On transfinite iteration

by

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1. Let $\mathfrak{D}$ be a class of sets closed under the union of chains and $f\colon \mathfrak{D}\to\mathfrak{D}$ an extensional operator on $\mathfrak{D}$, i. e., $fX \supseteq X$ holds for all $X \in \mathfrak{D}$. Then, the powers f^{α} of f for any ordinal α are defined, as usual, by

$$f^{\alpha}X = f\bigcup_{\beta<\alpha} f^{\beta}X\,.$$

These f^{α} may, but need not be, distinct for all α. Of course, if all f^{α} are distinct $\mathfrak{D}$ cannot be a set, and if $\mathfrak{D}$ is a set of cardinal $\mathfrak{d}$ one must have $f^{\alpha}=f^{\delta}$ for all $\alpha \geqslant \delta$ when δ denotes the first ordinal whose cardinal is greater than $\mathfrak{d}$. Less obvious criteria for the equality of all powers of f from some ordinal onwards can be based on suitable properties of f. A property of this kind was described in a recent paper by G. Schwarz [1] in the following way:

($S_{\mathfrak{t}}$) *If $\mathfrak{A}$ is a collection of $\mathfrak{t}$ sets $A \in \mathfrak{D}$, then there exists a set Φ of subcollections $\mathfrak{B} \subseteq \mathfrak{A}$, each consisting of less than $\mathfrak{t}$ sets, such that*

$$f\bigcup_{A\in\mathfrak{A}} A = \bigcup_{\mathfrak{B}\in\Phi} f\bigcup_{B\in\mathfrak{B}} B\,.$$

For any extensional isotonic [1] operator f satisfying ($S_{\mathfrak{t}}$) with some $\mathfrak{t}$ which is either denumerable or has an immediate predecessor (in the natural well-ordering of the cardinals) Schwarz proves the equation

$$f^{\alpha}X = \bigcup_{\eta<\xi} f^{\eta}X \qquad (\alpha \geqslant \xi) \tag{1}$$

where ξ denotes the first ordinal whose cardinal is $\mathfrak{t}$.

In the present note, a number of conditions for operators f will be considered which are similar to ($S_{\mathfrak{t}}$) and have the same effect on the powers of f as ($S_{\mathfrak{t}}$). Also, their relations to each other and to ($S_{\mathfrak{t}}$) will be discussed and some statements concerning products of operators will be deduced from them.

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(1) This means $fX \subseteq fY$ whenever $X \subseteq Y$.

2. The conditions for an extensional isotonic operator f to be studied here are (²):

($\mathrm{A}_\mathfrak{k}$) *If a chain* $\mathfrak{C}$ *of* $\mathfrak{k}$ *sets is such that any* $\mathfrak{B} \subseteq \mathfrak{C}$ *with* $|\mathfrak{B}| < \mathfrak{k}$ *has an upper bound in* $\mathfrak{C}$, *then* $f\bigcup_{X\in\mathfrak{C}} X = \bigcup_{X\in\mathfrak{C}} fX$.

($\mathrm{B}_\mathfrak{k}$) *If a chain* $\mathfrak{C}$ *of* $\mathfrak{k}$ *sets is such that any* $E \subseteq \bigcup_{X\in\mathfrak{C}} X$ *with* $|E| < \mathfrak{k}$ *is contained in some* $X \in \mathfrak{C}$, *then* $f\bigcup_{X\in\mathfrak{C}} X = \bigcup_{X\in\mathfrak{C}} fX$.

($\mathrm{T}_\mathfrak{k}$) $fX = \bigcup fE$ *(all* $E \subseteq X$ *with* $|E| < \mathfrak{k}$*)*.

Amongst these, ($\mathrm{A}_\mathfrak{k}$) comes closest to the formula (1) as is shown by the proof of the following (³)

PROPOSITION 1. *If an extensional isotonic operator* f *satisfies* ($\mathrm{A}_\mathfrak{k}$) *for some regular* $\mathfrak{k}$, *then* $f^\alpha X = \bigcup_{\eta<\xi} f^\eta X$ *for any* $\alpha \geqslant \xi$ *where* ξ *is the least ordinal of cardinal* $\mathfrak{k}$.

Proof. If the $f^\eta X$ are not all distinct for $\eta < \xi$ one has $f^\eta X = f^\gamma X$ for all $\eta \geqslant \gamma$ with some $\gamma < \xi$ and then $\bigcup_{\eta<\xi} f^\eta X = f^\gamma X = f^\xi X$. Otherwise, the $f^\eta X$, $\eta < \xi$, form a chain of $\mathfrak{k}$ sets and the mapping $\eta \to f^\eta X$, $\eta < \xi$, is an order isomorphism. Since $\mathfrak{k}$ is regular, any set of less than $\mathfrak{k}$ ordinals $\eta < \xi$ has an upper bound less than ξ and therefore the chain $\{f^\eta X \mid \eta < \xi\}$ satisfies the hypothesis of ($\mathrm{A}_\mathfrak{k}$). This leads to $f^\xi X = f\bigcup_{\eta<\xi} f^\eta X = \bigcup_{\eta<\xi} f^{\eta+1} X = \bigcup_{\eta<\xi} f^\eta X$, and (1) now follows by induction.

3. The relations between the four conditions stated above are listed in (⁴)

PROPOSITION 2. *For any* $\mathfrak{k}$, $(\mathrm{T}_\mathfrak{k}) \Rightarrow (\mathrm{S}_\mathfrak{k}) \Rightarrow (\mathrm{A}_\mathfrak{k}) \Leftarrow (\mathrm{B}_\mathfrak{k})$ *and if* $\mathfrak{k}$ *is regular, also* $(\mathrm{A}_\mathfrak{k}) \Rightarrow (\mathrm{S}_\mathfrak{k})$.

Proof. $(\mathrm{T}_\mathfrak{k}) \Rightarrow (\mathrm{S}_\mathfrak{k})$. If $\mathfrak{A}$ is a collection of $\mathfrak{k}$ sets and $E \subseteq \bigcup_{A\in\mathfrak{A}} A$ a set of less than $\mathfrak{k}$ elements, then E determines a $\mathfrak{B} \subseteq \mathfrak{A}$ consisting of less than $\mathfrak{k}$ sets such that $E \subseteq \bigcup_{B\in\mathfrak{B}} B$ if one chooses for each $e \in E$ some $B \in \mathfrak{A}$ with $e \in B$. Since f is isotonic, one obtains $fE \subseteq f\bigcup_{B\in\mathfrak{B}} B \subseteq f\bigcup_{A\in\mathfrak{A}} A$ for this $\mathfrak{B}$ and ($\mathrm{T}_\mathfrak{k}$) then leads to $f\bigcup_{A\in\mathfrak{A}} A = \bigcup_{\mathfrak{B}} f\bigcup_{B\in\mathfrak{B}} B$ with certain $\mathfrak{B} \subseteq \mathfrak{A}$ where $|\mathfrak{B}| < \mathfrak{k}$. This is ($\mathrm{S}_\mathfrak{k}$).

(²) $|A|$ denotes the cardinal of the set A.

(³) A cardinal $\mathfrak{k}$ is regular if the first ordinal ξ of cardinal $\mathfrak{k}$ has the property that only sets of $\mathfrak{k}$ ordinals $\eta < \xi$ have ξ as supremum.

(⁴) Implication is abbreviated by $\Rightarrow$ and each condition is regarded as applying to the same extensional isotonic operator f.

$(\mathrm{S}_\mathfrak{k}) \Rightarrow (\mathrm{A}_\mathfrak{k})$. If $\mathfrak{C}$ is a chain as described in ($\mathrm{A}_\mathfrak{k}$) and Φ a collection of $\mathfrak{B} \subseteq \mathfrak{A}$ as given by ($\mathrm{S}_\mathfrak{k}$) for $f\bigcup_{A\in\mathfrak{C}} X$, then each $\bigcup_{B\in\mathfrak{B}} B$, $\mathfrak{B} \in \Phi$, is contained in some $X \in \mathfrak{C}$ by the hypothesis concerning $\mathfrak{C}$ and $f\bigcup_{X\in\mathfrak{C}} X = \bigcup_{\mathfrak{B}\in\Phi} f\bigcup_{B\in\mathfrak{B}} B \subseteq \bigcup_{X\in\mathfrak{C}} fX$ gives ($\mathrm{A}_\mathfrak{k}$).

$(\mathrm{A}_\mathfrak{k}) \Rightarrow (\mathrm{B}_\mathfrak{k})$. If $\mathfrak{C}$ is a chain as described in ($\mathrm{B}_\mathfrak{k}$) and $\mathfrak{B} \subseteq \mathfrak{C}$ with $|\mathfrak{B}| < \mathfrak{k}$, then $\mathfrak{B}$ is bounded above in $\mathfrak{C}$; for otherwise, a well-ordered subchain $\mathfrak{B}^* \subseteq \mathfrak{B}$ of $\mathfrak{C}$, cofinal with $\mathfrak{C}$, could be chosen and by taking one element from each $B' - B$ (B and B' in $\mathfrak{B}^*$ and B' immediate successor of B) one would obtain a set $E \subseteq \bigcup_{X\in\mathfrak{C}} X$ of less than $\mathfrak{k}$ elements which by its definition cannot be contained in any $X \in \mathfrak{C}$. This shows that $\mathfrak{C}$ also satisfies the hypothesis of ($\mathrm{A}_\mathfrak{k}$) and thus ($\mathrm{A}_\mathfrak{k}$) implies ($\mathrm{B}_\mathfrak{k}$).

$(\mathrm{B}_\mathfrak{k}) \Rightarrow (\mathrm{A}_\mathfrak{k})$. If $\mathfrak{C}$ is a chain as described in ($\mathrm{A}_\mathfrak{k}$) and $E \subseteq \bigcup_{X\in\mathfrak{C}} X$ with $|E| < \mathfrak{k}$, then one can choose for each $e \in E$ some $X_e \in \mathfrak{C}$ such that $e \in X_e$ and this collection of less than $\mathfrak{k}$ sets X_e has an upper bound in $\mathfrak{C}$ which contains E. Therefore $\mathfrak{C}$ also fulfills the hypothesis of ($\mathrm{B}_\mathfrak{k}$) and hence ($\mathrm{B}_\mathfrak{k}$) implies ($\mathrm{A}_\mathfrak{k}$).

$(\mathrm{A}_\mathfrak{k}) \Rightarrow (\mathrm{S}_\mathfrak{k})$ for regular $\mathfrak{k}$. If $\mathfrak{A}$ is a collection of $\mathfrak{k}$ sets, let A_η, $\eta < \xi$, be a well-ordering of it, of ordinal type ξ where ξ is the first ordinal of cardinal $\mathfrak{k}$ and define the sets $X_\eta = \bigcup_{\eta'<\eta} A_{\eta'}$. These may not all be equal from some η onwards. If so, one has $X_\eta = \bigcup_{A\in\mathfrak{A}} A$ for this η, and since X_η is the union of less than $\mathfrak{k}$ sets ($\mathrm{S}_\mathfrak{k}$) holds trivially. If, however, there exists, for each η, some $\eta' > \eta$ such that $X_{\eta'} \supset X_\eta$, one can select a subsequence of ordinals $\eta' < \xi$ such that all sets $X_{\eta'}$ are distinct and $\sup \eta' = \xi$. The $X_{\eta'}$ then form a well-ordered chain, again of ordinal type ξ since $\mathfrak{k}$ was regular and by the regularity of $\mathfrak{k}$ this chain satisfies the hypothesis of ($\mathrm{A}_\mathfrak{k}$). In this case, one has $f\bigcup_{A\in\mathfrak{A}} A = f\bigcup X_{\eta'} = \bigcup fX_{\eta'}$ where each X_η is the union of less than $\mathfrak{k}$ sets in $\mathfrak{A}$, and this is ($\mathrm{S}_\mathfrak{k}$).

COROLLARY. *If an extensional isotonic operator* f *satisfies* ($\mathrm{T}_\mathfrak{k}$), ($\mathrm{S}_\mathfrak{k}$) *or* ($\mathrm{B}_\mathfrak{k}$) *for some regular* $\mathfrak{k}$, *then* (1) *holds.*

This follows immediately from propositions 1 and 2.

4. Given an extensional isotonic operator f on the subsets of a fixed set E (⁵), one is often interested in those $A \subseteq E$ which are *closed under* f, that is for which $fA = A$. These A form what is called a *closure system* $\mathfrak{F}$: the intersection of any family of sets $A_\alpha \in \mathfrak{F}$ again belongs to $\mathfrak{F}$ since $fA_\alpha = A_\alpha$ implies $f\bigcap A_\alpha \subseteq \bigcap fA_\alpha = \bigcap A_\alpha \subseteq f\bigcap A_\alpha$ and thus $f\bigcap A_\alpha = \bigcap A_\alpha$.

(⁵) This condition will be assumed throughout the present section.

Now, if $f^{\gamma}X = f^{\xi}X$ for all $\gamma \geqslant \xi$ with some ξ, then $f^{\xi}X$ is exactly the *closure of X with respect to $\mathfrak{F}$*, i. e., the smallest $Y \supseteq X$ such that $fY = Y$ (or: $Y \in \mathfrak{F}$), for obviously any such Y satisfies $Y \supseteq f^{\xi}X$ and $f^{\xi}X \in \mathfrak{F}$ since $ff^{\xi}X = f^{\xi+1}X = f^{\xi}X$. Therefore, conditions on f which ensure the equality of all powers of f from some ordinal onwards also give a description of the closure operator associated with $\mathfrak{F}$.

Sometimes, one is led to consider more than one operator simultaneously, say two operators f and g, and correspondingly the collection of sets closed under *both f and g*. In this case, one has the

PROPOSITION 3. *If two extensional isotonic operators f and g both satisfy* $(A_{\mathfrak{t}})$ *for some $\mathfrak{t}$, then their product fg also does and if $\mathfrak{t}$ is regular the closure of any X relative to the simultaneous closure system $\mathfrak{F} \cap \mathfrak{G}$, where $\mathfrak{F}$ and $\mathfrak{G}$ belong to f and g respectively, is $(fg)^{\xi}X$ with the least ordinal ξ of cardinal $\mathfrak{t}$.*

Proof. The first step is to show that fg (for which $(fg)X = f(gX)$) again satisfies $(A_{\mathfrak{t}})$. Let $\mathfrak{C}$ be a chain as given in the hypothesis of $(A_{\mathfrak{t}})$. Then, $(fg)\bigcup_{X\in\mathfrak{C}} X = f\bigcup_{X\in\mathfrak{C}} gX$ by $(A_{\mathfrak{t}})$ for g. Now, if $\mathfrak{B}'$ is a subchain of the chain $g\mathfrak{C} = \{gX \mid X \in \mathfrak{C}\}$ consisting of less than $\mathfrak{t}$ sets, one can choose a subchain $\mathfrak{B} \subseteq \mathfrak{C}$ with $|\mathfrak{B}| < \mathfrak{t}$ and $g\mathfrak{B} = \mathfrak{B}'$. This $\mathfrak{B}$ will have an upper bound B in $\mathfrak{C}$ and gB is then an upper bound for $\mathfrak{B}'$ in $g\mathfrak{C}$. If $g\mathfrak{C}$ itself occurs among these $\mathfrak{B}'$, one obtains $f\bigcup_{X\in\mathfrak{C}} gX = fgB \subseteq \bigcup_{X\in\mathfrak{C}} fgX$ or $fg\bigcup_{X\in\mathfrak{C}} X = \bigcup_{X\in\mathfrak{C}} fgX$. If, however, $|g\mathfrak{C}| = \mathfrak{t}$, $(A_{\mathfrak{t}})$ for f implies $f\bigcup_{X\in\mathfrak{C}} gX = \bigcup_{X\in\mathfrak{C}} fgX$. In all, $(A_{\mathfrak{t}})$ holds for fg.

The remaining part follows from the inequalities $fX \subseteq fgX$ and $gX \subseteq fgX$ which imply $f\big((fg)^{\xi}X\big) \subseteq fg(fg)^{\xi}X = (fg)^{\xi}X \subseteq f\big((fg)^{\xi}X\big)$ and the same for g, and from the fact that for any $Y \supseteq X$ such that $fY = gY = Y$ one has $Y \supseteq (fg)^{\xi}X$.

Since the rôles of f and g above are symmetrical, the same proposition is true for the product gf. Also, in view of proposition 2, similar statements hold concerning the conditions $(B_{\mathfrak{t}})$, $(S_{\mathfrak{t}})$ and $(T_{\mathfrak{t}})$.

The preceding can easily be extended to an arbitrary number of operators: If f_{ν}, $\nu \in I$, is a family of extensional isotonic operators, their product p relative to a fixed well-ordering of I can be defined by the induction formulae

$$\Big(\prod_{\nu'<\nu} f_{\nu'}\Big) X = \bigcup_{\nu'<\nu} \Big(\prod_{\nu''\leqslant\nu'} f_{\nu''}\Big) X \quad \text{and} \quad \prod_{\nu'\leqslant\nu} f_{\nu'} = f_{\nu} \prod_{\nu'<\nu} f_{\nu'}.$$

If $(A_{\mathfrak{t}})$ holds for all f_{ν} and is already proved for $\prod_{\nu'<\nu} f_{\nu'}$ it follows for $\prod_{\nu'\leqslant\nu} f_{\nu'}$ by what was just shown. Similarly, if $(A_{\mathfrak{t}})$ holds for all $p_{\nu'} = \prod_{\nu''\leqslant\nu'} f_{\nu''}$,

$\nu' < \nu$, and $\mathfrak{C}$ is a chain as described in the hypothesis of $(A_{\mathfrak{t}})$, then

$$\Big(\prod_{\nu'<\nu} f_{\nu'}\Big)\bigcup_{X\in\mathfrak{C}} X = \bigcup_{\nu'<\nu} p_{\nu'} \bigcup_{X\in\mathfrak{C}} X = \bigcup_{\nu'<\nu}\ \bigcup_{X\in\mathfrak{C}} p_{\nu'}X = \bigcup_{X\in\mathfrak{C}}\ \bigcup_{\nu'<\nu} p_{\nu'}X = \bigcup_{X\in\mathfrak{C}}\Big(\prod_{\nu'<\nu} f_{\nu'}\Big) X$$

and hence $(A_{\mathfrak{t}})$ also holds for $\prod_{\nu'<\nu} f_{\nu'}$. Thus, one sees by induction that $p = \prod_{\nu\in I} f_{\nu}$ satisfies $(A_{\mathfrak{t}})$ and therefore by proposition 1 $p(p^{\xi}X) = p^{\xi}X$ if $\mathfrak{t}$ is regular. In virtue of the general relation $f_{\nu}X \subseteq pX$, $\nu \in I$, one also has $f_{\nu}(p^{\xi}X) = p^{\xi}X$ for all f_{ν}, and since $Y \supseteq X$ and $f_{\nu}Y = Y$ for all f_{ν} implies $Y = p^{\xi}Y \supseteq p^{\xi}X$, this leads to the

COROLLARY. *If the extensional isotonic operators f_{ν}, $\nu \in I$, all satisfy* $(A_{\mathfrak{t}})$ *for some $\mathfrak{t}$, then any of their products p, based on a well-ordering of I, again satisfies* $(A_{\mathfrak{t}})$ *and if $\mathfrak{t}$ is regular the closure of any X relative to the closure system $\bigcap_{\nu} \mathfrak{F}_{\nu}$, $\mathfrak{F}_{\nu}$ the collection of sets closed under f_{ν}, is $p^{\xi}X$ with the least ordinal ξ of cardinal $\mathfrak{t}$.*

Reference

[1] G. Schwarz, *A note on transfinite iteration*, J. Symb. Logic 21 (1956), p. 265-266.

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