tau_i \pmod 1$ for each $i = 1,2,\dots,k$.

Now let us identify all the points which are on the boundary of Q^k. This identification leads to a continuous image of Q^k, which is topologically equivalent to the k-dimensional sphere S^k. We denote by X^{S^k} the linear space of all continuous X-valued functions on S^k. As in the preceding examples we can consider each member of X^{S^k} as a function from X^{Q^k} with the same values at all the points of the boundary of Q^k.

But each function from Q^k with the same value on the boundary of Q^k belongs to X^{T^k} and hence X^{S^k} is a linear subspace of X^{T^k} while on the other hand the last one is a linear subspace of X^{Q^k}. Clearly X^{S^k} is also a linear subspace of X^{E^k}, because X^{T^k} is such. Therefore a linear map-system can be introduced as follows

(1.13') $$\mathfrak{A}_{S^k,a} \overset{\text{def}}{=} \mathfrak{A}_{T^k,a} X^{S^k}.$$

Clearly

$$\mathfrak{A}_{S^k,a} \subset \mathfrak{A}_{T^k,a} \subset \begin{cases} \mathfrak{A}_{Q^k,a}, \\ \mathfrak{A}_{E^k,a}. \end{cases}$$

The construction of (1.13) and (1.13') can be unified in the following way. Let R and R' be two topological spaces and let f be a continuous mapping of R onto R'. In what follows let X^R and $X^{R'}$ be the sets of all continuous X-valued functions on R and R' respectively. It is easy to see that, as in the preceding examples, $X^{R'}$ can be considered as a linear subspace of X^R. Indeed, the linear mapping

$$(Ux)(t) = x(f(t)), \qquad x \in X^{R'},$$

is an imbedding of $X^{R'}$ into X^R and because of this we can consider X^R as a linear subspace of X^R.

If S is a semigroup of operators defined on some subspaces of X^R in such a way that the pair (S, X^R) is a linear map-system, then we can set

(1.14) $$(S, X^{R'}) = (S, X^R) X^{R'}$$

and this is a generalization of (1.12) and (1.13). The example (1.14) will be needed later for it leads to some map-systems associated closely with *variétés différentiables*.

We now consider some special operator-systems contained in $\mathfrak{A}_{Q^k,a}$ and $\mathfrak{A}_{T^k,a}$. First we notice that the operators $(l/a)_i$ are unambiguously defined in X^{Q^k} for $t^0 = (0,0,\dots,0)$ since it is a z-symmetric point of Q^k. Let

$$(P_i x)(t_1,t_2,\dots,t_k) \overset{\text{def}}{=} \frac{1}{\lambda_i}\left(\frac{l}{a}\right)_i x(t_1,\dots,t_{i-1},1,t_{i+1},\dots,t_k)$$

and let

$$L_i x \overset{\text{def}}{=} (I-P_i)(l/a)_i x\,,$$

where we set

$$\lambda_i \overset{\text{def}}{=} \int_0^1 \frac{d\tau}{a_i(\tau)} \quad (\neq 0)$$

and I is the identity operator.

It is easy to verify that

$$P_i(L_i x) = 0\,, \quad P_i(L_j x) = L_j(P_i x)\,,$$
$$P_i^2 x = P_i x\,, \quad P_i(P_j x) = P_j(P_i x)\,,$$
$$aD_i(L_i x) = x\,, \quad aD_i(L_j y) = L_j(aD_i y)$$

for $x \in X^{Q^k}$, $y \in G_i$ and $i \neq j$.

Since $P_i x = 0$ implies $P_i(L_j x) = 0$, we have

$$L_j(X^{Q^k,P}) \subset X^{Q^k,P}$$

where

$$X^{Q^k,P} \overset{\text{def}}{=} \{x \in X^{Q^k}\colon\ P_i x = 0 \text{ for } i = 1, 2, \dots, k\}\,.$$

Hence

(1.15) $$\mathfrak{A}^P_{Q^k,a} \overset{\text{def}}{=} \mathfrak{A}_{Q^k,a} X^{Q^k,P}$$

is a linear operator-system.

If $x \in X^{Q^k,P} \cap X^{T^k}$ then $L_i x \in X^{T^k}$ and hence

$$L_j(X^{T^k,P}) \subset X^{T^k,P}, \quad \text{where} \quad X^{T^k,P} \overset{\text{def}}{=} X^{Q^k,P} \cap X^{T^k}.$$

Therefore

(1.16) $$\mathfrak{A}^P_{T^k,a} \overset{\text{def}}{=} \mathfrak{A}_{Q^k,a} X^{T^k,P}$$

is an operator-system.

If $x \in X^{Q^k}$ then $y = L_1 L_2 \dots L_k x \in X^{Q^k,P}$ and if $y \in X^{Q^k,P}$ then $L_1 L_2 \dots L_k y \in X^{T^k,P}$.

Thus for each $x \in X^{Q^k}$

$$L_1^2 L_2^2 \dots L_k^2 x \in X^{T^k,P} \quad \text{for each} \quad x \in X^{Q^k}$$

and then $\mathfrak{A}^P_{T^k,a}$ is algebraically dense in $\mathfrak{A}^P_{Q^k,a}$, $\mathfrak{A}_{Q^k,a}$ and $\mathfrak{A}_{T^k,a}$.

An operator $D_1 D_2 \dots D_k$ is said to be an *annulator* of an element $x \in X^{Q^k}$ if and only if there is a permutation $(i_1, i_2, \dots, i_k)$ of numbers $1, 2, \dots, k$ such that

$$D_{i_1}\big(D_{i_2} \dots (D_{i_k} x) \dots\big) = 0\,.$$

Let $X^{Q^k,cP} \subset X^{Q^k}$ denote the least linear set containing all elements with annulator $D_1 D_2 \dots D_k$.

For each $x \in X^{Q^k}$ we have

$$x_0 = (I-P_1)(I-P_2) \dots (I-P_k) x \in X^{Q^k,P}$$

and if $x \in X^{T^k}$ then $x_0 \in X^{T^k,P}$. On the other hand $x - x_0 \in X^{Q^k,cP}$ and if $x \in X^{T^k}$, then also $x - x_0 \in X^{T^k}$. Therefore

$$X^{Q^k} = X^{Q^k,P} \oplus X^{Q^k,cP}, \quad X^{Q^k,P} \cap X^{Q^k,cP} = \{0\}$$

and

$$X^{T^k} = X^{T^k,P} \oplus X^{T^k,cP}, \quad X^{T^k,P} \cap X^{T^k,cP} = \{0\}\,,$$

where $X^{T^k,cP} \overset{\text{def}}{=} X^{Q^k,cP} \cap X^{T^k}$.

The operators aD_i are one-to-one in $X^{Q^k,P}$, since, if $aD_i x = 0$ then x does not depend on the variable t_i and

$$P_i x = 0 \quad \text{if and only if} \quad x = 0\,.$$

Finally we consider a Hilbert space $\boldsymbol{H}$. Let B_j, $j = 1, 2, \dots, k$, be selfadjoint operators that map $G_j \subset \boldsymbol{H}$ onto some closed subspaces of $\boldsymbol{H}$ in such a way that all the operators $(I + \varepsilon i B_j)^{-1}$, where $\varepsilon = \pm 1$ and $j = 1, 2, \dots, k$, are commutative (5).

Since B_j are selfadjoint, we have $B_j^{-1}(0) \perp B_j(G_j)$ where $B^{-1}(0)$ are closed and $\boldsymbol{H} = B_j^{-1}(0) \oplus B_j(G_j)$. By our assumptions the projectors P_j of $\boldsymbol{H}$ onto $B_j(0)$ are commutative. Further B_j maps in a one-to-one manner $G_j \cap \tilde{\boldsymbol{H}}$ onto $\tilde{\boldsymbol{H}}$, where $\tilde{\boldsymbol{H}} = (I-P_1)(I-P_2) \dots (I-P_k)\boldsymbol{H}$.

Therefore for $L_j = B_j^{-1}$ the conditions (1.4) are satisfied. Hence setting $\Delta = 1, 2, \dots, k$, $Z = \tilde{\boldsymbol{H}}$ we obtain a linear operator-system

(1.17) $$\mathfrak{A}_H \overset{\text{def}}{=} (\{B_j\}, \tilde{\boldsymbol{H}})\,.$$

The linear complement $\boldsymbol{H}_0$ of $\tilde{\boldsymbol{H}}$ in $\boldsymbol{H}$ is identical with the least linear set containing all the elements with an annulator $B_1 B_2 \dots B_k$, where $B_1 B_2 \dots B_k$ is an annulator of $x \in \boldsymbol{H}$ if and only if there is a permutation $i_1, i_2, \dots, i_k$ of the numbers $1, 2, \dots, k$ such that $B_{i_1}\big(B_{i_2} \dots (B_{i_k} x) \dots\big)$ exists and is equal to zero. Indeed,

$$x_0 = (I-P_1)(I-P_2) \dots (I-P_k) x \in \tilde{X}$$

and

$$x_0 - x = \sum_{i=1}^{k} P_i y_i \in \boldsymbol{H}_0$$

because $B_i(P_i y_i) = 0$ for $i = 1, 2, \dots, k$.

(5) As regards selfadjoint operators on a Hilbert space see [6], p. 307-338.

2. The notion of ideal of an operator-system [10]. We now introduce a notion of ideal of a map-system. This notion is an important one, very much like that used in the ring theory.

To this aim, let us consider the identity homomorphism of an algebraically closed map-system $\mathfrak{A}$ onto itself. It is not difficult to see that the function $\mathfrak{J}$,

$$\mathfrak{J}(A) \overset{\text{def}}{=} \{x\colon Ax = 0\}, \quad A \in l\mathfrak{A},$$

can be treated as the "kernel" of this homomorphism. This suggests that the ideal of a map-system should be a function defined on $l\mathfrak{A}$ with values in the set of all subgroups (or linear subspaces) of $r\mathfrak{A}$. On the other hand each value $\mathfrak{J}(A)$ of an "ideal" for fixed A can be thought of as the null-set of composition of A with elements of $r\mathfrak{A}/\mathfrak{J}$, where $\mathfrak{A}/\mathfrak{J}$ denotes the quotient map-system. Therefore, in order to extend a given map-system, not necessarily algebraically closed, to an algebraically closed one, we must associate with each $A \in l\mathfrak{A}$ a subset $\mathfrak{J}(A)$ of elements of $r\mathfrak{A}$ on which A will vanish after the extension is done, i. e. we must fix a certain ideal of $\mathfrak{A}$.

At first we introduce the notion of ideal for an operator-system and linear operator-systems only. In the general case it is much more difficult and will be considered later (see [13], [14], and [15]).

DEFINITION 2.1. An *ideal* of an operator-system (linear operator-system) $\mathfrak{A} = (\boldsymbol{S}, \boldsymbol{X})$ is a mapping $\mathfrak{J}$ defined on $\boldsymbol{S}$ such that its values $\mathfrak{J}(A)$ are subgroups (linear subspaces) of $\boldsymbol{X}$ and for $A, B \in \boldsymbol{S}$

$$\mathfrak{J}(B) \subset \mathfrak{J}(AB) \quad \text{and} \quad \boldsymbol{G}_A \cap \mathfrak{J}(AB) = A^{-1}(\mathfrak{J}(B)). \tag{2.1}$$

For instance if $\mathfrak{J}(A) = \boldsymbol{X}$ for each $A \in \boldsymbol{S}$, then $\mathfrak{J}$ is an ideal.

An ideal $\mathfrak{J}$ is called an *extensor* (6) if and only if $\mathfrak{J}(I) = \{0\}$, where $\{0\}$ is the linear set containing only the zero element of $\boldsymbol{X}$. The ideal $\mathfrak{J}(A)$ such that $\mathfrak{J}(A) = \{0\}$ for all $A \in \boldsymbol{S}$ is also an important one and is named *null-ideal.*

Clearly an operator-system $\mathfrak{A}$ admits the null-ideal provided all the operators $A \in l\mathfrak{A}$ are one-to-one. Conversely, if $\mathfrak{A}$ admits the null-ideal then all the operators $A \in l\mathfrak{A}$ are one-to-one.

If $\mathfrak{J}'$ and $\mathfrak{J}$ are ideals, we write $\mathfrak{J} \subset \mathfrak{J}'$ whenever $\mathfrak{J}(A) \subset \mathfrak{J}'(A)$ for every $A \in l\mathfrak{A}$. We say that $\mathfrak{J}$ is contained in $\mathfrak{J}'$. We denote by $\tilde{\mathfrak{J}}^M$ the *minimal ideal*, i. e. such an ideal which is contained in any other ideal.

Remark 2.1. It is easy to see that if $\mathfrak{J}'$ and $\mathfrak{J}''$ are ideals, then so is the function $\mathfrak{J}$, where $\mathfrak{J}(A) = \mathfrak{J}'(A) \cap \mathfrak{J}''(A)$; the same being true for an arbitrary family of ideals $\mathfrak{J}_\nu$, we infer that there is only one minimal ideal. In general, it is not identical with null-ideal, because it may happen for an arbitrary operator-system that the latter does not exist.

(6) In [10] extensors were named *zero-ideals.*

If $\mathfrak{A}$ is algebraically closed, then the function

$$\mathfrak{J}^0(A) \overset{\text{def}}{=} \{x\colon Ax = 0\}, \quad A \in l\mathfrak{A},$$

is the minimal ideal of $\mathfrak{A}$.

Indeed, for every ideal $\mathfrak{J}$ of $\mathfrak{A}$ we have

$$\mathfrak{J}(A) \supset \mathfrak{J}^0(A) \quad \text{for} \quad A \in l\mathfrak{A}.$$

THEOREM 2.1 ([10]). *If $\mathfrak{H} = (\boldsymbol{H}, \boldsymbol{h})$ is a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}'$ and $\mathfrak{J}'$ is an ideal of $\mathfrak{A}'$, then the formula*

$$\mathfrak{J}(A) = \{x\colon \boldsymbol{h}x \in \mathfrak{J}'(\boldsymbol{H}A)\}, \quad A \in \boldsymbol{S}, \tag{2.2}$$

defines an ideal of $\mathfrak{A}$ denoted by $\mathfrak{H}^{-1}\mathfrak{J}'$.

Proof. Clearly $\mathfrak{J}(A)$ are subgroups (linear subspaces) of $\boldsymbol{X}$. Further:

$$\begin{aligned} \boldsymbol{G}_A \cap \mathfrak{J}(AB) &= \{x \in \boldsymbol{G}_A\colon \boldsymbol{h}x \in \mathfrak{J}'(\boldsymbol{H}(AB))\} \\ &= \{x \in \boldsymbol{G}_A\colon \boldsymbol{h}x \in (\boldsymbol{H}A)^{-1}(\mathfrak{J}'(\boldsymbol{H}B))\} \\ &= \{x \in \boldsymbol{G}_A\colon (\boldsymbol{H}A)(\boldsymbol{h}x) \in \mathfrak{J}'(\boldsymbol{H}B)\} \\ &= \{x \in \boldsymbol{G}_A\colon \boldsymbol{h}(Ax) \in \mathfrak{J}'(\boldsymbol{H}B)\} \\ &= A^{-1}\{y\colon \boldsymbol{h}y \in \mathfrak{J}'(\boldsymbol{H}B)\} = A^{-1}(\mathfrak{J}(B)). \end{aligned}$$

Since $\mathfrak{J}(AB) \supset \mathfrak{J}(B)$, the theorem follows.

If $\mathfrak{A}' \supset \mathfrak{A}$ then we write

$$\mathfrak{J}'\boldsymbol{X} \overset{\text{def}}{=} (\boldsymbol{h})^{-1}\mathfrak{J}'$$

where $(\boldsymbol{h})$ is the identical imbedding of $\mathfrak{A}$ into $\mathfrak{A}'$ and then we have $(\boldsymbol{h})^{-1}\mathfrak{J}'(A) = \mathfrak{J}'(A) \cap \boldsymbol{X}$, and $\mathfrak{J}'$ is an arbitrary ideal of $\mathfrak{A}'$.

Let $\mathfrak{A} = (\boldsymbol{S}, \boldsymbol{X})$ be an operator-system (a linear operator-system). We say that an element $A \in \boldsymbol{S}$ is an *annulator* (7) of an element $x \in \boldsymbol{X}$ if and only if there exist elements $A_1, A_2, \ldots, A_k \in \boldsymbol{S}$ such that $A = A_1 A_2 \ldots A_k$ and $A_1(A_2 \ldots (A_k x) \ldots) = 0$.

An element $x \in \boldsymbol{X}$ belongs to $\mathfrak{J}^M(A)$ if and only if there are elements $B \in \boldsymbol{S}$, $x_1, x_2, \ldots, x_k \in \boldsymbol{X}$, where AB is an annulator of all x_i, with

$$x = B(x_1 + x_2 + \ldots + x_k).$$

THEOREM 2.2. *The function $\mathfrak{J}^M = \{\mathfrak{J}^M(A)\}$ is the minimal ideal of $\mathfrak{A}$* ([11]).

(7) The notion of annulator corresponds to that of "rank of an element" in [11].

Proof. Each $\mathfrak{I}^M(A)$, where $A \in \boldsymbol{S}$, is a subgroup (a linear subspace) of $\boldsymbol{X}$. In fact, let $x, y \in \mathfrak{I}^M(A)$. Then there are $B, C \in \boldsymbol{S}$, $u_1, u_2, \ldots, u_m \in \boldsymbol{X}$ and $v_1, v_2, \ldots, v_n \in \boldsymbol{X}$ such that

$$x = B(u_1+u_2+\ldots+u_m), \quad y = C(v_1+v_2+\ldots+v_n),$$

where AB and AC are annulators of u_i and v_j respectively. There are $u_i', v_j', u', v' \in \boldsymbol{X}$ with

$$u_i = Cu_i', \quad v_j = Bv_j',$$
$$x = BCu', \quad y = BCv',$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. If we set

$$u_{m+1}' \stackrel{\text{def}}{=} u' - (u_1'+u_2'+\ldots+u_m'),$$
$$v_{n+1}' \stackrel{\text{def}}{=} v' - (v_1'+v_2'+\ldots+v_n'),$$

then

$$B(Cu_{m+1}') = C(Bv_{n+1}') = 0,$$

$$x+y = BC(u_1'+u_2'+\ldots+u_m'+u_{m+1}'+v_1'+v_2'+\ldots+v_n'+v_{n+1}'),$$

where ABC is an annulator of $u_1', u_2', \ldots, u_m', u_{m+1}', v_1', v_2', \ldots, v_n', v_{n+1}'$. Hence

$$x+y \in \mathfrak{I}^M(A).$$

If $x \in \mathfrak{I}^M(A)$, then obviously $-x \in \mathfrak{I}^M(A)$ (and $\lambda x \in \mathfrak{I}^M(A)$) and hence $\mathfrak{I}^M(A)$ is a subgroup (a linear subspace) of $\boldsymbol{X}$.

Let $x \in \mathfrak{I}^M(AB)$ and $x \in \boldsymbol{G}_A$. We have

$$x = C(x_1+x_2+\ldots+x_k),$$

where ABC is an annulator of $x_1, x_2, \ldots, x_k$. Let $Ax = ACv$ and let $x_{k+1} = v-(x_1+x_2+\ldots+x_k)$. Then $A(Cx_{k+1}) = 0$ and ABC is an annulator of $x_1, x_2, \ldots, x_k, x_{k+1}$. Since

$$Ax = (AC)(x_1+x_2+\ldots+x_k+x_{k+1}),$$

we have $Ax \in \mathfrak{I}^M(B)$.

Conversely, let $Ax \in \mathfrak{I}^M(B)$. This means that

$$Ax = C(u_1+u_2+\ldots+u_k),$$

where BC is an annulator of $u_1, u_2, \ldots, u_k$. There are $x_i, y, z \in \boldsymbol{X}$ with $u_i = Ax_i$, $Ax = (AC)y$, and $x = Cz$, where $i = 1, 2, \ldots, k$, and let

$$x_0 \stackrel{\text{def}}{=} z-y, \quad x_{k+1} \stackrel{\text{def}}{=} y-(x_1+x_2+\ldots+x_k).$$

We have

$$A(Cx_0) = C(Ax_{k+1}) = 0$$

and

$$x = C(x_0+x_1+\ldots+x_k+x_{k+1}),$$

where ABC is an annulator of x_i $(i = 0, 1, 2, \ldots, k, k+1)$. Therefore $x \in \mathfrak{I}^M(AB)$.

Clearly $\mathfrak{I}^M(A) \subset \mathfrak{I}^M(AB)$ for each $A, B \in \boldsymbol{S}$. Now let $\mathfrak{I}$ be an arbitrary ideal and let $x \in \mathfrak{I}^M(A)$. There are $C \in \boldsymbol{S}$ and $x_1, x_2, \ldots, x_k \in \boldsymbol{X}$ with annulator AC such that

$$x = C(x_1+x_2+\ldots+x_k).$$

It is easy to see that $x_i \in \mathfrak{I}(AC)$ $(i = 1, 2, \ldots, k)$. In fact, if for instance $AC = B_1 B_2 \ldots B_n$ and

$$B_1\big(B_2 \ldots (B_n x_i) \ldots\big) = 0,$$

then

$$B_1\big(B_2 \ldots (B_n x) \ldots\big) \in \mathfrak{I}(I),\ B_2\big(\ldots (B_n x_i) \ldots\big) \in \mathfrak{I}(B_1), \ldots, B_n x_i \in \mathfrak{I}(B_1 B_2 \ldots B_{n-1}),$$

and $x_i \in \mathfrak{I}(B_1 B_2 \ldots B_n) = \mathfrak{I}(AC)$. Since $\mathfrak{I}(AC)$ is a group (a linear space), it follows that

$$x_1+x_2+\ldots+x_k \in \mathfrak{I}(AC)$$

and then

$$x = C(x_1+x_2+\ldots+x_k) \in \mathfrak{I}(A).$$

Hence $\mathfrak{I}^M \subset \mathfrak{I}$ and the theorem holds.

Now we are going to show a certain property of the minimal ideal. Let $\mathfrak{A} = (\boldsymbol{S}, \boldsymbol{X})$ be an operator-system (a linear operator-system) and let $\boldsymbol{S}_0$ and $\boldsymbol{S}_1$ be a decomposition of $\boldsymbol{S}$ into the simple product of sub-semigroups, i. e. each element of $\boldsymbol{S}$ can be decomposed in exactly one way into the product of elements from $\boldsymbol{S}_0$ and $\boldsymbol{S}_1$.

THEOREM 2.3. *If each $A \in \boldsymbol{S}_0$ is one-to-one in $\mathfrak{A}$, and if*

(*) *$C_0(C_1 x) = C_1(C_0 x)$, for $C_0 \in \boldsymbol{S}_0$, $C_1 \in \boldsymbol{S}_1$ whenever the right side exists,*

then

$$\mathfrak{I}^M(A) = \mathfrak{I}_1^M(\mathfrak{c}A) \quad \textit{for} \quad A \in \boldsymbol{S},$$

where $\mathfrak{I}^M$ and $\mathfrak{I}_1^M$ are the minimal ideals of $\mathfrak{A}$ and $\boldsymbol{S}_1\mathfrak{A}$ respectively, and $\mathfrak{c}A$ is the component of A from S_1.

Proof. The inclusion $\mathfrak{I}^M(A) \supset \mathfrak{I}_1^M(A)$ is obvious. In fact: $\mathfrak{I}^M(A) \supset \mathfrak{I}^M(\mathfrak{c}A) \supset \mathfrak{I}_1^M(\mathfrak{c}A)$.

Let $x \in \mathfrak{I}^M(A)$. Then $x = B(x_1+x_2+\ldots+x_k)$ where AB is an annulator of $x_1, x_2, \ldots, x_k$. By virtue of (*) the annulator AB admits decompositions

$$AB = C_{1j}^0 C_{2j}^0 \ldots C_{ljj}^0 C_{1j}^1 C_{2j}^1 \ldots C_{kjj}^1,$$

where

$$C^0_{ij} \in \boldsymbol{S}_0\,, \quad C^1_{ij} \in \boldsymbol{S}_1\,, \quad C^0_{1j}\big(\ldots C^0_{lj}\big(C^1_{1j} \ldots (C^1_{k_jj}x_j)\ldots\big)\ldots\big) = 0 \quad \text{for} \quad j = 1, 2, \ldots, k\,.$$

Since C^0_{ij} are one-to-one, we have

$$C^1_{1j}\big(\ldots (C^1_{k_jj}x_j)\ldots\big) = 0\,.$$

Let $A = A_0A_1$, $B = B_0B_1$, where $A_0, B_0 \in \boldsymbol{S}_0$ and $A_1, B_1 \in \boldsymbol{S}_1$. By virtue of the uniqueness of decomposition, $C^1_{1j} \ldots C^1_{k_jj} = A_1B_1$. Therefore $A_1B = A_1B_1B_0$ is an annulator of $x_1, x_2, \ldots, x_k$ and then $x \in \mathfrak{J}^M(A_1)$, which completes the proof.

Definition 2.2. $\mathfrak{A}$ is said to be a *proper operator-system* if and only if:

Whenever

$$A_{1i}A_{2i} \ldots A_{k_ii} = A\,, \qquad A_{ij}, A \in l\mathfrak{A}$$

and

$$y_i \overset{\text{def}}{=} A_{1i}\big(A_{2i} \ldots (A_{k_ii}x_i)\ldots\big)\,, \qquad x_i \in r\mathfrak{A}$$

exist for $i = 1, 2, \ldots, l$, then $x_1 + x_2 + \ldots + x_1 = 0$ implies $y_1 + y_2 + \ldots + y_1 = 0$ (cf. Theorem 3.2).

Theorem 2.4. *$\mathfrak{A}$ is a proper operator-system if and only if the minimal ideal of $\mathfrak{A}$ is an extensor* (cf. Theorem 3.1).

Proof. Let us suppose that $\mathfrak{A}$ is proper and let $y = B(x_1 + x_2 + \ldots + x_l)$, where B is an annulator of $x_1, x_2, \ldots, x_l$. We put $x_0 \overset{\text{def}}{=} -(x_1 + x_2 + \ldots + x_l)$ so that $x_0 + x_1 + \ldots + x_l = 0$. Since B is an annulator of $x_1, x_2, \ldots, x_l$, there are $B_{ij} \in \boldsymbol{S}$ $(i = 1, 2, \ldots, k_j,\ j = 1, 2, \ldots, l)$ such that

$$B = B_{1j}B_{2j} \ldots B_{k_jj} \quad \text{and} \quad y_j = B_{1j}\big(B_{2j} \ldots (B_{k_jj}x_j)\ldots\big) = 0\,.$$

Hence $y = y_0 + y_1 + \ldots + y_l$, where $y_0 = -Bx_0$, and since $\mathfrak{A}$ is proper, we have $y = 0$.

Now conversely, let us suppose that the minimal ideal of $\mathfrak{A}$ is a zero-ideal. Let $A_{1i}A_{2i} \ldots A_{k_ii} = A$ for $i = 1, 2, \ldots, l$ and let

$$y_i = A_{1i}\big(A_{2i} \ldots A_{k_ii}x_i \ldots\big)$$

exist. Setting $y_i = A\hat{x}_i$ we see that A is an annulator of $x_i - \hat{x}_i$ and

$$A\big((x_1 - \hat{x}_1) + (x_2 - \hat{x}_2) + \ldots + (x_l - \hat{x}_l)\big) = 0\,.$$

Hence

$$A(x_1 + x_2 + \ldots + x_l) = A\hat{x}_1 + A\hat{x}_2 + \ldots + A\hat{x}_l = y_1 + y_2 + \ldots + y_l\,,$$

but this is exactly what we have to show, because A maps zero onto zero.

Now we are going to show some ideals for a number of concrete operator-systems. At first let us consider the linear operator-system (1.6). We denote $G^0_\nu \overset{\text{def}}{=} \{x\colon A_\nu x = 0\}$.

Lemma 2.1. (Cf. [8], Proposition 2.) *Each element $x \in \boldsymbol{Z}$ of the rank not greater than A^p, where $p = (p_{\nu_1}, p_{\nu_2}, \ldots, p_{\nu_k})$, is of the form*

$$x = \sum_{i=1}^{k}\Big(\sum_{0 \leqslant j \leqslant p_{\nu_i}-1} L^j_{\nu_i}x_{ij}\Big)\,, \tag{2.3}$$

where $x_{ij} \in G^0_{\nu_i}$ for $i = 1, 2, \ldots, k$, $0 \leqslant j \leqslant p_{\nu_i}-1$ and k is an arbitrary natural number.

Proof. If $p = (0, 0, \ldots, 0)$ then it is all right. Let us assume that the lemma is valid for $p_{\nu_1} + p_{\nu_2} + \ldots + p_{\nu_k} \leqslant n$ and we will try to show that it is also valid for $p_{\nu_1} + p_{\nu_2} + \ldots + p_{\nu_l} = n+1$. Let

$$A_{\nu_{i_1}}\big(A_{\nu_{i_2}} \ldots \big(A_{\nu_{i_n}}(A_{\nu_{i_{n+1}}}x)\big)\ldots\big) = 0\,.$$

Setting $y = A_{\nu_{i_{n+1}}}x$ we see that $(p_{\nu_1}, \ldots, p_{\nu_{i_0-1}}, p_{\nu_{i_0}} - 1, p_{\nu_{i_0+1}}, \ldots, p_{\nu_l})$ is an annulator of y and hence by our assumption

$$y = A_{\nu_{i_0}}x = \sum_{i=1,\ldots,i_0-1,i_0+1,\ldots,l}\ \sum_{0 \leqslant j \leqslant p_{\nu_i}-1} L^j_{\nu_i}x_{ij} + \begin{cases} \displaystyle\sum_{0 \leqslant j \leqslant p_{\nu_{i_0}}-2} L^j_{\nu_{i_0}}x_{i_0j} & \text{if} \quad i_0 > 1\,, \\ 0 & \text{if} \quad i_0 = 1\,, \end{cases}$$

where $x_{ij} \in G^0_{\nu_i}$ for each i and j. For $i \neq i_0$ we set $x'_{ij} = L_{\nu_{i_0}}x_{ij} \in G^0_{\nu_i}$. Thus

$$L_{\nu_{i_0}}(A_{\nu_{i_0}}x) = \sum_{i=1,\ldots,i_0-1,i_0+1,\ldots,l}\ \sum_{0 \leqslant j \leqslant p_{\nu_i}-1} L^j_{\nu_i}x'_{ij} + \begin{cases} \displaystyle\sum_{1 \leqslant j \leqslant p_{\nu_0}-1} L^j_{\nu_{i_0}}x_{i_0j} & \text{if} \quad i_0 > 1\,, \\ 0 & \text{if} \quad i_0 = 1\,. \end{cases}$$

But $x_{i_00} = x - L_{\nu_{i_0}0}(Ax) \in G^0_{\nu_{i_0}}$ and hence

$$x = \sum_{i=1,\ldots,i_0-1,i_0+1,\ldots,l}\ \sum_{0 \leqslant j \leqslant p_{\nu_i}-1} L^j_{\nu_i}x'_{ij} + R\,,$$

where

$$R = \begin{cases} \displaystyle\sum_{0 \leqslant j \leqslant p_{\nu_i}-1} L^j_{\nu_{i_0}}x_{i_0j} & \text{if} \quad i_0 > 1 \\ x_{i_00} & \text{if} \quad i_0 = 1\,, \end{cases}$$

and so the proof is completed.

From the lemma we immediately obtain

Corollary 2.1. *The set of all elements of the form* (2.3) *is the least linear set containing all the elements of a rank not greater than A^p.*

And hence we have

THEOREM 2.5. *If $\mathfrak{J}^M$ denotes the minimal ideal of the operator-system* (1.6), *then*

$$\mathfrak{J}^M(A^p) = \Big\{ A^q \Big(\sum_{i=1}^{n} \sum_{0 \leqslant j \leqslant p_{\nu_i}+q_{\nu_i}-1} L^j_{\nu_i} x_{ij} \Big) \colon\; x_{ij} \in \boldsymbol{G}^0_{\nu_i},\; q \in \boldsymbol{N}^{\aleph_\xi} \Big\}. \tag{2.4}$$

Now we apply the result of Theorem 2.5 to an arbitrary linear operator-system $\mathfrak{A}_{\Omega,a}\boldsymbol{Y}$ contained in the operator-system (1.9), where $\boldsymbol{Y}$ is a linear subspace of $\boldsymbol{X}^\Omega$ such that $(l/a)_i \boldsymbol{Y} \subset \boldsymbol{Y}$ for $i = 1, 2, \ldots, k$. The relation $aD_i x = 0$ shows that $x \in \boldsymbol{Y}$ does not depend on variable t_i and hence

$$(l/a)_i^n x = {}^{(n)}a_i x = \{{}^{(n)}a_i(t_i)\, x(t)\}$$

where ${}^{(0)}a_i(\tau) \equiv 1$ and ${}^{(n+1)}a_i = (l/a)_i\, {}^{(n)}a_i$, i. e.

$${}^{(n)}a_i(t_i) = \int_{t_i^0}^{t_i} \frac{1}{a_i(\tau_n)} \Big(\ldots \int_{t_i^0}^{\tau_3} \frac{1}{a_i(\tau_2)} \Big(\int_{t_i^0}^{\tau_2} \frac{1}{a_i(\tau_1)}\, d\tau_1 \Big) d\tau_2 \ldots \Big) d\tau_n .$$

Thus by (2.4) the minimal ideal of $\mathfrak{A}_{\Omega,a}\boldsymbol{Y}$ has the form

$$\mathfrak{J}^M_{\Omega,a,\boldsymbol{Y}}(aD^p) = \Big\{ aD^q \Big(\sum_{i=1} \sum_{0 \leqslant j \leqslant p_i+q_i-1} {}^{(j)}a_i x_{ij} \Big) \colon\; x_{ij} \in \boldsymbol{Y} \text{ does not depend on variable } t_i,\; q \in \boldsymbol{N}^{\aleph_\xi} \Big\}. \tag{2.5}$$

In case $\boldsymbol{Y} = \boldsymbol{X}^\Omega$ we set

$$\mathfrak{J}^M_{\Omega,a} \overset{\mathrm{def}}{=} \mathfrak{J}^M_{\Omega,a,\boldsymbol{X}^\Omega}. \tag{2.5'}$$

The ideal (2.5) is an extensor. This can easily be shown by means of the functionals:

$$\varphi_\xi(aD^p, x) = (-1)^{|p|} \int \frac{(D^p \xi)\big({}^{(1)}a_1(t_1), {}^{(1)}a_2(t_2), \ldots, {}^{(1)}a_k(t_k)\big)}{a_1(t_1)\, a_2(t_2) \ldots a_k(t_k)}\, x(t)\, dt \tag{2.6}$$

where $|p| = p_1 + p_2 + \ldots + p_k$ and ξ is an infinitely derivable scalar-valued function defined on $\boldsymbol{E}^k$ and such that there is a compact subset $\Omega' \subset \Omega$ outside of which $\xi\big({}^{(1)}a_1(t_1), \ldots, {}^{(1)}a_k(t_k)\big)$ vanishes.

The proof is quite easy and is based on the following properties of the functionals (2.6)

$$\begin{aligned} &\varphi_\xi(aD^p, aD^q x) = \varphi_\xi(aD^{p+q}, x), \\ &\varphi_\xi(aD^{(0,0,\ldots,0)}, x) = 0 \text{ for each } \xi \text{ implies } x = 0. \end{aligned} \tag{2.6'}$$

We do not quote the whole proof because it will be given in a more general case in another section.

It is well known that in some particular cases the operators aD^q in (2.5) can be omitted, i. e. we can set $q = (0, \ldots, 0)$. For instance when $\boldsymbol{Y}$ coincides with the whole space $\boldsymbol{X}^\Omega$ this follows from the result of König [8] [4] and (1.9″). But in general it is not true, and this is shown by the following example. Let, in (1.9), $k = 3$, $\Omega = \boldsymbol{E}^3$, $t_i^0 = 0$, $i = 1, 2, 3$, and all $a_i(t_i) = 1$. The space $\boldsymbol{Y}$ is generated by the functions x_0, x^0_{pmn}, x^1_{pmn}, where $0 \leqslant p$, $n < +\infty$, $1 \leqslant m < +\infty$ and

$$x^0_{010}(t) = t_3 W(t_2) - W(t_3), \qquad x^1_{010}(t) = t_3 W(t_1) + W(t_3)$$

where $W(\tau)$ is a Weierstrass non-derivable function.

Further

$$l_3 x_0 = x^0_{010} + x^1_{010},$$

$$\begin{aligned} x^0_{p,m+1,n} &= l_3 x^0_{pmn}, & x^0_{p+1,m,n} &= l_1 x^0_{pmn}, & x^0_{p,m,n+1} &= l_2 x^0_{pmn}, \\ x^1_{p,m+1,n} &= l_3 x^1_{pmn}, & x^1_{p+1,m,n} &= l_2 x^1_{pmn}, & x^1_{p,m,n+1} &= l_1 x^1_{pmn} \end{aligned}$$

where we write shortly l instead of $(l/a)_i$, as it was in (1.7).

It can easily be verified that

$$x^0_{pmn} = \frac{t_1^p}{p!} \Big(\frac{t_3^m}{m!}\, {}^{(n)}W(t_2) - \frac{t_2^n}{n!}\, {}^{(m-1)}W(t_3) \Big),$$

$$x^1_{pmn} = \frac{t_2^p}{p!} \Big(\frac{t_3^m}{m!}\, {}^{(n)}W(t_1) + \frac{t_1}{n!}\, {}^{(m-1)}W(t_3) \Big),$$

where ${}^{(0)}W(s) = W(s)$ and ${}^{(n+1)}W(s) = \int_0^s {}^{(n)}W(\tau)\, d\tau$.

Clearly

$$x_0 = D_3(l_1^0 x^0_{010} + l_2^0 x^1_{010} + l_3 0)$$

where $D_1 x^0_{010} = D_2 x^1_{010} = 0$ and hence $x_0 \in \mathfrak{J}^M_{\Omega,a,\boldsymbol{Y}}(D_1 D_2)$. But $x_0(t) = W(t_1) + W(t_2)$ cannot be obtained as the sum of elements $y_1, y_2 \in \boldsymbol{Y}$ where $D_1 y_1 = D_2 y_2 = 0$. Therefore in this case aD^q cannot be omitted in (2.5).

The operator aD_1 is one-to-one in $\mathfrak{A}^1_{\Omega,a}$ (see (1.11)), and hence in virtue of Theorem 2.3 we obtain the minimal ideal of (1.11) in the form

$${}^{(1)}\mathfrak{J}^M_{\Omega,a}(aD^p) = \Big\{ \sum_{i=2}^{k} \sum_{0 \leqslant j \leqslant p_i - 1} {}^{(j)}a_i x_{ij} \colon\; x_{ij} \in \boldsymbol{X}^{\Omega,1} \text{ does not depend on the variable } t_i \Big\}. \tag{2.7}$$

Clearly (2.7) is not a null-ideal but

$${}^{(1)}\mathfrak{J}^M_{\Omega,a}(aD^{(p_1,0,\ldots,0)}) = \{0\}$$

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(8) This was noticed by S. Łojasiewicz.

for each non-negative integer p_1. This can immediately be verified by the use of the functionals (2.6).

Thus

$$ {}^{(1)}\mathfrak{I}^M_{\Omega,a} \subset \mathfrak{I}^M_{\Omega,a} \boldsymbol{X}^{\Omega,1} . \tag{2.8} $$

Indeed, setting

$$ x^* \overset{\text{def}}{=} {}^{(1)}a_1 {}^{(1)}a_2 \ldots {}^{(1)}a_k \hat{x} , \tag{2.9} $$

where $\hat{x} \neq 0$ is a fixed element of $\boldsymbol{X}$, we obtain

$$ x^* \in \mathfrak{I}^M_{\Omega,a}(aD^{(2,0,\ldots,0)}) \boldsymbol{X}^{\Omega,1} $$

and since $\hat{x} \neq 0$,

$$ x^* \notin {}^{(1)}\mathfrak{I}^M_{\Omega,a}(aD^{(2,0,\ldots,0)}) . $$

In the case of $\mathfrak{A}^0_{\Omega,a}$ we see that all the operators aD^p are one-to-one, and hence the minimal ideal of $\mathfrak{A}^0_{\Omega,a}$ is the null-ideal. Since

$$ x^* \in {}^{(1)}\mathfrak{I}^M_{\Omega,a}(aD^{(0,2,0,\ldots,0)}) \boldsymbol{X}^{\Omega,0} , $$

it follows that

$$ {}^{(0)}\mathfrak{I}^M_{\Omega,a} \subset {}^{(1)}\mathfrak{I}^M_{\Omega,a} \boldsymbol{X}^{\Omega,0} \tag{2.10} $$

where ${}^{(0)}\mathfrak{I}^M_{\Omega,a}$ is the minimal ideal of $\mathfrak{A}^0_{\Omega,a}$.

We note that in the case of the operator-systems given by (1.12) and (1.12′), where

$$ \mathfrak{A}^{\infty}_{\boldsymbol{E}^k,a} \subset \mathfrak{A}^{\infty,1}_{\boldsymbol{E}^k,a} \subset \mathfrak{A}_{\boldsymbol{E}^k,a} , $$

we have

$$ {}^{\infty}\mathfrak{I}^M_{\boldsymbol{E}^k,a} = \mathfrak{I}^M_{\boldsymbol{E}^k,a} \boldsymbol{X}^{\boldsymbol{E}^k,\infty} , \qquad {}^{\infty,1}\mathfrak{I}^M_{\boldsymbol{E}^k,a} = \mathfrak{I}^M_{\boldsymbol{E}^k,a} \boldsymbol{X}_1^{\boldsymbol{E}^k,\infty} . \tag{2.11} $$

Here ${}^{\infty}\mathfrak{I}^M_{\boldsymbol{E}^k,a}$ denotes the minimal ideal of $\mathfrak{A}^{\infty}_{\boldsymbol{E}^k,a}$ which, in this case, is the null-ideal. Similarly ${}^{\infty,1}\mathfrak{I}^M_{\boldsymbol{E}^k,a}$ is the minimal ideal of $\mathfrak{A}^{\infty,1}_{\boldsymbol{E}^k,a}$. It is not the null-ideal.

3. The Fundamental Theorem.

In this section we are going to prove a theorem similar to that concerning division of a ring by an ideal in the theory of commutative rings. This theorem, named the Fundamental Theorem, is divided into two parts. The first part concerns the construction of a quotient operator-system $\mathfrak{A}/\mathfrak{I}$ and its connexion with the operator-system $\mathfrak{A}$. The second part gives the necessary and sufficient conditions of reducing given homomorphism $\mathfrak{H}$ of $\mathfrak{A}'$ into $\mathfrak{A}''$ to a homomorphism $\mathfrak{H}^*$ of $\mathfrak{A}'/\mathfrak{I}'$ into $\mathfrak{A}''/\mathfrak{I}''$ where $\mathfrak{I}'$ and $\mathfrak{I}''$ are ideals of $\mathfrak{A}'$ and $\mathfrak{A}''$ respectively. Although the Fundamental Theorem will be proved later for arbitrary map-systems, we will show it now only for operator-systems; it is much more simple in this case. The reader is advised to follow the theorem very closely for it will be needed troughout the whole theory of map-systems and will be used later in the topological part of the theory as well. We should also note, that Theorems 1 and 2 of [8] are among most important particular cases of the Fundamental Theorem.

FUNDAMENTAL THEOREM [10]. *To every operator-system (linear operator system) $\mathfrak{A} = (\boldsymbol{S}, \boldsymbol{X})$ and every ideal $\mathfrak{I}$ of $\mathfrak{A}$ there corespond an algebraically closed operator-system (linear operator-system) $\mathfrak{A}/\mathfrak{I} = (\boldsymbol{S}, \boldsymbol{X}^*)$ named a quotient operator-system and a simple homomorphism $(\boldsymbol{h})$ of $\mathfrak{A}$ almost onto $\mathfrak{A}/\mathfrak{I}$ named a natural homomorphism such that*

(i) *$\mathfrak{I} = (\boldsymbol{h})^{-1}\mathfrak{I}^{*M}$ where $\mathfrak{I}^{*M}$ is the minimal ideal of $\mathfrak{A}/\mathfrak{I}$,*

(ii) *Let $\mathfrak{A}'$ and $\mathfrak{A}''$ be two operator-systems (linear operator-systems), let $\mathfrak{I}'$ and $\mathfrak{I}''$ be two ideals of $\mathfrak{A}'$ and $\mathfrak{A}''$ respectively, let $\mathfrak{A}'/\mathfrak{I}'$ and $\mathfrak{A}''/\mathfrak{I}''$ denote the quotient operator-systems (linear operator-systems) and $(\boldsymbol{h}')$, $(\boldsymbol{h}'')$ the corresponding natural homomorphisms.*

If $\mathfrak{H} = (\boldsymbol{H}, \boldsymbol{h})$ is a homomorphism of $\mathfrak{A}'$ into $\mathfrak{A}''$, then there is a homomorphism $\mathfrak{H}^* = (\boldsymbol{H}, \boldsymbol{h}^*)$ of $\mathfrak{A}'/\mathfrak{I}'$ into $\mathfrak{A}''/\mathfrak{I}''$ such that

$$ \mathfrak{H}^*(\boldsymbol{h}') = (\boldsymbol{h}'')\mathfrak{H} \tag{$*$} $$

if and only if

$$ \mathfrak{I}' \subset \mathfrak{H}^{-1}\mathfrak{I}'' . \tag{$**$} $$

The homomorphism $\mathfrak{H}^*$ is uniquely determined by $\mathfrak{H}$, $(\boldsymbol{h}')$, and $(\boldsymbol{h}'')$. The homomorphism $\mathfrak{H}^*$ is simple if and only if $\mathfrak{H}$ is simple; $\boldsymbol{h}^*$ is an isomorphism if and only if

$$ \mathfrak{H}^{-1}\mathfrak{I}'' = \mathfrak{I}' . \tag{$\overset{**}{*}$} $$

If $\mathfrak{H}$ maps almost onto $\mathfrak{A}''$, then $\mathfrak{H}^*$ maps onto $\mathfrak{A}''/\mathfrak{I}''$.

The second part of the Fundamental Theorem can easily be represented by the following commutative diagram:

$$ \begin{tikzcd} \mathfrak{A}' \arrow[r, "\mathfrak{H}"] \arrow[d, "(\boldsymbol{h}')"'] & \mathfrak{A}'' \arrow[d, "(\boldsymbol{h}'')"] \\ \mathfrak{A}'/\mathfrak{I}' \arrow[r, "\mathfrak{H}^*"] & \mathfrak{A}''/\mathfrak{I}'' \end{tikzcd} $$

where $\mathfrak{H}^*$ exists if and only if $\mathfrak{I}' \subset \mathfrak{H}^{-1}\mathfrak{I}''$.

Remark 3.1. If $\mathfrak{A}$ is algebraically closed and $\mathfrak{I}^M$ is the minimal ideal of $\mathfrak{A}$, then the natural homomorphism maps isomorphically $\mathfrak{A}/\mathfrak{I}$ onto $\mathfrak{A}$.

In fact, by virtue of condition (i) we have

$$ \mathfrak{I}^M = (\boldsymbol{h})^{-1}\mathfrak{I}^{*M} , $$

where $\mathfrak{I}^{*M}$ is the minimal ideal of $\mathfrak{A}/\mathfrak{I}^M$. On the other hand we have

$$\mathfrak{I}^M(A) = \{x \in r\mathfrak{A}\colon\ Ax = 0\},$$
$$\mathfrak{I}^{*M}(A) = \{\bar{x} \in r\mathfrak{A}/\mathfrak{I}\colon\ A\bar{x} = 0\}.$$

Hence $\{x\colon\ hx = 0\} = \{x\colon\ hx \in \mathfrak{I}^{*M}(I)\} = \big((h)^{-1}\mathfrak{I}^{*M}\big)(I) = \mathfrak{I}^M(I) = \{0\}$, and thus h is one-to-one.

Corollary 3.1. *Let $\mathfrak{A}$ be an operator-system (a linear operator-system) and $\mathfrak{I}$ an ideal of $\mathfrak{A}$. Let $\mathfrak{A}^*$ be an algebraically closed operator-system (a linear operator-system) and let $\mathfrak{H}$ be a homomorphism of $\mathfrak{U}$ almost onto $\mathfrak{A}^*$ such that*

$$\mathfrak{H}^{-1}\mathfrak{I}^{*M} \supset \mathfrak{I}$$

where $\mathfrak{I}^{*M}$ is the minimal ideal of $\mathfrak{A}^*$.

Then there is the uniquely determined homomorphism $\mathfrak{H}^*$ of $\mathfrak{A}/\mathfrak{I}$ onto $\mathfrak{A}^*$ such that

$$\mathfrak{H}^*(h') = \mathfrak{H},$$

where (h') is the natural homomorphism of $\mathfrak{A}$ into $\mathfrak{A}/\mathfrak{I}$; h^* is an isomorphism if and only if

$$\mathfrak{H}^{-1}\mathfrak{I}^{*M} = \mathfrak{I}.$$

Proof of Corollary 3.1. If we set $\mathfrak{A}' = \mathfrak{A}$, $\mathfrak{A}'' = \mathfrak{A}^*$, the existence of h^* follows from the second part of the Fundamental Theorem.

Remark 3.2. This is a generalization of the first isomorphism theorem for groups. Namely, let $\mathfrak{A}, \mathfrak{A}^*$ be such operator-systems that $l\mathfrak{A} = l\mathfrak{A}^*$ consists only of one element, the identity. In this case Corollary 3.1 states that if $X^* = r\mathfrak{A}^*$ is a homomorphic image of $X = r\mathfrak{A}$, then the groups $X/h^{-1}(0)$ and X^* are isomorphic. Hence, in the general case, if $\mathfrak{A}^*$ is a homomorphic almost image of $\mathfrak{A}$ (h): $\mathfrak{A} \overset{\text{almost onto}}{\to} \mathfrak{A}^*$, and if $\mathfrak{I}^M$ is the minimal ideal of $\mathfrak{A}$, then it is natural to call the ideal $(h)^{-1}\mathfrak{I}^M$ the *kernel* of the homomorphism (h).

Thus, the following formula holds (up to an isomorphism)

$$\mathfrak{A}/(h)^{-1}\mathfrak{I}^M = \mathfrak{A}^*,$$

where h is assumed to be a simple homomorphism of $\mathfrak{A}$ almost onto $\mathfrak{A}^*$.

Let $\mathfrak{A}_0$ be a fixed operator-system. We consider the pairs $\big(\mathfrak{A}, (h)\big)$ that consist of an algebraically closed operator-system $\mathfrak{A}$ and a simple homomorphism (h) of $\mathfrak{A}_0$ almost onto $\mathfrak{A}$. We say that two such pairs are equivalent, $\mathfrak{A}', (h) \sim \mathfrak{A}'', (h')$, if and only if there is a simple isomorphism (h^*) of $\mathfrak{A}'$ onto $\mathfrak{A}''$ such that

$$h^*h' = h''. \tag{ii}$$

This situation can be illustrated by the following commutative diagram

$$\begin{tikzcd} & \mathfrak{A}_0 \arrow[dl, "(h')"'] \arrow[dr, "(h'')"] & \\ \mathfrak{A}' \arrow[rr, leftrightarrow, "(h^*)"] & & \mathfrak{A}'' \end{tikzcd},$$

where $h^*h' = h''$, $h^{*-1}h'' = h'$.

Any two pairs, no matter in what way they are given, are considered to be identical if and only if they are equivalent, in the sense just defined. Now that this convention is made, each pair $\mathfrak{A}, (h)$ is uniquely determined by the kernel of the homomorphism (h), $(h)^{-1}\mathfrak{I}^M$, where $\mathfrak{I}^M$ is the minimal ideal of $\mathfrak{A}$. This is a simple consequence of Corollary 3.1.

Corollary 3.2. *If $\mathfrak{A}'$ and $\mathfrak{A}''$ are arbitrary operator-systems (linear operator-systems) and $\mathfrak{H}$ is a homomorphism of $\mathfrak{A}'$ into $\mathfrak{A}''$, then there is a homomorphism $\mathfrak{H}^*$ of $\mathfrak{A}'/\mathfrak{I}^{M'}$ into $\mathfrak{A}''/\mathfrak{I}^{M''}$ such that*

$$\mathfrak{H}^*(h') = (h'')\mathfrak{H}$$

where (h') and (h'') are the natural homomorphisms of $\mathfrak{A}'$ almost onto $\mathfrak{A}'/\mathfrak{I}^{M'}$ and $\mathfrak{A}''$ almost onto $\mathfrak{A}''/\mathfrak{I}^{M''}$.

Proof. This corollary also follows directly from the second part of the Fundamental Theorem.

Definition 3.1. $\mathfrak{A}/\mathfrak{I}$ is named an *extension* of $\mathfrak{A}$ if and only if the natural homomorphism (h) of $\mathfrak{A}$ into $\mathfrak{A}/\mathfrak{I}$ is an isomorphism. We often identify hx and x considering $\mathfrak{A} \subset \mathfrak{A}/\mathfrak{I}$.

Theorem 3.1. *$\mathfrak{A}/\mathfrak{I}$ is an extension of $\mathfrak{A}$ if and only if $\mathfrak{I}$ is an extensor.*

Proof. According to the first part of the Fundamental Theorem, $hx = 0$ is equivalent to $x \in \mathfrak{I}(I)$; the rest is evident.

Theorem 3.2. *An operator-system (a linear operator-system) $\mathfrak{A}$ admits an extension to algebraically closed operator-system (linear operator-system) if and only if $\mathfrak{A}$ is proper.*

Proof. If $\mathfrak{A}$ is proper, then in virtue of Theorem 2.4 the minimal ideal of $\mathfrak{A}$ is an extensor and then $\mathfrak{A}/\mathfrak{I}^M$ is an extension of $\mathfrak{A}$ to an algebraically closed operator-system (linear operator-system).

Conversely, if $\mathfrak{A}$ admits an extension to an algebraically closed operator-system (linear operator-system) the condition stated in Definition 2.2 holds in the extension of $\mathfrak{A}$ and hence also in $\mathfrak{A}$.

The connexion of the Fundamental Theorem with the theory of commutative rings shows the following two examples: Let R be an arbitrary commutative ring and let $\boldsymbol{S}$ be a multiplicative semigroup of elements of $\boldsymbol{R}$ which have inverses in $\boldsymbol{R}$. We set

$$\mathfrak{A} \stackrel{\text{def}}{=} (\boldsymbol{S}, \boldsymbol{R})$$

where the composition of $a \in \boldsymbol{S}$ and $b \in \boldsymbol{R}$ is the usual multiplication.

It is easy to see that if $\boldsymbol{I}$ is an ideal of $\boldsymbol{R}$, then $\mathfrak{I}$, where $\mathfrak{I}(a) = \boldsymbol{I}$ for every $a \in \boldsymbol{S}$, is an ideal of $\mathfrak{A}$ and we have

$$\mathfrak{A}/\mathfrak{I} = (\boldsymbol{S}, \boldsymbol{R}/\boldsymbol{I}).$$

If, as above, $\boldsymbol{R}$ is a commutative ring and $\boldsymbol{S}'$ is a multiplicative semigroup which consists of elements of $\boldsymbol{R}$ which are not divisors of zero, then we can set

$$\mathfrak{A}' \stackrel{\text{def}}{=} (\boldsymbol{S}', \boldsymbol{R})$$

where the compositional product of $a \in \boldsymbol{S}'$ and $b \in \boldsymbol{R}$ is b/a whenever it exists. It is easy to see that the function $\mathfrak{I}'(a) = \{0\}$ for $a \in \boldsymbol{S}'$ is an ideal of $\mathfrak{A}$. The operator-system

$$\mathfrak{A}/\mathfrak{I}' = (\boldsymbol{S}', \boldsymbol{R}^*)$$

consists of the same semigroup $\boldsymbol{S}'$ and the additive group $\boldsymbol{R}^*$ which is the least extension of the additive group of the ring $\boldsymbol{R}$ containing all the inverses of the elements of $\boldsymbol{S}'$. It is well known that $\boldsymbol{R}^*$ is also a ring with the multiplication extended from $\boldsymbol{R}$.

Now we give some applications of the Fundamental Theorem to the linear operator-systems (1.9), (1.10), (1.11), (1.12), (1.12′), (1.15), and (1.16) and their ideals (2.5′), (2.7), (2.8), (2.10). We have

$$\mathfrak{A}_{\Omega,a} \supset \mathfrak{A}^1_{\Omega,a} \supset \mathfrak{A}^0_{\Omega,a}$$

and we see that $\mathfrak{A}^1_{\Omega,a}$ is algebraically dense in $\mathfrak{A}_{\Omega,a}$ and so is $\mathfrak{A}^0_{\Omega,a}$ in $\mathfrak{A}^1_{\Omega,a}$. Hence in virtue of the Fundamental Theorem we obtain

$$\mathfrak{A}_{\Omega,a}/\mathfrak{I}^M_{\Omega,a} = \mathfrak{A}^1_{\Omega,a}/\mathfrak{I}^M_{\Omega,a}X^{\Omega,1} = \mathfrak{A}^0_{\Omega,a}/\mathfrak{I}^M_{\Omega,a}\boldsymbol{X}^{\Omega,0}, \tag{3.1}$$

$$\mathfrak{A}^1_{\Omega,a}/{}^{(0)}\mathfrak{I}^M_{\Omega,a} = \mathfrak{A}^0_{\Omega,a}/{}^{(1)}\mathfrak{I}^M_{\Omega,a}\boldsymbol{X}^{\Omega,0}. \tag{3.2}$$

If we denote by $(\boldsymbol{h}_1^*)$, $(\boldsymbol{h}_2^*)$, and $(\boldsymbol{h}_3^*)$ the extended identical imbeddings of $\mathfrak{A}^1_{\Omega,a}$ almost onto $\mathfrak{A}_{\Omega,a}$, $\mathfrak{A}^0_{\Omega,a}$ almost onto $\mathfrak{A}^1_{\Omega,a}$, and $\mathfrak{A}^0_{\Omega,a}$ almost onto $\mathfrak{A}_{\Omega,a}$ respectively, then $(\boldsymbol{h}_3^*) = (\boldsymbol{h}_1^*)(\boldsymbol{h}_2^*)$ and we have the commutative diagram

$$\begin{tikzcd} \mathfrak{A}^0_{\Omega,a}/{}^{(0)}\mathfrak{I}^M_{\Omega,a} \arrow[rr, "(\boldsymbol{h}_2^*)"] \arrow[dr, "(\boldsymbol{h}_3^*)"'] & & \mathfrak{A}^1_{\Omega,a}/{}^{(1)}\mathfrak{I}^M_{\Omega,a} \arrow[dl, "(\boldsymbol{h}_1^*)"] \\ & \mathfrak{A}_{\Omega,a}/\mathfrak{I}^M_{\Omega,a} & \end{tikzcd} \tag{3.3}$$

where all the mappings are onto.

Clearly each $\boldsymbol{h}$ identifies some points for

$${}^{(1)}\mathfrak{I}^M_{\Omega,a} \subset \mathfrak{I}^M_{\Omega,a}\boldsymbol{X}^{\Omega,1}, \qquad {}^{(0)}\mathfrak{I}^M_{\Omega,a} \subset {}^{(1)}\mathfrak{I}^M_{\Omega,a}\boldsymbol{X}^{\Omega,0}$$

and then

$${}^{(0)}\mathfrak{I}^M_{\Omega,a} \subset \mathfrak{I}^M_{\Omega,a}\boldsymbol{X}^{\Omega,0} \qquad \text{(see (2.8), (2.10))}$$

In the case of operator-systems (1.12) and (1.12′), in virtue of (2.11) and the second part the Fundamental Theorem

$$\mathfrak{A}^{\infty}_{\boldsymbol{E}^k,a}/{}^{\infty}\mathfrak{I}^M_{\boldsymbol{E}^k,a} \subset \mathfrak{A}^{\infty,1}_{\boldsymbol{E}^k,a}/{}^{\infty,1}\mathfrak{I}^M_{\boldsymbol{E}^k,a} \subset \mathfrak{A}_{\boldsymbol{E}^k,a}/\mathfrak{I}^M_{\boldsymbol{E}^k,a}. \tag{3.4}$$

Now we consider the linear operator-systems $\mathfrak{A}^P_{\boldsymbol{Q}^k,a}$ (see (1.15)) and $\mathfrak{A}^P_{\boldsymbol{T}^k,a}$ (see (1.16)). Since operators aD_i are one-to-one in these operator-systems, the minimal ideals are null-ideals. Clearly the ideal $\mathfrak{I}^M_{\boldsymbol{Q}^k,a}\boldsymbol{X}^{\boldsymbol{Q}^k,P}$ is not minimal in $\mathfrak{A}^P_{\boldsymbol{Q}^k,a}$. Indeed, the function

$$x_1^* = (I-P_1)(I-P_2)\ldots(I-P_k)x^*,$$

where x^* is defined in (2.9), belongs to $\mathfrak{I}^M_{\boldsymbol{Q}^k,a}(aD^{(2,0,\ldots,0)})\boldsymbol{X}^{\boldsymbol{Q}^k,P}$ and does not vanish identically. Similarly the ideal $\mathfrak{I}^M_{\boldsymbol{Q}^k,a}\boldsymbol{X}^{\boldsymbol{T}^k,P}$ is not minimal, because

$$L_1L_2\ldots L_kx_1^* \in \mathfrak{I}^M_{\boldsymbol{Q}^k,a}(aD^{(3,0,\ldots,0)})\boldsymbol{X}^{\boldsymbol{T}^k,P},$$

and does not vanish identically.

Now let us examine the case of inclusion

$$\mathfrak{A}^P_{\boldsymbol{T}^k,a} \subset \mathfrak{A}_{\boldsymbol{T}^k,a} \subset \mathfrak{A}_{\boldsymbol{E}^k,a}$$

where $\mathfrak{I}^M_{\boldsymbol{E}^k,a}X^{\boldsymbol{T}^k,P}$ is the null-ideal (⁹). Then, in virtue of the Fundamental Theorem, the extension of identical imbedding of $\mathfrak{A}_{\boldsymbol{T}^k,a}$ into $\mathfrak{A}_{\boldsymbol{E}^k,a}$ is one-to-one. Therefore, there is an intrinsic extension of $\mathfrak{A}^P_{\boldsymbol{T}^k,a}/{}^P\mathfrak{I}^M_{\boldsymbol{T}^k,a}$ in $\mathfrak{A}_{\boldsymbol{E}^k,a}/\mathfrak{I}^M_{\boldsymbol{E}^k,a}$ and we can set

$$\operatorname{ext}\mathfrak{A}_{\boldsymbol{T}^k,a} \stackrel{\text{def}}{=} \text{algebraic closure of } \mathfrak{A}_{\boldsymbol{T}^k,a} \text{ in } \mathfrak{A}_{\boldsymbol{E}^k,a}/\mathfrak{I}^M_{\boldsymbol{E}^k,a}. \tag{3.5}$$

Since $\mathfrak{I}^M_{\boldsymbol{E}^k,a}X^{\boldsymbol{T}^k,P}$ is the null-ideal, $\operatorname{ext}\mathfrak{A}_{\boldsymbol{T}^k,a}$ is the minimal extension of the linear space of periodic functions to that of distributions.

Now we must establish correspondence between the classical theories and our results. At first we translate the distribution theory into the operator-system language.

(⁹) On the basis of the result of König [4] S. Łojasiewicz proved that this is also true in the general case.

Let $\boldsymbol{D}_k$ be the space of k-dimensional distributions of finite order. From Schwartz' point of view $\boldsymbol{D}_k$ consists of all the functionals

$$T_{p,x}(\xi) = (-1)^{|p|}\int (D^p\xi)(\tau)x(\tau)\,d\tau\,,$$

$$p = (p_1, p_2, \ldots, p_k), \qquad |p| = p_1 + p_2 + \ldots + p_k\,,$$

where ξ runs over the space of infinitely derivable functions with compact carriers. The operation

$$(*) \qquad D^p T_{q,x} = T_{p+q,x}$$

leads to an algebraically closed linear operator-system $\mathfrak{D}_k$ that consists of the semigroup $\{D_i\}$ of differential operators D^p and the linear space of distributions $\boldsymbol{D}_k$ with composition given by $(*)$. Let $\boldsymbol{X}$ be the complex plane and let $a_i(t)$ be the functions of real argument which are equal to 1 identically, $a_i(t) = 1$ for all t and $i \leqslant k$. Let as in (1.9)

$$\mathfrak{A}_{\boldsymbol{E}^k} \overset{\text{def}}{=} \mathfrak{A}_{\boldsymbol{E}^k,\{1\}} = (\{D_i\}, \boldsymbol{X}^{\boldsymbol{E}^k})\,.$$

The transformation $\boldsymbol{h}(D^p x) = T_{p,x}$, where $D^p x \in r(\mathfrak{A}_{\boldsymbol{E}^k}/\mathfrak{J}^M_{\boldsymbol{E}^k})$, induces a simple homomorphism of $\mathfrak{A}_{\boldsymbol{E}^k}/\mathfrak{J}^M_{\boldsymbol{E}^k}$ onto $\mathfrak{D}_k$. By virtue of a result of König [4] the kernel of $(\boldsymbol{h})$ is the minimal-ideal of $\mathfrak{A}_{\boldsymbol{E}^k}/\mathfrak{J}^M_{\boldsymbol{E}^k}$ and thus $(\boldsymbol{h})$ is an isomorphism. Therefore we can identify $\mathfrak{A}_{\boldsymbol{E}^k}/\mathfrak{J}^M_{\boldsymbol{E}^k}$ with $\mathfrak{D}_k$ and we shall name each of the operator-systems $\mathfrak{A}_{\Omega,a}/\mathfrak{J}^M_{\Omega,a}$, where $\boldsymbol{X}$ is an arbitrary linear space, a *distributional operator-system*.

As the next step we show that, if $\boldsymbol{X}$ is the complex plane, then for $a_1(t) = 1$ and $k = 1$, $r(\mathfrak{A}^\infty_{\boldsymbol{E}^k}/{}^\infty\mathfrak{J}^M_{\boldsymbol{E}^k})$ (we omit the index a, see (1.12)) has a non-void common part with the field of operators of Mikusiński [5]. This common part consists of all the operators of the form $a/h_\lambda l^{n-1}$, where a is an arbitrary complex valued continuous function on $\langle 0, +\infty)$, $l = \{1\}$, i. e. l is the function which is equal to 1 identically, and

$$h_\lambda(t) = \begin{cases} 1 & \text{for} \quad t \geqslant \lambda\,, \\ 0 & \text{for} \quad t < \lambda\,. \end{cases}$$

The quotient la/h_λ can be interpreted as a translation of a, i. e. the function

$$b(t) = \begin{cases} 0 & \text{for} \quad t < -\lambda\,, \\ a(t+\lambda) & \text{for} \quad t \geqslant -\lambda\,. \end{cases}$$

Let $\boldsymbol{D}^\infty$ be the linear space that consists of all the quotients of the form b/l, where b is a continuous function on $(-\infty, +\infty)$ with a carrier bounded from the left.

We denote by $\mathfrak{C}$ the algebraically closed linear map-system that consists of the whole field of Mikusiński's operators considered as a linear space $\boldsymbol{C}$ and the semigroup of all the operators of the form $1/l^n \in \boldsymbol{C}$. The usual multiplication in $\boldsymbol{C}$, i. e. that generated by the convolution, is taken as the composition.

Let

$$\boldsymbol{h}(x/l^n) = D^n x$$

be a mapping where x is a continuous function with a carrier bounded from the left, and $x/l^n \in \boldsymbol{D}^\infty$. This mapping induces a simple isomorphism of $\mathfrak{C}\boldsymbol{D}^\infty$ into $\mathfrak{A}^\infty_{\boldsymbol{E}}/{}^\infty\mathfrak{J}^M_{\boldsymbol{E}}$. Since $\mathfrak{A}_{\boldsymbol{E}}/\mathfrak{J}^M_{\boldsymbol{E}} \supset \mathfrak{A}^\infty_{\boldsymbol{E}}/{}^\infty\mathfrak{J}^M_{\boldsymbol{E}}$, Mikusiński's operators of the form b/l^n, where the carrier of b is bounded from the left, can be treated as one-dimensional distributions with carriers bounded from the left. The chief difference between the linear space of Schwartz distributions and the field of Mikusiński's operators is that the operator-system determined by the latter has a one-to-one composition, whereas the former generates an operator-system with the composition which is of the many-to-one type. Therefore, in the general case, the operator-systems $\mathfrak{A}^0_{\boldsymbol{E},a}/{}^{(0)}\mathfrak{J}^M_{\boldsymbol{E},a}$ and $\mathfrak{A}^\infty_{\boldsymbol{E}^k,a}/{}^\infty\mathfrak{J}^M_{\boldsymbol{E}^k,a}$ are called *operational* (see (3.1)-(3.4)).

The operational operator-systems $\mathfrak{A}^0_{\boldsymbol{E}^k,a}/{}^{(0)}\mathfrak{J}^M_{\boldsymbol{E}^k,a}$ and $\mathfrak{A}^\infty_{\boldsymbol{E}^k,a}/{}^\infty\mathfrak{J}^M_{\boldsymbol{E}^k,a}$ are very similar. However they are quite different in relation to $\mathfrak{A}_{\boldsymbol{E}^k,a}/\mathfrak{J}^M_{\boldsymbol{E}^k,a}$. In fact, the simple homomorphism $(\boldsymbol{h}^*_3)$ defined in (3.3) does not imbed $\mathfrak{A}^0_{\boldsymbol{E}^k,a}/{}^{(0)}\mathfrak{J}^M_{\boldsymbol{E}^k,a}$ into $\mathfrak{A}_{\boldsymbol{E}^n,a}/\mathfrak{J}^M_{\boldsymbol{E}^k,a}$ isomorphically, while on the other hand $\mathfrak{A}_{\boldsymbol{E}^k,a}/\mathfrak{J}^M_{\boldsymbol{E}^k,a}$ contains $\mathfrak{A}^\infty_{\boldsymbol{E}^k,a}/{}^\infty\mathfrak{J}^1_{\boldsymbol{E}^k,a}$ as a proper subsystem. The periodic distributions can be obtained in the way determined by (3.5) for $a_i(t) = 1$. The extension (3.5) is the minimal extension and, as will be shown later on in the section on general map-systems, the operator-system yielded by (3.5) can be obtained as a quotient map-system of $\mathfrak{A}_{\boldsymbol{T}^k,a}$ and its minimal ideal.

Now let us consider the operator-systems $\mathfrak{A}^1_{\Omega,a}/{}^{(1)}\mathfrak{J}^M_{\Omega,a}$ and $\mathfrak{A}^{\infty,1}_{\boldsymbol{E}^k,a}/{}^{\infty,1}\mathfrak{J}^M_{\boldsymbol{E}^k,a}$. It can easily be seen that the elements of $r(\mathfrak{A}^1_{\Omega,a}/{}^{(1)}\mathfrak{J}^M_{\Omega,a})$ as well as $r(\mathfrak{A}^{\infty,1}_{\boldsymbol{E}^k,a}/{}^{\infty,1}\mathfrak{J}^M_{\boldsymbol{E}^k,a})$ are either like Schwartz distributions or like Mikusiński's operators, according to the variable chosen. That is why we shall call them briefly the *mixed operator-systems*.

Proof of the Fundamental Theorem. We notice at first that every linear operator system can be considered as an ordinary operator-system with an extended semigroup of operators by joining the operators of multiplication by scalars different from zero. The Fundamental Theorem proved for such an operator-system can immediately be transposed on the corresponding linear operator-system.

Hence it is sufficient to prove the Fundamental Theorem only for an ordinary operator-system.

Let $\mathfrak{A}=(\boldsymbol{S},\boldsymbol{X})$ be an operator-system and $\mathfrak{I}$ an ideal of $\mathfrak{A}$. We introduce in the set-theoretical product $\boldsymbol{S}\times\boldsymbol{X}$ a relation of equivalence as follows:

$$(A,x)\equiv(B,y)$$

if and only if there are $A',B'\in\boldsymbol{S}$ and $u,v\in\boldsymbol{X}$ such that

$$x=A'u,\quad y=B'v,\quad C\overset{\text{def}}{=}AA'=BB',$$
$$u-v\in\mathfrak{I}(C).$$

Let $(A,x)\equiv(B;y)$ and $(B,y)\equiv(C,z)$. Then there are $A',B',B'',C'\in\boldsymbol{S}$ and $u,v,w,s\in\boldsymbol{X}$ such that

$$x=A'u,\quad y=B'v,\quad D\overset{\text{def}}{=}AA'=BB',$$
$$u-v\in\mathfrak{I}(D)$$

and

$$y=B''w,\quad z=C's,\quad E\overset{\text{def}}{=}BB''=CC',$$
$$w-s\in\mathfrak{I}(E).$$

Let

$$x=(A'E)u',\quad y=(B''D)w',$$
$$y=(B'E)v',\quad z=(C'D)s'.$$

Then

$$\begin{aligned}A(Eu'-u)&=0,\\ B'(Ev'-v)&=0,\\ B''(Dw'-w)&=0,\\ C'(Ds'-s)&=0,\end{aligned}\qquad\text{and hence}\qquad\begin{aligned}Eu'-u&\in\mathfrak{I}(A')\subset\mathfrak{I}(D),\\ Ev'-v&\in\mathfrak{I}(B')\subset\mathfrak{I}(D),\\ Dw'-w&\in\mathfrak{I}(B'')\subset\mathfrak{I}(E),\\ Ds'-s&\in\mathfrak{I}(C')\subset\mathfrak{I}(E),\end{aligned}$$

and further

$$E(u'-v')-(u-v)\in\mathfrak{I}(D),\quad D(w'-s')-(w-s)\in\mathfrak{I}(E).$$

Thus

$$u'-v'\in\mathfrak{I}(ED),\quad w'-s'\in\mathfrak{I}(ED)$$

and hence

$$(u'-s')+(w'-v')\in\mathfrak{I}(ED).$$

Since $(B''D)w'=(B'E)v$ and $B''D=B'E=BB'B''$, we have $BB'B''(w'-v')=0$ and $w'-v'\in\mathfrak{I}(BB'B'')\subset\mathfrak{I}(ED)$.

Therefore we obtain

$$u'-s'\in\mathfrak{I}(ED).$$

Since

$$x=(A'E)u'\quad\text{and}\quad z=(C'D)s'$$

and $A(A'E)=C(C'D)=ED$, we finally have

$$(A,x)\equiv(C,z).$$

And so the relation $\equiv$ is transitive. Clearly it is also reflexive and symmetric.

LEMMA. 1. *If* $(C,x)\equiv(C',x')$ *and* $(c,y)\equiv(C',y')$ *then* $(C,x+y)\equiv(C',x'+y')$.

2. *If* $(C,z+x)\equiv(C,z+y)$, *then* $(C,x)\equiv(C,y)$.

Proof. 1. There are $C'',C''',D'',D'''\in\boldsymbol{S}$ and $u,u',v,v'\in\boldsymbol{X}$ such that

$$\begin{aligned}x&=C''u, & A&\overset{\text{def}}{=}CC''=C'C''',\\ x'&=C'''u', & u-u'&\in\mathfrak{I}(A),\\ y&=D''v, & B&\overset{\text{def}}{=}CD''=C'D''',\\ y'&=D'''v', & v-v'&\in\mathfrak{I}(B).\end{aligned}$$

We set

$$x=(BC'')w,\quad y=(AD'')s,$$
$$x'=(BC''')w',\quad y'=(AD''')s'.$$

Then

$$C''(Bw-u)=0,\quad C'''(Bw'-u')=0,$$
$$D''(As-v)=0,\quad D'''(As'-v')=0,$$

and thus

$$\begin{aligned}Bw-u&\in\mathfrak{I}(C'')\subset\mathfrak{I}(A),\\ Bw'-u'&\in\mathfrak{I}(C''')\subset\mathfrak{I}(A),\\ As-v&\in\mathfrak{I}(D'')\subset\mathfrak{I}(B),\\ As'-v'&\in\mathfrak{I}(D''')\subset\mathfrak{I}(B),\end{aligned}$$

which yields

$$B(w-w')-(u-u')\in\mathfrak{I}(A),$$
$$A(s-s')-(v-v')\in\mathfrak{I}(B).$$

Since $u-u'\in\mathfrak{I}(A)$ and $v-v'\in\mathfrak{I}(B)$, it follows that

$$w-w'\in\mathfrak{I}(AB),\quad s-s'\in\mathfrak{I}(AB).$$

Hence

$$w+s-(w'+s')\in\mathfrak{I}(AB).$$

If $E\overset{\text{def}}{=}BC''=AD''$ and $E'\overset{\text{def}}{=}BC'''=AD'''$ then

$$x+y=E(w+s),$$
$$x'+y'=E'(w'+s'),$$

and $(w+s)-(w'+s')\in\mathfrak{I}(AB)$ where $AB=EC=E'C'$. Hence $(C,x+y)\equiv(C',x'+y')$.

2. If $(C, z+x) \equiv (C, z+y)$, then there are $A, B \in S$ and $u, v \in X$ such that

$$z+x = Au, \quad D \overset{\text{def}}{=} AC = BC,$$
$$z+y = Bv, \quad u-v \in \mathfrak{I}(D).$$

Let $Au = Du'$ and $Bv = Dv'$. Then $A(u-Cu')=0$ and $B(v-Cv')=0$, which implies

$$u-Cu' \in \mathfrak{I}(A) \subset \mathfrak{I}(D) \quad\text{and}\quad v-Cv' \in \mathfrak{I}(B)\subset\mathfrak{I}(D).$$

Hence $u-v-C(u'-v') \in \mathfrak{I}(D)$ and since $u-v\in\mathfrak{I}(D)$, it follows that

$$C(u'-v')\in\mathfrak{I}(D) \quad\text{and}\quad u'-v'\in\mathfrak{I}(CD).$$

Since $z+x=Du'$ and $z+y=Dv'$, it follows that $x-y=D(u'-v')$ and hence $x-y\in\mathfrak{I}(C)$, which is equivalent to $(C,x)\equiv(C,y)$. And thus the proof of the lemma is completed.

Let X^* denote the set of classes of $S\times X$ modulo $\equiv$. We denote by $\overline{Ax}$ the class containing the element $(A,x)\in S\times X$. We introduce in X^* the operation of addition as follows. Let $\overline{Ax}, \overline{Bx}\in X^*$. There are $C\in S$ and $u,v\in X$ such that $\overline{Ax}=\overline{Cu}$ and $\overline{By}=\overline{Cv}$. We set

$$\overline{Ax}+\overline{Ay}=\overline{C(u+v)}.$$

By the first part of the lemma this definition is unambiguous. Clearly it is commutative and associative. We prove that the cancellation law holds. Let

$$\overline{Ax}+\overline{Cz}=\overline{By}+\overline{Cz}.$$

We choose $D\in S$ and $u,v,w\in X$ in such a way that $\overline{Ax}=\overline{Du}$, $\overline{By}=\overline{Dv}$ and $\overline{Cz}=\overline{Dw}$. Then

$$\overline{C(u+w)}=\overline{C(v+w)}$$

and by the second part of the lemma we obtain $\overline{Cu}=\overline{Cv}$ or $\overline{Ax}=\overline{By}$.

The equation $\overline{Ax}+\bar z=\overline{By}$ has a solution for each $\overline{Ax},\overline{By}\in X^*$. In fact, there are $C\in S$ and $u,v\in X$ such that $\overline{Ax}=\overline{Cu}$ and $\overline{By}=\overline{Cv}$. Then $\bar z=\overline{C(v-u)}$. Hence X^* is a group.

We introduce in $\mathfrak{A}/\mathfrak{I}\overset{\text{def}}{=}(S,X^*)$ the operation of composition as follows:

$$A\bar z=\overline{(AB)x}$$

where $A\in S$ and $\bar z=\overline{Bx}\in X^*$ and $x\in X$. This definition is unambiguous, and

$$A(\bar z_1+\bar z_2)=A\bar z_1+A\bar z_2,$$
$$(AB)\bar z_1=A(B\bar z_1)\quad\text{for}\quad A,B\in S\quad\text{and}\quad \bar z_1,\bar z_2\in X^*.$$

Hence $\mathfrak{A}/\mathfrak{I}$ is an algebraically closed operator-system.

The natural homomorphism of $\mathfrak{A}$ into $\mathfrak{A}/\mathfrak{I}$ is defined as follows:

$$hx=\overline{Ix}\quad\text{for}\quad x\in X.$$

The function $\mathfrak{I}^{*M}$ where $\mathfrak{I}^{*M}(A)=\{\bar x\colon A\bar x=0\}$ is the minimal ideal of $\mathfrak{A}/\mathfrak{I}$

$$\mathfrak{H}^{-1}\mathfrak{I}^{*M}=\{x\colon hx\in\mathfrak{I}^M(A)\}=\{x\colon \overline{Ix}\in\mathfrak{I}^{*M}(A)\}=\{x\colon x\in\mathfrak{I}(\mathfrak{A})\}=\mathfrak{I}(A).$$

Thus the first part of the Fundamental Theorem is proved. Now we are going to prove the second part. Let us assume that the condition of the second part of the Fundamental Theorem is fulfilled, i. e. that $\mathfrak{I}'\subset\mathfrak{H}^{-1}\mathfrak{I}''$. If

$$A(h'x)=B(h'y)$$

for $A,B\in S'$ and $x,y\in r\mathfrak{A}'$, then

$$(AB)h'(u-v)=0$$

where $x=Bu$, $y=Av$, and $u-v\in\mathfrak{I}'(AB)\subset(\mathfrak{H}^{-1}\mathfrak{I})(AB)$. Hence

$$h(u-v)\in\mathfrak{I}''\big(H(AB)\big)$$

or

$$H(AB)\big(h''h(u-v)\big)=0.$$

Thus

$$(HA)(HB)(h''hu)=(HA)(HB)(h''hv)$$

and since $x=Bu$ and $y=Av$,

$$(HA)(h''hx)=(HB)(h''hy).$$

Then we can define h as follows:

$$h^*\bar z=(HA)(h''hx)\in r\mathfrak{A}''/\mathfrak{I}'',\quad\text{where}\quad \bar z=A(h'x)\in r\mathfrak{A}'/\mathfrak{I}'.$$

Since h' is almost onto, h^* is defined on the whole $r\mathfrak{A}'/\mathfrak{I}'$. If $\bar z_1,\bar z_2\in r\mathfrak{A}'/\mathfrak{I}'$ then there are $C\in l\mathfrak{A}'$ and $x_1,x_2\in r\mathfrak{A}'$ such that

$$\bar z_1=C(h'x_1)\quad\text{and}\quad \bar z_2=C(h'x_2)$$

and then

$$h^*(\bar z_1+\bar z_2)=(HC)(h''hx_1)+(HC)(h''hx_2)=h^*\bar z_1+h^*\bar z_2.$$

Thus h^* is a homomorphism. We set

$$\mathfrak{H}^*\overset{\text{def}}{=}(H^*,h^*),$$

where $H^*A\overset{\text{def}}{=}HA$ for $A\in S'$.

Clearly $\mathfrak{H}^*$ is simple if and only if $\mathfrak{H}$ is simple. If $A\in l(\mathfrak{A}'/\mathfrak{I}')=l\mathfrak{A}'$ [10] and $\bar z\in r(\mathfrak{A}'/\mathfrak{I}')$, then there are $B\in l\mathfrak{A}'$ and $x\in r\mathfrak{A}'$ such that $\bar z=Bh'x$.

[10] In the sequel we will always write $l\mathfrak{A}'$ instead of $l(\mathfrak{A}'/\mathfrak{I}')$.

Hence

$$h^*(A\bar{z}) = H(AB)(h''hx) = (HA)(HB)(h''hx) = (H^*A)(HB)(h''hx) = (H^*A)(h^*\bar{z}).$$

Therefore $\mathfrak{H}^*$ is a homomorphism of $\mathfrak{A}'/\mathfrak{J}'$ into $\mathfrak{A}''/\mathfrak{J}''$.

Clearly $\mathfrak{H}^*$ satisfies condition (**).

Conversely, let $\mathfrak{H}^*$ be an arbitrary homomorphism of $\mathfrak{A}'/\mathfrak{J}'$ into $\mathfrak{A}''/\mathfrak{J}''$ satisfying the condition (**) of the second part of the Fundamental Theorem.

Clearly $H^*A = HA$ for each $A \in l\mathfrak{A}'$. If $x \in \mathfrak{J}'(A)$, then $A(h'x) = 0$. Hence

$$0 = h^*A(h'x) = (H^*A)(h^*h'x) = (HA)(h^*h'x) = (HA)(h''hx)$$

and then

$$hx \in \mathfrak{J}(HA) \quad \text{ or } \quad x \in (\mathfrak{H}^{-1}\mathfrak{J})(A).$$

And thus (*) holds.

Now let $\mathfrak{H}^* = (H, h^*)$ and $\mathfrak{H}_1^* = (H, h_1^*)$ be homomorphisms which both satisfy the condition (**). Then for arbitrary $x \in r\mathfrak{A}'$ we have

$$h^*h'x = h_1^*h'x.$$

Since for each $\bar{z} \in r\mathfrak{A}'/\mathfrak{J}'$ there are $A \in l\mathfrak{A}'$ and $x \in r\mathfrak{A}'$ such that $\bar{z} = A(h'x)$, we have

$$h^*\bar{z} = h^*A(h'x) = (HA)(h^*h'x) = (HA)(h_1^*h'x) = h_1^*A(h'x) = h_1^*\bar{z}.$$

Hence $\mathfrak{H}^* = \mathfrak{H}_1^*$.

If $\mathfrak{H}$ maps almost onto, then $\mathfrak{H}^*$ maps $\mathfrak{A}'/\mathfrak{J}'$ onto $\mathfrak{A}''/\mathfrak{J}''$. In fact let

$$\bar{z} = A''(h''x'')$$

be an arbitrary element of $\mathfrak{A}''/\mathfrak{J}''$. If $\mathfrak{H}$ maps almost onto $\mathfrak{A}''$, then there are $A', B' \in l\mathfrak{A}'$ and $y' \in r\mathfrak{A}'$ such that $A'' = HA'$ and

$$x'' = (HB')(hy').$$

Then

$$\bar{z} = (HA')h''\big((HB')(hy')\big) = \big(H(A'B')\big)(h''hy') = \big(H(A'B')\big)(h^*h'y') = h^*\Big(\big(H(A'B')\big)(h'y')\Big),$$

and so $\mathfrak{H}^*$ is a mapping onto.

Finally, we notice that $\mathfrak{J}' \subset \mathfrak{H}^{-1}\mathfrak{J}''$ if and only if h^* is one-to-one. In fact, let $\bar{z} \in r\mathfrak{A}'/\mathfrak{J}'$. Then there are $A \in l\mathfrak{A}'$ and $x \in r\mathfrak{A}'$ such that $\bar{z} = A(h'x)$.

$h^*\bar{z} = 0$ is equivalent to $(HA)(h^*h'x) = 0$ or $(HA)(h''hx) = 0$, or $hx \in \mathfrak{J}''(HA)$, or $x \in (\mathfrak{H}^{-1}\mathfrak{J}'')(A)$. On the other hand

$$\bar{z} = 0 \quad \text{if and only if} \quad x \in \mathfrak{J}'(A).$$

Therefore $h^*\bar{z} = 0$ implies $\bar{z} = 0$ if and only if $x \in (\mathfrak{H}^{-1}\mathfrak{J}'')(A)$ implies $x \in \mathfrak{J}'(A)$ or, which is equivalent, if and only if $(\mathfrak{H}^{-1}\mathfrak{J}'')(A) \subset \mathfrak{J}'(A)$.

This completes the proof of the Fundamental Theorem.

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