A connected subset of the plane

by

M. E. Rudin (Rochester)

A subset of a topological space is said to be *connected* if it is not the union of two non-empty disjoint sets, neither of which contains a limit point of the other. A connected set is *degenerate* if it consists of a single point. The object of this paper is the construction of a connected set which has, in a certain sense, few connected subsets:

THEOREM. *If the continuum hypothesis is true, then there exists a non-degenerate connected subset M of the plane with the following property: if N is a non-degenerate connected subset of M, then $M-N$ is at most countable.*

This disproves a conjecture made by Erdös ([1], p. 443). It might be of interest to note that the result cannot be strengthened, since every non-degenerate connected set M contains a non-degenerate connected subset N, such that $M-N$ is infinite ([1], p. 443).

We shall first construct a certain indecomposable continuum I in the plane, discuss some of its properties, and then construct M as a subset of I, by means of a transfinite induction process; M will contain at most one point on any composant of I.

1. The indecomposable continuum I

1.1. A 2-cell is a homeomorphic image of a closed square in the plane. A *chain* Γ is a collection of 2-cells $L_1, \ldots, L_n$ such that (1) L_i intersects L_j if and only if $|i-j| \leqslant 1$; (2) $L_m \cap L_{m+1}$ is an arc, for $m = 1, \ldots, n-1$. The sets L_i will be called the *links* of Γ; L_1 and L_n are the end-links, $L_2, \ldots, L_{n-1}$ are middle links. The union of the links of Γ will be denoted by Γ^*.

1.2. Fix three points a, b, c in the plane. A chain Γ is of type (a, b, c) if each of its end-links contains one of the points a and c in its interior and if b is in the interior of some middle link. Let $\{\Gamma_n\}$ be a sequence of chains with the following properties:

(1) Γ_1 is of type (a, b, c), Γ_2 is of type (b, c, a), Γ_3 is of type (c, a, b), Γ_4 is of type (a, b, c), and so on, permuting the letters a, b, c cyclically.

(2) Γ_n^* lies in the interior of Γ_{n-1}^*.

(3) Each link of Γ_n lies in some link of Γ_{n-1}.

(4) If L is a link of Γ_{n-1} which does not contain a, b, or c, and if L_i and L_j $(i<j)$ are links of Γ_n which lie in L, then either L_k lies in the interior of L for all $L_k \in \Gamma_n$ such that $i<k<j$, or one such L_k contains one of the points a, b, or c.

(5) The diameter of each link of Γ_n is less than $1/n$.

Put

$$I=\bigcap_{n=1}^{\infty}\Gamma_n^*.$$

1.3. It is evident that I is compact and connected (i. e., I is a continuum). To see that I is indecomposable, suppose I is the union of two continua H_1 and H_2, neither of which is equal to I. Then $I-H_1$ and $I-H_2$ are open (relative to I), and there is a chain Γ_n of type (a,b,c) which contains two links L_i and L_j such that $I\cap L_i\cap H_1=0$ and $I\cap L_j\cap H_2=0$. The connectedness of H_1 and H_2 now implies that a and c do not belong to the same set H_i. But the same argument applies to the pairs (a,b) and (b,c), and a contradiction is reached.

1.4. We now introduce some additional terminology.

(a) Let $L_1,\dots,L_n$ be the links of one of the above-mentioned chains Γ_k. Choose i,j such that $1<i\leqslant j<n$, and such that none of the links L_m $(i\leqslant m\leqslant j)$ contains a, b, or c. Let G be the interior of $L_i\cup\dots\cup L_j$. If $U=I\cap G$, we call U a *section* of I.

(b) Observe that for any section U of I, the closure of U is homeomorphic to the plane set E described as follows: let K be a Cantor set on the x-axis, and let E be the set of all points (x,y) such that $x\in K$ and $0\leqslant y\leqslant 1$. The section U itself corresponds to the subset of E with $0<y<1$. The points of I which, under the above homeomorphism of $\bar U$ onto E, map into points with $y=0$, form one *end* of U; the points of I which correspond to the points of E with $y=1$ form the other end of U.

(c) If U is a section, formed as above by means of the links $L_i,\dots,L_j$, and if $i\leqslant p\leqslant q\leqslant j$, the section V formed by means of $L_p,\dots,L_q$ will be called a *block* of U.

(d) By a *subsection* of a section U we mean any section which is a subset of U. A *strip* of U is a subsection of U whose ends are subsets of the ends of U.

(e) A closed subset S of I is a *separating set* if $I-S$ is not connected. If S is a separating set, $\boldsymbol{G}(S)$ is the collection of all sections U such that S does not intersect the ends of U and $U-S=A\cup B$, where $A\cap B=0$, A and B are open (relative to I), $\bar A$ contains one end of U, and $\bar B$ contains the other end of U.

(f) If $p\in I$, we let $C(p)$ be the set of all $q\in I$ such that I contains a proper subcontinuum which contains p and q. $C(p)$ is the *composant* of I which contains p.

1.5. The following properties of the concepts just introduced will be needed in the sequel.

(a) *If S is a separating set, then $\boldsymbol{G}(S)$ is not empty.*

(b) *The intersection of a composant C and a section U is the union of countably many of the arc segments (i. e., arcs minus end points) which are the components of U* (compare 1.4 (b)).

Proof. Since C is dense in I ([2], p. 147), C intersects infinitely many components of U. If S is a component of U such that $S\cap C\neq 0$, then there is a proper subcontinuum X of I such that $X\subset C$ and $X\cap S\neq 0$. Since I is indecomposable and $\bar S$ is an arc, $X\cup\bar S$ is also a proper subcontinuum of I. Consequently, $S\subset C$.

It is known ([2], p. 147) that C is the union of countably many proper subcontinua of I. Hence the proof will be complete if we show that *every proper non-degenerate subcontinuum of I is an arc.*

Let X be a proper non-degenerate subcontinuum of I. For each positive integer n, let $\Gamma_n(X)$ denote the subchain of Γ_n consisting of those links of Γ_n which intersect X; let $\Gamma_n(a)$, $\Gamma_n(b)$, and $\Gamma_n(c)$ denote the links of Γ_n which contain a, b, and c, respectively.

If X contains none of the points a, b, or c, then, for some n, none of the links $\Gamma_n(a)$, $\Gamma_n(b)$, $\Gamma_n(c)$ belong to $\Gamma_n(X)$. Hence X is a subset of a section, and 1.4 (b) shows that X is an arc.

So suppose $a\in X$. For some k, the chain Γ_k is of type (a,b,c) and $\Gamma_k(X)\neq\Gamma_k$, so that $c\notin X$. Similarly, $b\notin X$. Hence there is an integer n (held fixed during the rest of this proof), such that $\Gamma_n(b)\notin\Gamma_n(X)$ and $\Gamma_n(c)\notin\Gamma_n(X)$. The following three statements are then true:

(1) *If $i>n$, $\Gamma_i(a)$ is an end-link of $\Gamma_i(X)$.*

(2) *If $j>i>n$, if $Q\in\Gamma_j(X)$ and $Q\subset\Gamma_j(a)$, then every link of Γ_j between Q and $\Gamma_j(a)$ is a subset of $\Gamma_i(a)$.*

(3) *If $j>i>n$, $L\in\Gamma_i(X)$, $H\in\Gamma_j(X)$, $K\in\Gamma_j(X)$, and $H\cup K\subset L$, then every link of $\Gamma_j(X)$ between H and K is a subset of L.*

From (3) it is evident that X is an arc.

If (1) is false, then $\Gamma_i(a)$ lies between two links A and B of $\Gamma_i(X)$, and Γ_i is of type (c,a,b). One of A and B, say A, is between $\Gamma_i(c)$ and $\Gamma_i(a)$ in Γ_i. But Γ_{i+1} is of type (a,b,c) and $\Gamma_{i+1}(b)\notin\Gamma_{i+1}(X)$; by 1.2 (4) this implies that X intersects no link of Γ_i between $\Gamma_i(a)$ and $\Gamma_i(c)$, so that $X\cap A=0$, a contradiction.

If (2) is false, 1.2 (4) implies that some link of Γ_j between Q and $\Gamma_j(a)$ lies in either $\Gamma_i(b)$ or $\Gamma_i(c)$. Since Q and $\Gamma_j(a)$ belong to $\Gamma_j(X)$ and X is connected, this is impossible.

If (3) is false, 1.2 (4) and our choice of n imply that there is a link $Q \in \Gamma_j(X)$ between H and K, such that $Q \subset \Gamma_i(a)$. By (2), every link of $\Gamma_j(X)$ between Q and $\Gamma_j(a)$ lies in $\Gamma_i(a)$. Since $\Gamma_j(a)$ is an end-link of Γ_j, either H or K, say K, lies in $\Gamma_i(a)$. But $K \subset L$. Hence $L = \Gamma_i(a)$, $H \subset \Gamma_i(a)$, and by (2) every link of $\Gamma_j(X)$ between H and $\Gamma_j(a)$ lies in $\Gamma_i(a)$.

This completes the proof.

(c) *If a section U belongs to some $\boldsymbol{G}(S)$, then every strip of U belongs to $\boldsymbol{G}(S)$, and if U is a block of a section V, then V belongs to $\boldsymbol{G}(S')$, where $S' = S \cap U$.*

(Since S does not intersect the ends of U, S' is closed, so that $\boldsymbol{G}(S')$ is defined.)

(d) *If U and V are sections, then V has disjoint strips $V_1, \ldots, V_n$ with the following properties:*

(1) $V_1 \cup \ldots \cup V_n \supset U \cap V$;

(2) *if $l \leqslant m \leqslant n$, if Y is a strip of V_m, if Z is a strip of U, and if every strip of Z intersects Y, then $Z \cap V_m = Z \cap Y$.*

Proof. By 1.4 (a), there are chains Γ_k and Γ_h and sets G and H, such that G is the interior of the union of some links of Γ_k, H is the interior of the union of some links of Γ_h, and such that $U = I \cap G$, $V = I \cap H$. Let $H_1, \ldots, H_n$ be the components of $\Gamma_k^* \cap H$, and put $V_m = H_m \cap I$ $(1 \leqslant m \leqslant n)$. These sets V_m have the desired properties.

Note that if $k \leqslant h$, then $H \subset \Gamma_k^*$, so that $n = 1$ and $V_1 = V$. If $k > h$, then the sets H_m are the "straight pieces" which Γ_k^* cuts out of H.

2. Preparatory lemmas

LEMMA 1. *Let S be a separating set and let $\boldsymbol{K}$ be a countable collection of subsets of I with the following property: If $K \in \boldsymbol{K}$, then either (1) K is a composant of I, or (2) every section $U \in \boldsymbol{G}(S)$ contains a section $V \in \boldsymbol{G}(S)$ such that $S \cap V \cap K = 0$.*

Then S contains a point which belongs to no member of $\boldsymbol{K}$.

Proof. Fix a section $W \in \boldsymbol{G}(S)$, and replace each composant $K \in \boldsymbol{K}$ by the countable collection of arc-segments whose union is $K \cap W$. Since we will operate entirely within W, we may therefore assume without loss of generality that $\boldsymbol{K}$ is a sequence $\{K_n\}$ $(n = 1, 2, \ldots)$, each of whose members satisfies (2) (each arc-segment clearly satisfies (2)). Using (2), we can construct a sequence $\{U_n\}$ such that $W \supset U_1$, $U_{n-1} \supset U_n$, $U_n \in \boldsymbol{G}(S)$, and $S \cap \overline{U}_n \cap K_n = 0$. Any point of the non-empty set $S \cap \bigcap_{n=1}^{\infty} \overline{U}_n$ has the desired property.

LEMMA 2. *Suppose S is a separating set, K is a subset of a section V, and every strip X of V has a strip Y such that $Y \cap K \cap S = 0$. Then if $U \in \boldsymbol{G}(S)$, U contains a section $W \in \boldsymbol{G}(S)$ such that $W \cap K \cap S = 0$.*

Proof. We can assume without loss of generality that $S \subset U$, and that every strip W of U intersects K, hence V. Choose strips $V_1, \ldots, V_n$ as in 1.5 (d), and select corresponding members $W_1 \supset \ldots \supset W_{n+1}$ of $\boldsymbol{G}(S)$ as follows:

Put $W_1 = U$. If W_i has been selected, let X be a strip of V_i such that every strip of X intersects W_i. (If no such X exists, put $W_{i+1} = W_i$.) Then there exists a strip Y of X such that $Y \cap K \cap S = 0$, and there is a strip Z of W_i, every strip of which intersects Y. Let $W_{i+1} = Z$. Then

$$W_{i+1} \cap V_i = W_{i+1} \cap Y$$

and we see that $W_{i+1} \cap V_i \cap K \cap S = 0$.

The section $W = W_{n+1}$ then has the desired property.

LEMMA 3. Hypothesis. (1) *A and B are disjoint subsets of I, no composant of I intersects both A and B, $A \cup B$ is at most countable, and $B - (a \cup b \cup c) \neq 0$.*

(2) *If W is a section which intersects B, there is a strip X of W and a separating set S such that $S \cap A = 0$, $X \in \boldsymbol{G}(S)$, and every term of $\boldsymbol{G}(S)$ intersects B.*

Conclusion. *There is a separating set T such that $T \cap A = 0$, and the following property holds: if S is a separating set such that $S \cap B = 0$, then every section $U \in \boldsymbol{G}(S)$ contains a section $V \in \boldsymbol{G}(S)$ such that $V \cap S \cap T = 0$.*

Proof. Order the points of $A \cup B$ in a simple countable sequence.

We will say a *section U has property* λ if there is a separating set S such that $S \cap A = 0$, $U \in \boldsymbol{G}(S)$, and every term of $\boldsymbol{G}(S)$ intersects B.

Note that, if U has property λ, every strip of U has property λ.

For each U such that $A \cap \overline{U} = 0$, let $Y(U)$ denote U.

For each U having property λ such that $A \cap \overline{U} \neq 0$, there is a first point p of A in $\overline{U}$ and we will associate with U a section $Y(U)$ such that (i) $Y(U)$ is a subsection of U having property λ, (ii) $p \notin \overline{Y(U)}$, (iii) p is in the closure of the strip of U having $Y(U)$ as a block. That such a section exists can be shown from the definition of property λ as follows. Assume that U is a section having property λ. If S is a separating set such that $U \in \boldsymbol{G}(S)$ and $S \cap A = 0$, there is a section V such that $p \in V$ if $p \in U$, $V \cap S = 0$, $p \in \overline{V}$ and $V \subset U$. So if Z is the strip of U having V as a block, one of the (at most two) blocks of Z which is maximal with respect to the property of not intersecting V, must belong to $\boldsymbol{G}(S)$ and hence satisfy the conditions desired for $Y(U)$.

With each section U which intersects B, associate sections $P(U)$ and $Q(U)$ such that (i) $P(U)$ is a strip of U having property λ, (ii) $Q(U)$ is a subsection of U containing the first point of B in U, (iii) the diameter of $Q(U)$ is less than half the diameter of U, (iv) $Q(U) \cap P(U) = 0$. For each positive integer i, define sections $P_i(U)$ and $Q_i(U)$ intersecting B as follows. Let $P_1(U) = P(U)$ and $Q_1(U) = Q(U)$. If $Q_{i-1}(U)$ has been defined, then $P_i(U) = P(Q_{i-1}(U))$ and $Q_i(U) = Q(Q_{i-1}(U))$. Observe that any open set which contains the first point of B in U also contains $P_i(U)$ for some i.

Now choose a fixed section V having property λ and arrange the strips of V in a sequence $V_1, V_2, \ldots$ By induction, we define for each positive integer n a collection $\boldsymbol{F}_n$ of subsections of V with the following properties:

(a) No component of V intersects two members of $\boldsymbol{F}_n$.

(b) Every strip of V intersects some member of $\boldsymbol{F}_n$.

(c) If $U \in \boldsymbol{F}_n$, then U has property λ.

Let $\boldsymbol{F}_1$ consist of the single section V.

Suppose $\boldsymbol{F}_{n-1}$ is defined and n is even. Let $\boldsymbol{F}_n$ be the collection of all subsections W of V such that

(i) $W \subset V_n$ or $W \subset V - V_n$;

(ii) for some $U \in \boldsymbol{F}_{n-1}$, W is a strip of $Y(U)$ or W is a strip of U which does not intersect $Y(U)$;

(iii) W is maximal with respect to properties (i) and (ii).

Suppose $\boldsymbol{F}_{n-1}$ is defined and n is odd. Let $\boldsymbol{F}_n$ be the collection of all subsections W of V such that

(i) $W \subset V_n$ or $W \subset V - V_n$;

(ii) for some $U \in \boldsymbol{F}_{n-1}$, W is a strip of $P_i(U)$ for some i, or W is a strip of U which intersects no one $P_i(U)$.

(iii) W is a maximal with respect to properties (i) and (ii).

Since V is a section, there is a chain Γ_k whose links are $L_1, \ldots, L_n$ such that V is the intersection of I with the interior of $L_i \cup \ldots \cup L_j$, where $1 < i \leqslant j < n$ (compare 1.4 (a)). Let R_1 and R_2 be the intersection of I with the interiors of $L_1 \cup \ldots \cup L_{i-1}$ and $L_{j+1} \cup \ldots \cup L_n$, respectively. Let D be the union of R_1 and of all sections E which intersect R_1 and which, for some n, intersect no member of $\boldsymbol{F}_n$.

Finally, define

$$T = I \cap (\bar{D} - D) .$$

We will now show that T has the desired properties.

First D is open with respect to I, and the fact that every strip of V intersects some member of $\boldsymbol{F}_n$ (for every n) implies that $D \cap R_2 = 0$, and hence T is a separating set.

Secondly, let us prove that $T \cap A = 0$. If J is one of the arc segments which are the components of V, then, for each n, either J contains a point of B or J intersects some member of $\boldsymbol{F}_n$. This follows from the following facts: (1) every strip of V intersects some member of $\boldsymbol{F}_n$, (2) $\boldsymbol{F}_1 = \{V\}$, (3) for n even, each term of $\boldsymbol{F}_{n-1}$ contains only a finite number of terms of $\boldsymbol{F}_n$, (4) for n odd, the terms of $\boldsymbol{F}_n$ lying in any one term W of $\boldsymbol{F}_{n-1}$ intersect every arc-segment-component of W except the one containing the first point of B in W, (5) no component of V intersects two members of $\boldsymbol{F}_n$, and (6) every term of $\boldsymbol{F}_n$, for $n > 1$, is a subset of some term of $\boldsymbol{F}_{n-1}$.

Suppose there is a point $p \in T \cap A$. Since no point of B is on the composant containing p (since $p \in A$), for each n there is a term W_n of $\boldsymbol{F}_n$ such that $p \in \overline{W}_n$. Let p_n be the first point of A in $\overline{W}_n$. The definition of $\boldsymbol{F}_n$ for n even shows that W_n is either a strip of $Y(W_{n-1})$ or W_n is a strip of W_{n-1} which does not intersect $Y(W_{n-1})$. So by the definition of $Y(U)$, $p_{n-1} \notin W_n$ if n is even. Consequently, $p = p_n$ for some n; but then $p \notin W_{n+2}$, and this contradiction shows that $T \cap A = 0$.

Thirdly, let S be a separating set such that $S \cap B = 0$; the proof will be completed by an appeal to Lemma 2 with T in place of K. If X is a strip of V, then $X = V_n$ for some n; let p be the first point of B which lies in some member W of $\boldsymbol{F}_n$ such that $W \subset V_n$; since $p \notin S$, there is an open set G such that $p \in G$ and $\bar{G} \cap S = 0$. The definition of $P_i(W)$ shows that $P_i(W) \subset G$ for some i. If $m > n+1$ the definition of $\boldsymbol{F}_n$ for n odd shows that F_m contains a section Z which lies in $P_i(W)$ and hence in G. Hence $\bar{Z} \cap S = 0$, and if Y is the strip of X which has Z as a block, $Y \cap S \cap T = 0$. This shows that the hypothesis of Lemma 2 is satisfied; consequently the conclusion of Lemma 2 holds and the proof of Lemma 3 is complete.

LEMMA 4. *There is a collection* $\boldsymbol{S}$ *of separating sets such that* (1) *if* R *is a separating set, some subset of* R *belongs to* $\boldsymbol{S}$ *and* (2) *if* $S \in \boldsymbol{S}$, $I - S$ *is the union of two mutually separated sets* D *and* E *such that, if* U *is a section and* $U \cap S \neq 0$, *then* $D \cap U \neq 0$ *and* $E \cap U \neq 0$.

Proof. For each separating set R, $I - R$ is the union of two separated sets D' and E'. Order all sections in a simple countable sequence and, for each positive integer n, define mutually separated sets D_n and E_n as follows. Let $D_1 = D'$ and $E_1 = E'$. If D_{n-1} and E_{n-1} have been defined, consider the nth section U. If $U \cap D_{n-1} = 0$, let $D_n = D_{n-1}$ and $E_n = E_{n-1} \cup U$. And if $U \cap D_{n-1} \neq 0$ but $U \cap E_{n-1} = 0$, let $D_n = D_{n-1} \cup U$ and $E_n = E_{n-1}$. Otherwise, let $D_n = D_{n-1}$ and $E_n = E_{n-1}$. Then let $D = \bigcup_{n=1}^{\infty} D_n$ and $E = \bigcup_{n=1}^{\infty} E_n$ and $S = I - (E \cup D)$. Then D and E are mutually separated and S is a subset of R such that, if U is a section and $U \cap S \neq 0$, then $D \cap U \neq 0$ and $E \cap U \neq 0$. Hence if $\boldsymbol{S}$ is the set of all S derived from separating sets R in the manner described, $\boldsymbol{S}$ has the desired properties.

3. The construction of the set M

Let $\boldsymbol{B}$ be the collection of all countable subsets of I which have at most one point on any composant. Choose $\boldsymbol{S}$ as in Lemma 4. Both $\boldsymbol{B}$ and $\boldsymbol{S}$ have the power of the continuum. Hence, if the continuum hypothesis is true, there is a function f, defined on the set Ω of all countable ordinals, such that $f(\alpha) \in \boldsymbol{B} \cup \boldsymbol{S}$ for each $\alpha \in \Omega$, and such that every member of $\boldsymbol{B} \cup \boldsymbol{S}$ is $f(\alpha)$ for some $\alpha \in \Omega$.

For each $\alpha \in \Omega$, we will define subsets A_α and Z_α of I (M will be the union of the sets A_α) and a collection $\boldsymbol{K}_\alpha$ of subsets of I.

Let $A_1 = 0$, let $Z_1 = C(a) \cup C(b) \cup C(c)$ (compare 1.4 (f)), and let $\boldsymbol{K}_1$ be the collection whose elements are $C(a)$, $C(b)$ and $C(c)$.

Let $\beta \in \Omega$, and suppose that, for each $\alpha < \beta$, the following induction hypotheses are satisfied:

H1: $A_\alpha \cap Z_\alpha = 0$, $A_\gamma \subset A_\alpha$, $Z_\gamma \subset Z_\alpha$ if $\gamma < \alpha$.

H2: If $p \in A_\alpha$, then $C(p) - \{p\} \subset Z_\alpha$.

H3: A_α is at most countable.

H4: $\boldsymbol{K}_\alpha$ is at most countable, and Z_α is a subset of the union of the members of $\boldsymbol{K}_\alpha$.

H5: *At least one of the following three statements is true for every choice of* $S \in \boldsymbol{S}$ *and* $K \in \boldsymbol{K}_\alpha$:

(a) $S \cap A_\alpha \neq 0$.

(b) K *is a composant of* I.

(c) *Every section* $U \in \boldsymbol{G}(S)$ *contains a section* $V \in \boldsymbol{G}(S)$ *such that* $S \cap V \cap K = 0$.

If β is a limit ordinal, we put

$$A_\beta = \bigcup_{\alpha<\beta} A_\alpha, \quad Z_\beta = \bigcup_{\alpha<\beta} Z_\alpha, \quad \boldsymbol{K}_\beta = \bigcup_{\alpha<\beta} \boldsymbol{K}_\alpha,$$

and the induction hypotheses clearly hold with β in place of α.

If β is not a limit ordinal, we choose α such that $\beta = \alpha + 1$, and consider two cases.

Case 1. Suppose $f(\alpha) = S \in \boldsymbol{S}$.

We assert that $S - Z_\alpha \neq 0$. By H1, $A_\alpha \subset I - Z_\alpha$, and if H5 (a) holds, the assertion is evident. If H5 (b) or H5 (c) hold, then Lemma 1 shows that S contains a point which does not lie in any $K \in \boldsymbol{K}_\alpha$, and our assertion follows from H4.

Hence there is a point $p \in S - Z_\alpha$, and we put

$$A_\beta = A_\alpha \cup \{p\}, \quad Z_\beta = Z_\alpha \cup (C(p) - \{p\}), \quad \boldsymbol{K}_\beta = \boldsymbol{K}_\alpha \cup \{C(p)\},$$

so that the induction hypotheses again hold, with β in place of α.

Case 2. *Suppose* $f(\alpha) \in \boldsymbol{B}$.

Let $B = f(\alpha) - Z_\alpha$ and $A = A_\alpha - f(\alpha)$. We consider two situations which cover all possibilities.

Case 2 (i). *Suppose* $B \neq 0$, *and that, if* W *is a section which intersects* B, *there is a strip* X *of* W *and a separating set* S *such that* $X \in \boldsymbol{G}(S)$, $S \cap A = 0$, *and every term of* $\boldsymbol{G}(S)$ *intersects* B.

In this case the hypothesis of Lemma 3 is satisfied (since $Z_1 \supset (a \cup b \cup c)$). Choose T as in the conclusion of Lemma 3. Put

$$A_\beta = A_\alpha \cup B, \quad Z_\beta = Z_\alpha \cup \bigcup_{p \in B} (C(p) - \{p\}) \cup (T - B),$$

$$\boldsymbol{K}_\beta = \boldsymbol{K}_\alpha \cup \bigcup_{p \in B} \{C(p)\} \cup \{T\}.$$

From the properties of T the induction hypotheses may easily be proved with β in place of α.

Case 2 (ii). *Suppose either:*

(a) $B = 0$,

or there exists a section W *such that* $W \cap B = 0$ *and that*

(b) *if* X *is any strip of* W *and* S *is any separating set such that* $X \in \boldsymbol{G}(S)$ *and* $S \cap A = 0$, *then there is a term of* $\boldsymbol{G}(S)$ *not intersecting* B.

Put $A_\beta = A_\alpha$, $Z_\beta = Z_\alpha \cup [(W \cap \bar{B}) - A_\alpha]$, $\boldsymbol{K}_\beta = \boldsymbol{K}_\alpha \cup \{W \cap \bar{B}\}$.

It should be noticed that, if X is a section and $X \cap B = 0$, then $X \cap \bar{B} = 0$.

The induction hypotheses certainly follow if $B = 0$; so we can assume case 2 (ii) (b).

Clearly H1 to H4 hold with β in place of α. To prove that H5 also holds, suppose $K = W \cap \bar{B}$, $S \in \boldsymbol{S}$, and $S \cap A = 0$. We will prove that H5 (c) holds. For U in $\boldsymbol{G}(S)$ we can assume without loss of generality that $S \subset U$. Clearly H5 (c) follows from Lemma 2 if every strip X of W has a strip Y such that $Y \cap S \cap K = 0$, so we can assume that there is a strip X of W such that every strip of X intersects S.

By Lemma 4, since $S \in \boldsymbol{S}$, $I - S$ is the union of mutually separated sets D and E such that every section intersecting S intersects both D and E. Since $X \cap S \neq 0$ and $U \supset S$, some subsection of $X \cap U$ intersects S; hence there is a section L such that $\bar{L} \subset X \cap U \cap D$. Let Y be the strip of W having L as a block. Similarly $Y \cap U \cap S \neq 0$, and there is a section N such that $\bar{N} \subset Y \cap U \cap E$. Let Z be the strip of W having N as a block and let Q be the minimal block of Z containing N and $Z \cap L$.

Notice that $Q \subset U$ since the ends of Q are in U. Since $\bar{L} \subset D$ and $\bar{N} \subset E$, $Q \in G(S)$ and $Z \in G(S \cap Q)$. But since Z is a strip of W and $Z \in G(S \cap Q)$ and $S \cap A = 0$, and 2 (ii) (b) is the case, there is a term J of $G(S \cap Q)$ such that $J \cap B = 0$ (and hence $J \cap \bar{B} = 0$).

At least one of the sets which is maximal with respect to being a section contained in $J \cap Q$ must belong to $G(S)$; let V be such a section. Then $V \in G(S)$, $V \subset Q \subset U$, and $V \cap K \subset V \cap \bar{B} \subset J \cap \bar{B} = 0$. So V has the desired properties to give us H5 (c).

Let $M = \bigcup_{\alpha \in \Omega} A_\alpha$.

4. Proof that M is connected and that, if N is a non-degenerate connected subsed of M, then $M-N$ is at most countable

To see that M is connected we observe that M intersects every separating set. If R is a separating set, by the description of S in Lemma 4, there is an $S \in S$ such that $R \supset S$. For some α, $S = f(\alpha)$, and, by Case 1, $A_{\alpha+1}$, and hence M, contains a point of S.

From the definition of M and H1, we see that, for $\alpha \in \Omega$, $M \cap Z_\alpha = 0$, and, by H2, M has at most one point on any composant.

Let B be a countable dense subset of the set of points of condensation of $M-N$ such that $B \subset M-N$. If $M-N$ is uncountable, then $\bar{B} \cap (M-N)$ is uncountable and if W is open with respect to I and $B \cap W \neq 0$, then $\bar{B} \cap (M-N) \cap W$ is uncountable.

For some α, $f(\alpha) = B$. So consider Case 2 with $\beta = \alpha+1$. Either

(i) $Z_\beta \supset (T-B)$, where T is a separating set,

or

(ii) $Z_\beta \supset \big((W \cap \bar{B}) - A_\alpha\big)$, where W is open with respect to I and $W \cap B \neq 0$ (since $B \neq 0$) and A_α is countable.

If (i), then $N \subset I-T$ and hence, since I is indecomposable, N is a subset of one composant of I. But one composant of I intersects M in at most one point; this contradicts the non-degeneraty of N.

If (ii), then $\big(\bar{B} \cap W \cap (M-N)\big) - A_\alpha \neq 0$ since A_α is countable. But then $(M-N) \cap Z_\beta \neq 0$ and this impossible since $M \cap Z_\beta = 0$.

References

[1] P. Erdös, *Some remarks on connected sets*, Bull. Amer, Math. Soc. 50 (1944), p. 442-446.

[2] C. Kuratowski, *Topologie II*, Warszawa 1950.

UNIVERSITY OF ROCHESTER

Reçu par la Rédaction le 15. 7. 1957

A note on Kosiński's r-spaces *

by

M. L. Curtis (Athens, Georgia)

Following Kosiński [1] we call a point x in a space X an *r-point* if x has arbitrarily-small neighborhoods U such that for each $y \in U$ there is a deformation retraction of $\bar{U}-y$ onto $\bar{U}-U$. A space X is an *r-space* if it is finite dimensional, compact metric and each point is an r-point. Problem 7 of [1] asks if (a, b) being an r-point of $A \times B$ implies that a and b are r-points of A and B respectively. We answer this question in the negative by giving a 4-dimensional finite polyhedron P^4 which is not an r-space but is such that its Cartesian product $P^4 \times S^1$ with a 1-sphere S^1 is an r-space. This example also furnishes a negative answer to Problem 6 of [1]. The polyhedron P^4 is the suspension of a Poincaré space M^3; i. e. M^3 is a polyhedral orientable closed 3-manifold such that $H_1(M^3, Z) = 0$ but $\pi_1(M^3) \neq 0$. It is not known if $P^4 \times S^1$ is a topological manifold.

One can readily show that P^4 has the homotopy type of the 4-sphere S^4. This fact also follows from Lemma 9 of [1]. Let $P^4 = M^3 \vee (a \cup b)$ where a and b are points and $\vee$ denotes the join. Clearly, $P^4 - a$ and $P^4 - b$ are contractible. Since P^4 is locally Euclidean at all other points and has the homotopy type of S^4, it follows that $P^4 - x$ is contractible for any $x \in P^4$. It follows from Theorem 6 of [1] that P^4 is not an r-space. We will show that $P^4 \times S^1$ is an r-space. We note that $P^4 \times S^1$ is an r-space if and only if the double suspension M^5 of M^3 is an r-space. (Indeed, for any space X we may represent the suspension X' of X as $X \vee (a \cup b)$ and the double suspension X'' as $X' \vee (c \cup d)$ where a, b, c, d are points. Then any point y in $P = (a \cup b) \times S^1$ in $X' \times S^1$ and any point z in $Q = (a \cup b) \vee (c \cup d)$ in X'' have homeomorphic neighborhoods. Similarly any point y in $(X' \times S^1) - P$ and any point z in $X'' - Q$ have homeomorphic neighborhoods.)

Let $\tilde{X} = X \vee p$ be the cone over X. Each point of $\tilde{X}$ can be represented as (x, r) with $x \in X$ and $r \in I$. The representation is unique except for p which can be written as $(x, 1)$ for any $x \in X$.

* Research supported by National Science Foundation Grant NSF-G4211.