On successive settings of an arc on the circumference of a circle

by

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Summary of results. Let (ϱ, φ) be a system of polar coordinates on the plane and $\varrho = \varrho_0$ the equation of a circle C. We consider an arbitrary natural N and such an angle ψ that all points $p_x = (\varrho_0, x\psi)$, where $x = 0, 1, \ldots, N$, are different. Let us suppose that C is a directed circle, say in the counter-clock-wise sense. Thus each two points $p_x, p_y \in C$ define an arc with the initial point p_x and the endpoint p_y. We shall denote this (open) arc by $\langle x, y\rangle$. For $1 \leqslant x, y \leqslant N$ let us say that p_y *immediately follows* p_x if $p_z \notin \langle x, y\rangle$ for $1 \leqslant z \leqslant N$. Let p_{a_r} be the point which immediately follows p_0 and p_{a_k} that immediately followed by p_0.

THEOREM. *The difference $y-x$, where p_y immediately follows p_x, takes the values a_r, $-a_k$, a_r-a_k (the last one only in the case $N < a_r+a_k-1$).*

This theorem is a conjecture of H. Steinhaus. Let us denote by (x, y) the length of $\langle x, y\rangle$.

COROLLARY 1. *The lengths (x, y), where p_y immediately follows p_x, take the values $(a_k, 0)$, $(0, a_r)$, (a_k, a_r) and the last value is attained only for $N < a_r+a_k-1$.*

This corollary easily follows from the above theorem. We shall deduce from it the following corollary, conjectured by J. Oderfeld:

COROLLARY 2. *If the length of C is 1, $(0, 1) = z = \frac{1}{2}(\sqrt{5}-1)$ and f_m is the greatest Fibonacci number* (1) *which does not exceed N, then z^m, z^{m-1}, z^{m-2} are the possible values of (x, y), where p_y immediately follows p_x.*

In the proof of our theorem we shall apply some ideas due to P. Erdös and V. Sós Turán, who have proved it independently. Another proof (based on the theory of continued fractions) has been obtained by P. Szüsz. These proofs, however, have not been published.

1. In this section we shall prove our theorem. For convenience let us write $[x, y]$ instead of "p_y immediately follows p_x" and $[x, z, y]$ for $p_z \in \langle x, y\rangle$.

(1) $f_1 = f_2 = 1$ and $f_n = f_{n-1}+f_{n-2}$ for $n = 3, 4, \ldots$

We denote by $a_1, a_2, \ldots$ the finite increasing sequence of such numbers $n \leqslant N$ that $[0, z, n]$ holds for $0 < z < n$ or $[0, n, z]$ holds for $0 < z < n$. The numbers a_j such that $[0, z, a_j]$ holds for $0 < z < a_j$ will be said to be of the *first kind*, the others — of the *second kind*.

LEMMA. *If a_l, a_t are of different kinds, $a_l < a_t$, $a_t + a_l \leqslant N$ and no $a_q < a_t$ satisfies*

(a) $[a_l, a_q, a_t]$ *if a_l is of the first kind,*

(b) $[a_t, a_q, a_l]$ *if a_l is of the second kind,*

then

$$a_{t+1} = a_t + a_l .$$

Proof of the Lemma. Suppose that a_l is of the first kind. Then $[0, a_t, a_t - a_l]$ holds, which obviously implies $[a_l, a_t + a_l, a_t]$. Thus $a_{t+1} \leqslant a_t + a_l$. Suppose that $a_{t+1} < a_t + a_l$. Then it follows that $[a_l, a_{t+1}, a_t + a_l]$ or $[a_t + a_l, a_{t+1}, a_t]$. Consequently $[0, a_{t+1} - a_l, a_t]$ or $[a_l, a_{t+1} - a_t, 0]$. But by $a_{t+1} - a_l < a_t$, $a_{t+1} - a_t < a_l$ both obtained relations contradict the definition of a_l, a_t.

If a_l is of the second kind then the proof is analogous.

Proof of the Theorem. Suppose first that $a_k < a_r$. Let $n = 0, 1, \ldots, a_k - 1$. Then $n + a_r - a_k \leqslant N$ but for some $n < a_k$ we also have $n + a_r \leqslant N$. We now take all those $n < a_k$ for which $n + a_r \leqslant N$ holds. (If $N \geqslant a_r + a_k - 1$ then those are all $n < a_k$.) Let us prove that for those n we have $[n, n + a_r]$. Indeed, suppose that a certain l satisfies $[n, l, n + a_r]$. If $n < l$ then from $(n, l) < (n, n + a_r)$ follows $(0, l - n) < (0, a_r)$, which is evidently a contradiction. Now if $l < n$, then $(l, n + a_r) < (n, n + a_r)$ and thus $(0, n + a_r - l) < (0, a_r)$. This is also impossible.

We consider now the remaining n's, i. e. those for which $n < a_k$ and $N < n + a_r$ holds. We assert that they satisfy $[n, n + a_r - a_k]$. Indeed, suppose that $[n, l, n + a_r - a_k]$ for a certain l. If $n < l$, then $(n, l) < (n, n + a_r - a_k)$ implies $(0, l - n) < (0, a_r - a_k)$. Now let a_s be the greatest number of the second kind that satisfies $a_s < a_r$. a_s exists since $a_1 = 1$ is a number of both kinds and $a_1 \leqslant a_k < a_r$. From our Lemma follows $a_r = a_s + a_k$. Thus $(0, l - n) < (0, a_s)$ which implies $l - n = a_r$. This contradicts $n + a_r > N$. If $l < n$, then $(l, n + a_r - a_k) < (0, a_r - a_k)$ implies $(0, n - l + a_r - a_k) < (0, a_s)$. It follows as previously that $n - l + a_r - a_k = a_r$, which contradicts $n < a_k$.

Now let $n = a_k, a_k + 1, \ldots, N$. We shall prove $[n, n - a_k]$. Indeed, suppose that $[n, l, n - a_k]$ for a certain l. Thus $n < l$ implies $(l - n + a_k, 0) < (a_k, 0)$ and $l < n$ implies $(n - l, 0) < (a_k, 0)$. These inequalities contradict the definition of a_k.

Thus we have proved that if $a_k < a_r$, then $[n, n + a_r]$ or $[n, n + a_r - a_k]$ (only in the case $N < a_r + a_k - 1$) or $[n, n - a_k]$ for $n \leqslant N$. If $a_r < a_k$, then we obtain by considerations quite similar to the above that $[n + a_k, n]$ or $[n + a_k - a_r, n]$ or $[n - a_r, n]$ for $n \leqslant N$. Our theorem follows.

2. In order to prove Corollary 2 let us observe the formula

$$f_{n+1} z - f_n = (-1)^n z^{n+1}$$

(which can be obtained by induction). Since now $(0, x) = xz \pmod 1$, the above formula implies

$$(*) \qquad z^n = \begin{cases} (f_n, 0) & \text{if } n \text{ is even}, \\ (0, f_n) & \text{if } n \text{ is odd}, \end{cases}$$

Let us prove by induction that $a_n = f_{n+1}$. This equality can easily be verified for $n = 1, 2$. We assume $a_n = f_{n+1}$, $a_{n-1} = f_n$ for some n. Then from $(*)$ it follows that a_{n-1}, a_n are of different kinds. Thus our Lemma (section 1) implies $a_{n+1} = a_n + a_{n-1}$. It remains to observe that $f_{n+2} = f_{n+1} + f_n$.

It is evident that f_m is the greater of the numbers a_k, a_r and the other is f_{m-1}. If $f_m = a_r$, then m is odd by $(*)$ and thus $(0, a_r) = (0, f_m) = z^m$, $(a_k, 0) = (f_{m-1}, 0) = z^{m-1}$, $(a_k, a_r) = z^{m-2}$. If $f_m = a_k$, then m is even and $(a_k, 0) = (f_m, 0) = z^m$, $(0, a_r) = (0, f_{m-1}) = z^{m-1}$, $(a_k, a_r) = z^{m-2}$. It remains to apply Corollary 1.

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