On a theorem of Urbanik

by

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1. Suppose that we have in a space E a countably additive field M of certain subsets of E (including $\emptyset$ and E), and n countably additive finite measures $\mu_1, \mu_2, \ldots, \mu_n$ defined for each of the subsets of M. These measures are assumed to have the following properties:

(i) They are non-atomic, i. e. for each $i = 1, 2, \ldots n$, and for each $X \in M$, the inequality $\mu_i(X) > 0$ implies the existence of a $Y \subset X$ such that $\mu_i(X) > \mu_i(Y) > 0$.

(ii) They are non-proportional, i. e. there exists at least one pair of indices i and j and a set $X \in M$, such that $\mu_i(X)/\mu_i(E) \neq \mu_j(X)/\mu_j(E)$.

2. Then, utilising a theorem of Urbanik [2], it is easy to prove (1), provided that the μ_i are non-negative:

THEOREM 1. *The assumptions* (i) *and* (ii) *imply the existence of a partition* $E = E_1 \cup E_2 \cup \ldots \cup E_n$ *such that* $\mu_i(E_1) \geqslant \mu_i(E)/n$ *for each* $i = 1, 2, \ldots, n$, *with strict inequality holding for at least one* i.

Remark. Unless (ii) is satisfied, normalisation of the measures to make $\mu_i(E) = 1$ for each i, will make each measure identical. The most that can then be stated is that a partition exists such that $\mu_i(E_i) = \mu_i(E)/n$ for each i.

3. Urbanik [2] actually proved the stronger result, that given (i), (ii) and

(iii) The family of sets of measure zero is the same for each μ_i, then we have

THEOREM 2. *The assumptions* (i)-(iii) *imply the existence of a partition* $E = E_1 \cup E_2 \cup \ldots \cup E_n$ *such that* $\mu_i(E_i) > \mu_i(E)/n$ *for each* $i = 1, 2, \ldots, n$.

This result has application to the quasi-economic problem of fair division, proposed by Steinhaus [1].

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(1) The use of the Radon-Nikodym theorem does not require condition (iii), provided that the measures are non-negative.

4. The assumption that the functions μ_i are measures is extremely restrictive from an economic point of view, since their additivity property is very special. On economic grounds, it is desirable to replace the measures μ_i by more general sub-additive finite set functions v_i, which are assumed to posses a property analogous to (i), and the following additional properties:

(iv) $v_i(X) \geqslant 0$ for any $X \in M$, and for each $i = 1, 2, \ldots, n$.

(v) For any two subsets X and Y in M such that $X \subset Y$, then

$$v_i(X) \leqslant v_i(Y) \quad \text{for each} \quad i = 1, 2, \ldots, n,$$

with *strict* inequality if $v_i(Y-X) > 0$.

(vi) For any two subsets X and Y in M

$$v_i(X \cup Y) \leqslant v_i(X) + v_i(Y) \quad \text{for each} \quad i = 1, 2, \ldots, n$$

with *strict* inequality if either $v_i(X)$ or $v_i(Y)$ is non-zero.

5. It is impossible to prove Theorem 1 for the new functions v_i using Urbanik's method, since this makes essential use, *via* the Radon-Nikodym theorem, of the additivity property of the measures. It is possible to give a simple constructive proof of Theorem 1, however, for the new conditions. We utilise Steinhaus's picturesque but accurate description of the problem as that of dividing a cake among n persons. The "cake" represents the space E and the set functions v_i are the valuations which the persons place upon the cake, i. e. $v_i(X)$ is the value to the ith person of the portion X of the cake. The cake is then divided among the n persons by a step by step process as follows:

The first person cuts from the cake a "slice" which he values at $v_1(E)/n$ (assuming (i)) and passes the slice to the second person. If the v_2 value of the slice is not greater than $v_2(E)/n$, he passes it on untouched to the third person. If such is not the case, he diminishes the slice to a smaller slice of v_2 value equal to $v_2(E)/n$ (restoring the "crumbs" so removed to the original cake), and *then* passes it on to the third person. A similar process is adopted for the 3rd, 4th, ..., nth person. After the slice has been passed around to each of the n persons, it is awarded to the last person who has diminished it; if there is no such person, it is awarded to the person who originally cut the slice. The process then recommences with the remainder of the cake and $(n-1)$ persons, treating it as an entirely new cake; and so on for $(n-2)$, $(n-3)$, ... When there are only two persons left, say i and j, person i cuts off a piece equal to one half the v_i value of the remaining cake, and j selects that one of the two pieces which he values more highly, leaving i to take the remaining piece. We shall show that such a process leads to a partition of the desired type.

6. THEOREM 3. *The assumptions* (i), (iv)-(vi) *imply the existence of a partition* $E = E_1 \cup E_2 \cup \ldots \cup E_n$ *such that*

$$v_i(E_i) \geqslant v_i(E)/n \quad \textit{for each} \quad i = 1, 2, \ldots, n$$

with strict inequality holding for at least one i.

Proof. Without loss of generality, we may normalise the v_i so that $v_i(E) = 1$ for each i. If assumption (ii) is false, then this normalisation will make each v_i identical to a common valuation, say v. We may then by (i), divide the cake into n pieces $E_1, E_2, \ldots, E_n$, such that $v(E_1) = v(E_2) = \ldots = v(E_n)$ and $v(E_i) > 1/n$, by (vi). Thus Theorem 3 is true in this case. If (ii) is true, then we proceed by an inductive proof, first proving it true for $n = 2$.

Since v_1 is non-atomic, person 1 may select a set X such that

$$v_1(X) = \tfrac{1}{2}.$$

From (vi)

$$v_1(E-X) > v_1(X) > \tfrac{1}{2}.$$

Suppose now that

(a) $$v_2(X) \leqslant \tfrac{1}{2}.$$

From (vi)

$$v_2(E-X) > \tfrac{1}{2}.$$

Alternatively suppose that

(b) $$v_2(X) > \tfrac{1}{2}.$$

In either (a) or (b) the partition $E = X \cup (E-X)$ is a partition satisfying Theorem 3.

7. Now consider the case of general n. Let person 1 cut a piece X of v_1 value $1/n$. Let this be reduced — if possible — to v_2 value $1/n$ and so on. Let the kth person receive the slice, and then renumber the persons involved so that he becomes person number 1. Then

$$v_1(X) = 1/n$$

and

$$v_i(X) \leqslant 1/n \quad \text{for} \quad i = 2, 3, \ldots, n.$$

From (vi)

(1) $$v_i(E-X) > (n-1)/n, \quad i = 2, 3, \ldots, n.$$

Now define $F \equiv E-X$, $M_1 \equiv M \cap F$, and w_i by

(2) $$w_i(Y) \equiv v_i(Y)/v_i(E-X), \quad i = 2, 3, \ldots, n.$$

Then each w_i satisfies our assumptions, and may be normalised. By the inductive hypothesis, there exists a partition $F = Y_2 \cup Y_3 \cup \ldots \cup Y_n$ such that

(3) $$w_i(Y_i) \geqslant 1/(n-1), \quad i = 2, 3, \ldots, n$$

with *strict* inequality holding for least one i. From (2),

$$v_i(Y_i) = w_i(Y_i)v_i(E-X), \quad i = 2, 3, \dots, n.$$

From (1) and (3),

$$v_i(Y_i) \geqslant \frac{1}{n-1}\cdot\frac{n-1}{n} = \frac{1}{n}, \quad i = 2, 3, \dots, n.$$

Therefore the partition $E = X\cup Y_2\cup Y_3\cup\dots\cup Y_n$ is a partition satisfying Theorem 3. Since the theorem is true for $n = 2$, it is proved.

8. By adding an assumption corresponding to (iii), viz.

(iii)′ For each set $X \in M$ such that $v_i(X) = 0$ for one i, then $v_j(X) = 0$, for $j = 1, 2, \dots, n$ $(i \neq j)$

we obtain

THEOREM 4. *The assumptions* (i), (iii)′, (iv)-(vi) *imply the existence of a partition* $E = E_1\cup E_2\cup\dots\cup E_n$ *such that*

$$v_i(E_i) > v_i(E)/n \quad \textit{for each} \quad i = 1, 2, \dots, n.$$

Proof. From Theorem 3; (a) if (ii) is false we have Theorem 4 directly, or (b) if (ii) is true, there will exist at least one k such that $v_k(E_k) > 1/n$. From (i) we may find a set $F_k \subset E_k$ such that $v_k(F_k) = 1/n$. Let $v_k(E_k - F_k) = v_k(G_k) = \delta > 0$. By (i) again, we may find an n-fold partition G_{kj} of G_k such that $v_k(G_{kj}) = \delta/n > 0$.

By (iii)′ none of these sets G_{kj} can be of zero value to any of the other "persons". Hence, by (v), the partition

$$E = (E_1\cup G_{k1})\cup(E_2\cup G_{k2})\cup\dots\cup(F_k\cup G_{kk})\cup\dots\cup(E_n\cup G_{kn})$$

is a partition satisfying Theorem 4.

9. It may be possible to weaken (vi) by dropping the requirement of *strict* subadditivity, and adding (ii), but a constructive proof does not appear to be available.

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References

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[2] K. Urbanik, *Quelques théorèmes sur les mesures*, Fund. Math. 41 (1954), p. 150-162.

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Sur la vitesse de la croissance des suites infinies d'entiers positifs I

(Echelle des vitesses)

par

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1. Soit $\mathfrak{M}$ l'espace composé de toutes les suites infinies d'entiers positifs convergentes au sens large, c'est-à-dire des suites qui soit tendent vers l'infini, soit deviennent stationnaires à partir d'un certain terme.

$s = (n_1, n_2, \dots, n_i, \dots)$ et $t = (m_1, m_2, \dots, m_i, \dots)$ étant deux éléments de $\mathfrak{M}$, nous écrirons

(1.1) $$s \gg t$$

lorsque

(1.2) $$\lim_{i\to\infty}\frac{n_i}{m_i} = \infty,$$

et dans ce cas seulement. Nous dirons alors que la vitesse de la croissance de la suite s dépasse celle de t.

Deux suites s et $s' = (n_1', n_2', \dots, n_i', \dots)$ seront dites équivalentes lorsque

$$0 < \varliminf_{i\to\infty}\frac{n_i}{n_i'} \leqslant \varlimsup_{i\to\infty}\frac{n_i}{n_i'} < \infty.$$

On voit que la relation (1.1) subsiste lorsqu'on y remplace s ou t, ou s et t simultanément, par des suites équivalentes [1]; de même, on aperçoit que toutes les suites stationnaires sont équivalentes et que la formule (1.2), donc aussi (1.1), est remplie par toute suite s tendant vers l'infini, lorsque t est stationnaire. (Comparer [2], p. 309—310.)

La relation (1.1), transitive et non-refléxive, établit dans $\mathfrak{M}$ un ordre partiel; démontrons qu'elle jouit de la propriété suivante:

(a) *M étant un ensemble au plus dénombrable $\subset\mathfrak{M}$, soit t un élément de $\mathfrak{M}$ tel que $M \gg t$* [2]. *Alors il existe un $s^* \in \mathfrak{M}$ satisfaisant à la condition*

(1.3) $$M \gg s^* \gg t.$$

[1] E. Marczewski a aperçu que l'implication inverse est aussi vraie: lorsque $y \gg s$ entraîne $y \gg s'$ et réciproquement, les suites s et s' remplissent l'inégalité du texte.

[2] C'est-à-dire que $p \gg t$ pour tout $p \in M$.