A connected subset of the plane

by

M. E. Rudin (Rochester)

A subset of a topological space is said to be connected if it is not the union of two non-empty disjoint sets, neither of which contains a limit point of the other. A connected set is degenerate if it consists of a single point. The object of this paper is the construction of a connected set which has, in a certain sense, few connected subsets:

**Theorem.** If the continuum hypothesis is true, then there exists a non-degenerate connected subset $M$ of the plane with the following property: if $N$ is a non-degenerate connected subset of $M$, then $M - N$ is at most countable.

This disproves a conjecture made by Erdős ([1], p. 443). It might be of interest to note that the result cannot be strengthened, since every non-degenerate connected set $M$ contains a non-degenerate connected subset $N$, such that $M - N$ is infinite ([1], p. 443).

We shall first construct a certain indecomposable continuum $I$ in the plane, discuss some of its properties, and then construct $M$ as a subset of $I$, by means of a transfinite induction process; $M$ will contain at most one point on any component of $I$.

1. The indecomposable continuum $I$

1.1. A 2-cell is a homeomorphic image of a closed square in the plane. A chain $I$ is a collection of 2-cells $L_1, ..., L_n$ such that (1) $L_i$ intersects $L_j$ if and only if $|i - j| \leq 1$; (2) $L_m \cap L_{m+1}$ is an arc, for $m = 1, ..., n-1$. The sets $L_i$ will be called the links of $I$; $L_1$ and $L_n$ are the end-links, $L_2, ..., L_{n-1}$ are middle links. The union of the links of $I$ will be denoted by $I$.

1.2. Fix three points $a, b, c$ in the plane. A chain $I$ is of type $[a, b, c]$ if each of its end-links contains one of the points $a$ and $c$ in its interior and if $b$ is in the interior of some middle link. Let $(I^n_m)$ be a sequence of chains with the following properties:

1. $I_1$ is of type $(a_1, b_1, c_1)$, $I_2$ is of type $(b_2, c_2, a_2)$, $I_3$ is of type $(c_3, a_3, b_3)$, $I_4$ is of type $(a_4, b_4, c_4)$, and so on, permuting the letters $a, b, c$ cyclically.

2. $I^n_m$ lies in the interior of $I^{n-1}_m$. 
(3) Each link of $\Gamma_n$ lies in some link of $\Gamma_{n-1}$.
(4) If $L$ is a link of $\Gamma_{n-1}$ which does not contain $a$, $b$, or $c$, and if $L_a$ and $L_b$ ($i < j$) are links of $\Gamma_n$ which lie in $L$, then either $L_a$ lies in the interior of $L$ for all $L_a \in \Gamma_n$ such that $i < k < j$, or one such $L_a$ contains one of the points $a$, $b$, or $c$.
(5) The diameter of each link of $\Gamma_n$ is less than $1/n$.

Put
$$I = \bigcap_{n=1}^{\infty} \Gamma_n.\]

1.3. It is evident that $I$ is compact and connected (i.e., $I$ is a continuum). To see that $I$ is indecomposable, suppose $I$ is the union of two continua $H_1$ and $H_2$, neither of which is equal to $I$. Then $I - H_1$ and $I - H_2$ are open (relative to $I$), and there is a chain $\Gamma_n$ of type $(a, b, c)$ which contains two links $L_a$ and $L_b$ such that $I \cap L_a \cap H_2 = 0$ and $I \cap L_b \cap H_1 = 0$. The connectedness of $H_1$ and $H_2$ now implies that $a$ and $c$ do not belong to the same set $H_i$. But the same argument applies to the pairs $(a, b)$ and $(b, c)$, and a contradiction is reached.

1.4. We now introduce some additional terminology.

(a) Let $L_1, \ldots, L_n$ be the links of one of the above-mentioned chains $\Gamma_n$. Choose $i, j$ such that $1 < i < j$, and such that none of the links $L_m$ ($i \leq m \leq j$) contains $a$, $b$, or $c$. Let $G$ be the interior of $L_1 \cup \cdots \cup L_j$. If $U = I - G$, we call $U$ a section of $I$.

(b) Observe that for any section $U$ of $I$, the closure of $U$ is homeomorphic to the plane set $E$ described as follows: let $K$ be a Cantor set on the $z$-axis, and let $E$ be the set of all points $(x, y, z)$ such that $x \in K$ and $0 \leq y < 1$. The section $U$ itself corresponds to the subset of $E$ with $0 < y < 1$. The points of $I$ which, under the above homeomorphism of $U$ onto $E$ map into points with $y = 0$, form one end of $U$; the points of $I$ which correspond to the points of $E$ with $y = 1$ form the other end of $U$.

(c) If $U$ is a section, formed as above by means of the links $L_1, \ldots, L_j$, and if $i \leq p < q \leq j$, the section $V$ formed by means of $L_p, \ldots, L_q$ will be called a block of $U$.

(d) By a subsection of a section $U$ we mean any section which is a subset of $U$. A strip of $U$ is a subsection of $U$ whose ends are subsets of the ends of $U$.

(e) A closed subset $S$ of $I$ is a separating set if $I-S$ is not connected. If $S$ is a separating set, $G(S)$ is the collection of all sections $U$ such that $S$ does not intersect the ends of $U$ and $U-S = A \cup B$, where $A \cap B = 0$, $A$ and $B$ are open (relative to $I$), $A$ contains one end of $U$, and $B$ contains the other end of $U$.

1.5. The following properties of the concepts just introduced will be needed in the sequel.

(a) If $S$ is a separating set, then $G(S)$ is not empty.
(b) The intersection of a component $C$ and a section $U$ is the union of countably many of the arc segments (i.e., area minus end points) which are the components of $U$ (compare 1.4 (b)).

Proof. Since $C$ is dense in $I$ ([2], p. 147), $C$ intersects infinitely many components of $U$. If $S$ is a component of $U$ such that $S \cap C \neq \emptyset$, then there is a proper subcontinuum $X$ of $I$ such that $X \subseteq C$ and $X \cap S \neq \emptyset$. Since $I$ is indecomposable and $S$ is an arc, $X \cup S$ is also a proper subcontinuum of $I$. Consequently, $S \subseteq C$.

It is known ([2], p. 147) that $C$ is the union of countably many proper subcontinua of $I$. Hence the proof will be complete if we show that every proper non-degenerate subcontinuum of $I$ is an arc.

Let $X$ be a proper non-degenerate subcontinuum of $I$. For each positive integer $n$, let $\Gamma_n(X)$ denote the subchain of $\Gamma_n$ consisting of those links of $\Gamma_n$ which intersect $X$. Let $\Gamma_n(a)$, $\Gamma_n(b)$, and $\Gamma_n(c)$ denote the links of $\Gamma_n$ which contain $a$, $b$, or $c$, respectively.

If $X$ contains none of the points $a$, $b$, or $c$, then, for some $n$, none of the links $\Gamma_n(a)$, $\Gamma_n(b)$, $\Gamma_n(c)$ belong to $\Gamma_n(X)$. Hence $X$ is a subset of a section, and 1.4 (b) shows that $X$ is an arc.

So suppose $a \in X$. For some $k$, the chain $\Gamma_k$ is of type $(a, b, c)$ and $\Gamma_k(X) \neq \Gamma_k$, so that $a \notin X$. Similarly, $b \notin X$. Hence there is an integer $n$ (held fixed during the rest of this proof), such that $\Gamma_n(b) \notin \Gamma_n(X)$ and $\Gamma_n(c) \notin \Gamma_n(X)$. The following three statements are then true:

1. If $i < n$, $\Gamma_i(a)$ is an end-link of $\Gamma_i(X)$.
2. If $j > i < n$, if $Q \in \Gamma_j(X)$ and $Q \subseteq \Gamma_i(a)$, then every link of $\Gamma_i$ between $Q$ and $\Gamma_i(a)$ is a subset of $\Gamma_i(a)$.
3. If $j > i > n$, $L \in \Gamma_j(X)$, $H \in \Gamma_i(X)$, $K \in \Gamma_j(X)$, and $H \cap K \subseteq L$, then every link of $\Gamma_j(X)$ between $H$ and $K$ is a subset of $L$.

From (3) it is evident that $X$ is an arc.

If (1) is false, then $\Gamma_i(a)$ lies between two links $A$ and $B$ of $\Gamma_i(X)$, and $\Gamma_i$ is of type $(a, b, c)$. One of $A$ and $B$, say $A$, is between $\Gamma_i(a)$ and $\Gamma_i(b)$ in $\Gamma_i$. But $\Gamma_i(c)$ is of type $(a, b, c)$ and $\Gamma_i(c) \notin \Gamma_i(a)$; by 1.2 (4) this implies that $X$ intersects no link of $\Gamma_i$ between $\Gamma_i(a)$ and $\Gamma_i(c)$, so that $X \cap A = \emptyset$, a contradiction.

Fundamenta Mathematicae. T. XLVI.
If (2) is false, 1.2 (4) implies that some link of $\Gamma_1$ between $Q$ and $\Gamma_2(a)$ lies in either $\Gamma_1(b)$ or $\Gamma_2(c)$. Since $Q$ and $\Gamma_2(a)$ belong to $\Gamma_2(X)$ and $X$ is connected, this is impossible.

If (3) is false, 1.2 (4) and our choice of $n$ imply that there is a link $Q \in \Gamma_2(X)$ between $H$ and $K$, such that $Q \not\subset \Gamma_2(a)$. By (5), every link of $\Gamma_2(X)$ between $Q$ and $\Gamma_2(a)$ lies in $\Gamma_2(a)$. Since $\Gamma_2(a)$ is an end-link of $\Gamma_2$, either $H$ or $K$, say $K$, lies in $\Gamma_2(a)$. But $K \subset L$. Hence $L = \Gamma_2(a)$, $H \subset \Gamma_2(a)$, and by (2) every link of $\Gamma_2(X)$ between $H$ and $\Gamma_2(a)$ lies in $\Gamma_2(a)$.

This completes the proof.

(c) If a section $U$ belongs to some $G(S)$, then every strip of $U$ belongs to $G(S)$, and if $U$ is a block of a section $V$, then $V$ belongs to $G(S')$, where $S' = S \cup U$.

(Since $S$ does not intersect the ends of $U$, $S'$ is closed, so that $G(S')$ is defined.)

(d) If $U$ and $V$ are sections, then $V$ has disjoint strips $V_1, \ldots, V_n$ with the following properties:

(1) $V_1 \cup \cdots \cup V_n = U \cup V$;

(2) if $1 \leq m \leq n$, if $Y$ is a strip of $V_m$, if $Z$ is a strip of $U$, and if every strip of $Z$ intersects $Y$, then $Z \cap V_m = Z \cap Y$.

Proof. By 1.4 (a), there are chains $I_1$ and $I_2$ and sets $G$ and $H$, such that $G$ is the interior of the union of some links of $I_2$, $H$ is the interior of the union of some links of $I_2$, and such that $U = I \cap G$, $V = I \cap H$. Let $H_1, \ldots, H_n$ be the components of $I_2 \cap H$, and put $V_m = H_m \cap I$ ($1 \leq m \leq n$). These sets $V_m$ have the desired properties.

Note that if $k \leq h$, then $H \cap I_k$, so that $n = 1$ and $V_1 = Y$. If $k > h$, then the sets $H_m$ are the “straight pieces” which $I_2$ cuts out of $H$.

2. Preparatory lemmas

LEMMA 1. Let $S$ be a separating set and let $K$ be a countable collection of subsets of $I$ with the following property: If $K \subset K$, then either (1) $K$ is a component of $I$, or (2) every section $U \subset G(S)$ contains a section $V \subset G(S)$ such that $S \cap V \cap K = 0$.

Then $S$ contains a point which belongs to no member of $K$.

Proof. Fix a section $W \subset G(S)$, and replace each component $K \subset K$ by the countable collection of arc-segments whose union is $K \cap W$. Since we will operate entirely within $W$, we may therefore assume without loss of generality that $K$ is a sequence $(K_n)$ ($n = 1, 2, \ldots$), each of whose members satisfies (2) (each arc-segment clearly satisfies (2)). Using (3), we can construct a sequence $(U_n)$ such that $W \supset U_n \supset U_{n-1} \supset \cdots \supset U_0 \subset G(S)$, and $S \cap U_n \cap K_{n-1} = 0$. Any point of the non-empty set $S \cap \bigcap U_n$ has the desired property.

LEMMA 2. Suppose $S$ is a separating set, $K$ is a subset of a section $V$, and every strip $X$ of $V$ has a strip $Y$ such that $S \cap X \cap Y = 0$. Then if $U \subset G(S)$, $U$ contains a section $W \subset G(S)$ such that $S \cap W \cap K = 0$.

Proof. We can assume without loss of generality that $S \subset U$, and that every strip $W$ of $U$ intersects $K$, hence $S$. Choose strips $V_1, \ldots, V_n$ as in 1.5 (d), and select corresponding members $W_1, \ldots, W_n$ of $G(S)$ as follows:

Put $W_1 = U$. If $W_1$ has been selected, let $X$ be a strip of $V_1$ such that every strip of $X$ intersects $W_1$. (If no such $X$ exists, put $W_{n+1} = W_1$.) Then there exists a strip $Y$ of $X$ such that $S \cap K \cap Y = 0$, and there is a strip $Z$ of $W_1$, every strip of which intersects $Y$. Let $W_{n+1} = Z$. Then

$W_{n+1} \cap V_1 = W_{n+1} \cap Y$

and we see that $W_{n+1} \cap V_1 \cap K = 0$.

The section $W = W_{n+1}$ then has the desired property.

LEMMA 3. Hypothesis. (1) $A$ and $B$ are disjoint subsets of $I$, no component of $I$ intersects both $A$ and $B$, $A \cup B$ is at most countable, and $B - (A \cup B - c) \neq \emptyset$.

(2) If $W$ is a section which intersects $B$, there is a strip $X$ of $W$ and a separating set $S$ such that $S \cap X = 0$, $X \subset G(S)$, and every term of $G(S)$ intersects $B$.

Conclusion. There is a separating set $T$ such that $T \cap A = 0$, and the following property holds: If $S$ is a separating set such that $S \cap B = 0$, then every section $U \subset G(S)$ contains a section $V \subset G(S)$ such that $V \cap S \cap T = 0$.

Proof. Order the points of $A \cup B$ in a simple countable sequence. We will say a section $U$ has property $\lambda$, if there is a separating set $S$ such that $S \cap A = 0$, $U \subset G(S)$, and every term of $G(S)$ intersects $B$. Note that, if $U$ has property $\lambda$, every strip $U$ of $U$ has property $\lambda$.

For each $U$ such that $A \cap U = 0$, let $Y(U)$ denote $U$.

For each $U$ having property $\lambda$ such that $A \cap U \neq 0$, there is a first point $p$ of $A$ in $U$ and we will associate with $U$ a section $Y(U)$ such that (i) $Y(U)$ is a subsequence of $U$ having property $\lambda$, (ii) $p \in \overline{Y(U)}$, (iiii) $p$ is in the closure of the strip of $U$ having $Y(U)$ as a block. That such a section exists can be shown from the definition of property $\lambda$ as follows.

Assume that $U$ is a section having property $\lambda$. If $S$ is a separating set such that $U \subset G(S)$ and $S \cap A = 0$, there is a section $V$ such that $p \in V$ if $p \in U$, $V \cap B = 0$, and $V \cap C$. So if $Z$ is the strip of $U$ having $V$ as a block, one of the (at most two) blocks of $Z$ which is maximal with respect to the property of not intersecting $V$, must belong to $G(S)$ and hence satisfy the conditions desired for $Y(U)$. 2
With each section \( U \) which intersects \( B_i \), associate sections \( P(U) \) and \( Q(U) \) such that \( (i) \) \( P(U) \) is a strip of \( U \) having property \( \lambda \), \( (ii) \) \( Q(U) \) is a subset of \( U \) containing the first point of \( B \) in \( U \), \( (iii) \) the diameter of \( Q(U) \) is less than half the diameter of \( U \), \( (iv) \) \( Q(U) \cap P(U) = 0 \). For each positive integer \( i \), define sections \( P_i(U) \) and \( Q_i(U) \) intersecting \( B \) as follows. Let \( P_1(U) = P(U) \) and \( Q_1(U) = Q(U) \). If \( Q_{i-1}(U) \) has been defined, then \( P_i(U) = P(Q_{i-1}(U)) \) and \( Q_i(U) = Q(Q_{i-1}(U)) \). Observe that any open set which contains the first point of \( B \) in \( U \) also contains \( P_i(U) \) for some \( i \).

Now choose a fixed section \( V \) having property \( \lambda \) and arrange the strips of \( V \) in a sequence \( V_1, V_2, \ldots \). By induction, we define for each positive integer \( n \) a collection \( F_n \) of subsectors of \( V \) with the following properties:

(a) No component of \( V \) intersects two members of \( F_n \).
(b) Every strip of \( V \) intersects some member of \( F_n \).
(c) If \( U \in F_n \), then \( U \) has property \( \lambda \).

Let \( F_0 \) consist of the single section \( V \).

Suppose \( F_{n-1} \) is defined and \( n \) is even. Let \( F_n \) be the collection of all subsectors \( W \) of \( V \) such that

(i) \( W \subset V \) or \( W \subset V - V_n \);
(ii) for some \( U \in F_{n-1} \), \( W \) is a strip of \( Y(U) \) or \( W \) is a strip of \( U \) which does not intersect \( Y(U) \);
(iii) \( W \) is maximal with respect to properties (i) and (ii).

Suppose \( F_{n-1} \) is defined and \( n \) is odd. Let \( F_n \) be the collection of all subsectors \( W \) of \( V \) such that

(i) \( W \subset V \) or \( W \subset V - V_n \);
(ii) for some \( U \in F_{n-1} \), \( W \) is a strip of \( P_i(U) \) for some \( i \), or \( W \) is a strip of \( U \) which intersect no one \( P_i(U) \).
(iii) \( W \) is maximal with respect to properties (i) and (ii).

Since \( V \) is a section, there is a chain \( I_1 \) whose links are \( L_1, \ldots, L_n \) such that \( V \) is the intersection of \( I \) with the interior of \( L_1 \cup \cdots \cup L_n \), where \( 1 \leq i \leq j < n \) (compare 1.4 (a)). Let \( B_1 \) and \( B_2 \) be the intersection of \( I \) with the interiors of \( L_1 \cup \cdots \cup L_{n-1} \) and \( L_{n+1} \cup \cdots \cup L_n \), respectively. Let \( D \) be the union of \( B_1 \) and of all sections \( E \) which intersect \( R_i \) and which, for some \( n \), intersect no member of \( F_n \).

Finally, define

\[ T = I \cap (D - D) \]

We will now show that \( T \) has the desired properties.
First \( D \) is open with respect to \( I \), and the fact that every strip of \( V \) intersects some member of \( F_n \) (for every \( n \)) implies that \( D \cap B_1 = 0 \), and hence \( T \) is a separating set.

Secondly, let us prove that \( T \cap A = 0 \). If \( J \) is one of the arc segments which are the components of \( V \); then, for each \( n \), either \( J \) contains a point of \( B \) or \( J \) intersects some member of \( F_n \). This follows from the following facts: (1) every strip of \( V \) intersects some member of \( F_n \), (2) \( F_1 = (V) \), (3) for \( n \) even, each term of \( F_{n-1} \) contains only a finite number of terms of \( F_n \), (4) for \( n \) odd, the terms of \( F_n \) lying in any one term \( W \) of \( F_{n-1} \) intersect every arc-segment-component of \( W \) except the one containing the first point of \( B \) in \( W \), (5) no component of \( V \) intersects two members of \( F_n \), and (6) every term of \( F_n \), for \( n > 1 \), is a subset of some term of \( F_{n-1} \).

Suppose there is a point \( p \in T \cap A \). Since no point of \( B \) is on the component containing \( p \) (since \( p \in A \)), for each \( n \) there is a term \( W_n \) of \( F_n \) such that \( p \in W_n \). Let \( p_n \) be the first point of \( A \) in \( W_n \). The definition of \( F_n \) for \( n \) even shows that \( W_n \) is either a strip of \( Y(W_{n-1}) \) or \( W_n \) is a strip of \( W_{n-1} \) which does not intersect \( Y(W_{n-1}) \). So by the definition of \( Y(U) \), \( p_{n-1} \notin W_n \) if \( n \) is even. Consequently, \( p = p_n \) for some \( n \); but then \( p \notin W_{n+1} \), and this contradiction shows that \( T \cap A = 0 \).

Thirdly, let \( S \) be a separating set such that \( S \cap B = 0 \); the proof will be completed by an appeal to Lemma 2 with \( T \) in place of \( X \). If \( X \) is a strip of \( V \), then \( X = V_n \) for some \( n \); let \( p \) be the first point of \( B \) which lies in some member \( W \) of \( F_n \) such that \( W \subset V_n \); since \( p \in S \), there is an open set \( G \) such that \( p \in G \) and \( G \cap S = 0 \). The definition of \( P_i(W) \) shows that \( P_i(W) \subset G \) for some \( i \). If \( m > n + 1 \) the definition of \( F_n \) for \( n \) odd shows that \( F_n \) contains a section \( Z \) which lies in \( P_i(W) \) and hence in \( G \). Hence \( Z \cap S = 0 \), and if \( Y \) is the strip of \( X \) which has \( Z \) as a block, \( Y \cap S = T = 0 \). This shows that the hypothesis of Lemma 2 is satisfied; consequently the conclusion of Lemma 2 holds and the proof of Lemma 3 is complete.

**Lemma 4.** There is a collection \( S \) of separating sets such that (1) if \( R \) is a separating set, some subset of \( R \) belongs to \( S \) and (2) if \( S \in S \), \( I \in S \) is the union of two mutually separated sets \( D \) and \( E \) such that, if \( U \) is a section and \( U \cap D = 0 \), then \( D \cap U = 0 \) and \( E \cap U = 0 \).

**Proof.** For each separating set \( R \), \( I \) in \( R \) is the union of two separated sets \( D \) and \( E \). Order all sections in a simple countable sequence and, for each positive integer \( n \), define mutually separated sets \( D_n \) and \( E_n \) as follows. Let \( D_0 = D' \) and \( E_0 = E' \). If \( D_{n-1} \) and \( E_{n-1} \) are defined, consider the \( n \)-th section \( U \). If \( U \cap D_{n-1} = 0 \), let \( D_n = D_{n-1} \) and \( E_n = E_{n-1} \cap U \). If \( U \cap D_{n-1} \neq 0 \) but \( U \cap E_{n-1} = 0 \), let \( D_n = D_{n-1} \cup U \) and \( E_n = E_{n-1} \). Otherwise, let \( D_n = D_{n-1} \) and \( E_n = E_{n-1} \). Then let \( D = \bigcup_{n=1}^{\infty} D_n \) and \( E = \bigcup_{n=1}^{\infty} E_n \) and \( S = I \cup (D \cup E) \). Then \( D \) and \( E \) are connected subsets of the plane.
mutually separated and $S$ is a subset of $R$ such that, if $U$ is a section and $U \cap S \neq \emptyset$, then $B \cap U \neq \emptyset$ and $E \cap U \neq \emptyset$. Hence if $S$ is the set of all $\tilde{S}$ derived from separating sets $\tilde{E}$ in the manner described, $S$ has the desired properties.

3. The construction of the set $M$

Let $B$ be the collection of all countable subsets of $I$ which have at most one point on any component. Choose $S$ as in Lemma 4. Both $B$ and $S$ have the power of the continuum. Hence, if the continuum hypothesis is true, there is a function $f$ defined on the set $\Omega$ of all countable ordinals, such that $f(a) \in B \cup S$ for each $a \in \Omega$, and that every member of $B \cup S$ is $f(a)$ for some $a \in \Omega$.

For each $a \in \Omega$, we will define subsets $A_a$ and $Z_a$ of $I$ ($M$ will be the union of the sets $A_a$) and a collection $K_a$ of subsets of $I$.

Let $A_a = 0$, let $Z_a = C(a) \cup C(b) \cup C(c)$ (compare 1.4 (ii)), and let $K_a$ be the collection whose elements are $C(a), C(b)$, and $C(c)$.

Let $\beta \in \Omega$, and suppose that, for each $a \prec \beta$, the following induction hypotheses are satisfied:

$H_1$: $A_0 \cap Z_0 = 0$, $A_0 \subseteq C_0$, $Z_0 \subseteq Z_0$ if $a < \alpha$.

$H_2$: If $a \in A_0$, then $C(p) - (p) \subseteq Z_0$.

$H_3$: $K_0$ is at most countable.

$H_4$: $K_0$ is at most countable, and $Z_0$ is a subset of the union of the members of $K_0$.

$H_5$: At least one of the following three statements is true for every choice of $S \subseteq S$ and $K \subseteq K_0$:

(a) $S \cap A_0 \neq 0$.

(b) $K$ is a component of $I$.

(c) Every section $U \subseteq G(S)$ contains a section $V \subseteq G(S)$ such that $S \cap V \cap K = 0$.

If $\beta$ is a limit ordinal, we put

$$A_\beta = \bigcup_{a < \beta} A_a, \quad Z_\beta = \bigcup_{a < \beta} Z_a, \quad K_\beta = \bigcup_{a < \beta} K_a,$$

and the induction hypotheses clearly hold with $\beta$ in place of $a$.

If $\beta$ is not a limit ordinal, we choose $a$ such that $\beta = a + 1$, and consider two cases.

Case 1. Suppose $f(a) = S \cap S$.

We assert that $S \cap Z_0 \neq 0$. By $H_1$, $A_0 \subseteq I - Z_0$, and if $H_5$ (a) holds, the assertion is evident. If $H_5$ (b) or $H_5$ (c) hold, then Lemma 1 shows that $S$ contains a point which does not lie in any $K \subseteq K_0$, and our assertion follows from $H_4$.

Hence there is a point $p \in S - Z_0$, and we put

$$A_p = A_p \cup \{p\}, \quad Z_p = Z_p \cup \{C(p) - (p)\}, \quad K_p = K_p \cup \{C(p)\},$$

so that the induction hypotheses again hold, with $\beta$ in place of $a$.

Case 2. Suppose $f(a) = B$.

Let $B = f(a) - Z_0$ and $A = A_a - f(a)$. We consider two situations which cover all possibilities.

Case 2 (i). Suppose $B \neq 0$, and that, if $W$ is a section which intersects $B$, there is a strip $X$ of $W$ and a separating set $S$ such that $X \subseteq G(S)$, $S \cap A = 0$, and every term of $G(S)$ intersects $B$.

In this case the hypothesis of Lemma 3 is satisfied (since $Z \cap (a \cup c) = 0$).

Choose $T$ as in the conclusion of Lemma 3. Put

$$A_0 = A_a \cup B, \quad Z_0 = Z_a \cup \bigcup_{p \in B} \{C(p) - (p)\} \cup \{T - B\},$$

$$K_0 = K_a \cup \bigcup_{p \in B} \{C(p)\} \cup \{T\}.$$

From the properties of $T$ the induction hypotheses may easily be proved with $\beta$ in place of $a$.

Case 2 (ii). Suppose either:

(a) $B = 0$,

or there exists a section $W$ such that $W \cap B = 0$ and that

(b) if $X$ is any strip of $W$ and $S$ is any separating set such that $X \subseteq G(S)$ and $S \cap A = 0$, then there is a term of $G(S)$ not intersecting $B$.

Put $A_p = A_p$, $Z_p = Z_p \cup \{W \cap B\} - \{A_p\}$, $K_p = K_p \cup \{W \cap B\}$.

It should be noticed that, if $X$ is a section and $X \cap B = 0$, then $X \cap B = 0$.

The induction hypotheses certainly fail if $B = 0$; so we can assume case 2 (ii) (b).

Clearly $H_1$ to $H_4$ hold with $\beta$ in place of $a$. To prove that $H_5$ also holds, suppose $K = W \cap B$, $S \subseteq S$, and $S \cap A = 0$. We will prove that $H_5$ (c) holds. For $U \subseteq G(S)$ we can assume without loss of generality that $S \subseteq U$. Clearly $H_5$ (c) follows from Lemma 2 if every strip $X$ of $W$ has a strip $Y$ such that $Y \cap S \cap K = 0$.

By Lemma 4, since $S \subseteq S$, $I - S$ is the union of mutually separated sets $D$ and $E$ such that every section intersecting $S$ intersects both $D$ and $E$. Since $X \cap S \neq 0$ and $U \subseteq S$, some subsection of $X \cap U$ intersects $S$; hence there is a section $L$ such that $L \subseteq X \cap U \cap D$. Let $Y$ be the strip of $W$ having $L$ as a block. Similarly $Y \cap U \cap S = 0$, and there is a section $N$ such that $N \subseteq Y \cap U \cap E$. Let $Z$ be the strip of $W$ having $N$ as a block and let $Q$ be the minimal block of $Z$ containing $N$ and $Z \cap L$. 
Notice that $Q \subseteq U$ since the ends of $Q$ are in $U$. Since $L \subset D$, and $N \subset E$, $Q \in G(S)$ and $Z \in G(S \cap Q)$. But since $Z$ is a strip of $W$ and $Z \in G(S \cap Q)$ and $S \cap A = 0$, and if (ii) (b) is the case, there is a term $J$ of $G(S \cap Q)$ such that $J \cap B = 0$ (and hence $J \cap B = 0$).

At least one of the sets which is maximal with respect to being a section contained in $J \cap B$ must belong to $G(S)$; let $V$ be such a section. Then $V \in G(S)$, $V \subseteq Q \subseteq U$, and $V \cap E \subset V \cap B \cap J \cap B = 0$. So $V$ has the desired properties to give us $H(S)(c)$.

Let $M = \bigcup_{a \in A} A_a$.

4. Proof that $M$ is connected and that, if $N$ is a non-degenerate connected subset of $M$, then $M \cap N$ is at most countable

To see that $M$ is connected, we observe that $M$ intersects every separating set. If $R$ is a separating set, by the description of $S$ in Lemma 4, there is an $S \in S$ such that $R \subset S$. For some $a, S = f(a)$, and, by Case 1, $A_{a+1}$, and hence $M$, contains a point of $S$.

From the definition of $M$ and $H_1$, we see that, for $a \in Q$, $M \cap Z_a = 0$, and, by $H_2$, $M$ has at most one point on any component.

Let $B$ be a countable dense subset of the set of points of $M$ such that $B \subset M \cap N$. If $M \cap N$ is uncountable, then $B \cap (M \cap N)$ is uncountable and if $W$ is open with respect to $I$ and $B \cap W \neq 0$, then $B \cap (M \cap N) \cap W$ is uncountable.

For some $a$, $f(a) = B$. So consider Case 2 with $\beta = \alpha + 1$. Either (i) $Z_0 \cup (T-B)$, where $T$ is a separating set, or (ii) $Z_0 \cap (W \cap B) - A_0$, where $W$ is open with respect to $I$ and $W \cap B \neq 0$ (since $B \neq 0$) and $A_0$ is countable.

If (i), then $N \subset I-T$ and hence, since $I$ is indecomposable, $N$ is a subset of one component of $I$. But one component of $I$ intersects $M$ in at most one point; this contradicts the non-degeneracy of $N$.

If (ii), then $(B \cap W \cap (M-N)) - A_0 \neq 0$ since $A_0$ is countable. But then $(M-N) \cap Z_0 \neq 0$ and this impossible since $M \cap Z_0 = 0$.

References


A note on Kosiński's $r$-spaces

by

M. L. Curtis (Athens, Georgia)

Following Kosiński [1] we call a point $x$ in a space $X$ an $r$-point if $x$ has arbitrarily-small neighborhoods $U$ such that for each $y \in U$ there is a deformation retract of $U - y$ onto $U - T$. A space $X$ is an $r$-space if it is finite dimensional, compact metric and each point is an $r$-point. Problem 7 of [1] asks if $(a, b)$ being an $r$-point of $A \times B$ implies that $a$ and $b$ are $r$-points of $A$ and $B$ respectively. We answer this question in the negative by giving a 4-dimensional finite polyhedron $P^4$ which is not an $r$-space but is such that its Cartesian product $P^4 \times S^k$ with a 1-sphere $S^k$ is an $r$-space. This example also furnishes a negative answer to Problem 6 of [1]: The polyhedron $P^4$ is the suspension of a Poicaré space $M^4$; i.e., $M^4$ is a polyhedral orientable closed 3-manifold such that $\pi_1(M^4, x) = 0$ but $\pi_1(M^4 \times S^k) \neq 0$. It is not known if $P^4 \times S^k$ is a topological manifold.

One can readily show that $P^4$ has the homotopy type of the 4-sphere $S^4$. This fact also follows from Lemma 9 of [1]. Let $P^4 = M^4 \cup (a \cup b)$ where $a$ and $b$ are points and $\cup$ denotes the join. Clearly, $P^4 - a$ and $P^4 - b$ are contractible. Since $P^4$ is locally Euclidean at all other points and has the homotopy type of $S^4$, it follows that $P^4 - x$ is contractible for any $x \in P^4$. It follows from Theorem 6 of [1] that $P^4$ is not an $r$-space. We will show that $P^4 \times S^k$ is an $r$-space. We note that $P^4 \times S^k$ is an $r$-space if and only if the double suspension $M^4$ of $M^4$ is an $r$-space. (Indeed, for any space $X$ we may represent the suspension $X'$ of $X$ as $X' \vee (a \cup b)$ and the double suspension $X''$ as $X'' \vee (a \cup d)$ where $a$, $b$, $d$, $l$ are points. Then any point $y$ in $P = (a \cup b) \cup (a \cup d)$ in $X''$ have homomorphic neighborhoods. Similarly any point $y$ in $(X' \times S^k) - P$ and any point $x$ in $X' - Q$ have homomorphic neighborhoods.)

Let $X = X' \cup P$ be the cone over $X$. Each point of $X$ can be represented as $(x, r)$ with $x \in X$ and $r \in I$. The representation is unique except for $P$ which can be written as $(x, 1)$ for any $x \in X$. 

* Research supported by National Science Foundation Grant NSF-6431.