pour un \( j = 1, 2, ..., n(i) \), où \( B_{ij} \in T^g(p) \) conformément à (37). L'identité
\[(A_i \cdot A_j) + A_i = A_j\] entraîne d'après (42) que
\[h(A_i \cdot A_j) + g(A_i) = h(A_j),\] d'où
\[(B_{ij} + B) \cdot B_i = h(A_i)\] d'après (24) et (40), c'est-
à-dire
\[(44)\]
\[B_{ij} = h(A_i).\]

En même temps, \( p \cdot A \) entraîne \( p \cdot A_i \cdot A_j \) pour les mêmes valeurs de \( i \) et \( j \) en vertu de (40), d'où \( p \in \text{Int}(A_j) \) en vertu des propriétés (7) et (9) des \( A_{ij} \), établies dans la démonstration du lemme 1. Il existe donc dans \( X \) un entourage ouvert \( G \) de \( p \) tel que \( G \subset A_i \), ce qui entraîne
\[h(G) \subset h(A_i) = H\] en vertu de (43) et (44). L'ensemble ouvert \( H \subset X \)
etant arbitraire, l'existence d'un tel \( G \subset X \) équivalent à la continuité de \( h \) en \( p \) (cf. par exemple [2], I, p. 73), c. q. f. d.

**Travaux cités**


**On successive settings of an arc on the circumference of a circle**

by

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**Summary of results.** Let \((\varrho, \varphi)\) be a system of polar coordinates on the plane and \( \varrho = \rho_0 \) the equation of a circle \( C \). We consider an arbitrary natural \( N \) and such an angle \( \gamma \) that all points \( \rho = (\rho_0, \varphi) \), where \( x = 0, 1, ..., N \), are different. Let us suppose that \( C \) is a directed circle, say in the counter-clock-wise sense. Thus each two points \( p_1, p_2 \in C \) define an arc with the initial point \( p_1 \) and the endpoint \( p_2 \). We shall denote this (open) arc by \( (x, y) \). For \( 1 \leq x, y \leq N \) let us say that \( p_1 \) immediately follows \( p_2 \) if \( p_2 \in (x, y) \). Let \( p_0 \) be the point which immediately follows \( p_1 \) and \( p_N \) that immediately followed by \( p_0 \).

**Theorem.** The difference \( y-x \), where \( p_0 \) immediately follows \( p_1 \), takes the values \( a_1, a_2, a_7-a_8 \) (the last one only in the case \( N < a_1 + a_8 - 1 \)).

This theorem is a conjecture of H. Steinhaus. Let us denote by \( (x, y) \) the length of \( (x, y) \).

**Corollary 1.** The lengths \( (z, y) \), where \( p_0 \) immediately follows \( p_1 \), take the values \( (a_1, 0), (0, a_2), (a_8, a_7) \) and the last value is attained only for \( N < a_8 + a_7 - 1 \).

This corollary easily follows from the above theorem. We shall deduce from it the following corollary, conjectured by J. Oderfeld:

**Corollary 2.** If the length of \( C \) is \( 1 \), \( (0, 1) = x = \frac{1}{2}(|y| - 1) \) and \( f_m \) is the greatest Fibonacci number \((\dagger)\) which does not exceed \( N \), then \( a_m, a_{m+1}, a_{m+2} \) are the possible values of \( (x, y) \), where \( p_0 \) immediately follows \( p_1 \).

In the proof of our theorem we shall apply some ideas due to P. Erdős and V. Sós Turán, who have proved it independently. Another proof (based on the theory of continued fractions) has been obtained by P. Szücs. These proofs, however, have not been published.

In this section we shall prove our theorem. For convenience let us write \( [x, y] \) instead of \( p_0 \) immediately follows \( p_1 \) and \( [x, z, y] \) for \( p_3 \in (x, y) \).

\[\text{Notes:} \quad f_1 = f_2 = 1 \text{ and } f_{n+1} = f_n + f_{n-1} \text{ for } n = 3, 4, ...\]

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We denote by $a_1, a_2, \ldots$ the finite increasing sequence of such numbers $n \leq N$ that $[0, z, n]$ holds for $0 < z < n$ or $[0, n, z]$ holds for $0 < z < n$. The numbers $a_i$ such that $[0, z, a_i]$ holds for $0 < z < a_i$ will be said to be of the first kind, the others — of the second kind.

**Lemma.** If $a_1, a_2$ are of different kinds, $a_1 < a_2$, $a_1 + a_2 \leq N$ and so $a_2 < a_1$, satisfies

(a) $[a_1, a_2, a_1]$ if $a_1$ is of the first kind,

(b) $[a_1, a_2, a_2]$ if $a_1$ is of the second kind,

then $a_{i+1} = a_i + a_1$.

**Proof of the Lemma.** Suppose that $a_1$ is of the first kind. Then $[0, a_1, a_1 - a_1] = [0, a_1, 0]$ holds, which obviously implies $[a_1, a_1 + a_1]$. Thus $a_{i+1} < a_1 + a_1$. Suppose that $a_{i+1} < a_1 + a_1$. Then it follows that $[a_1, a_1 + a_1, a_1 + a_1]$ or $[a_1 + a_1, a_1 + a_1, a_1]$. Consequently $[0, a_1 + a_1 - a_1, a_1]$ or $[a_1 + a_1 - a_1, a_1]$. But by $a_{i+1} < a_1$, $a_{i+1} - a_1 < a_1$, both obtained relations contradict the definition of $a_{i+1}, a_1$.

If $a_1$ is of the second kind then the proof is analogous.

**Proof of the Theorem.** Suppose first that $a_0 < a_1$. Let $n = 0, 1, \ldots, a_0 - 1$. Then $n + a_0 - a_0 < N$ but for some $n < a_0$ we also have $n + a_0 < N$. We now take all those $n < a_0$ for which $n + a_0 < N$ holds. (If $N > a_0 + a_0 - 1$ then those are all $a_0$.) Let us prove that for those $n$ we have $[n, n, a_0]$. Indeed, suppose that a certain $l$ satisfies $[n, l, n + a_0]$ if $n < l$ then from $\{n, l, a_0\} < (n + a_0, n + a_0)$ follows $(0, l - n) < (0, a_0)$, which is evidently a contradiction. Now if $l < n$ then $(l, n + a_0) < (n, n + a_0)$ and thus $(0, n + a_0 - l) < (0, a_0)$. This is also impossible.

We consider now the remaining $n$'s, i.e., those for which $n < a_0$ and $N < n + a_0$ holds. We assert that they satisfy $[n, n + a_0 - a_0]$. Indeed, suppose that $[n, l, n + a_0 - a_0]$ for a certain $l$. If $n < l$, then $(n, l) < (n, n + a_0 - a_0)$ implies $(0, l - n) < (0, a_0 - a_0)$. Now let $a_0$ be the greatest number of the second kind that satisfies $a_0 < a_{i+1}$, $a_0$ exists since $a_0 - 1$ is a number of both kinds and $a_0 < a_0$. From our Lemma follows $a_0 = a_0 + a_0$. Thus $(0, l - n) < (0, a_0)$ which implies $l - n = a_0$. This contradicts $n + a_0 > N$. If $l < n$, then $(l, n + a_0 - a_0) < (0, a_0 - a_0)$ implies $(0, l - l + a_0 - a_0) < (0, a_0)$, which is also impossible.

Now let $n = a_0, a_0 + 1, \ldots, N$. We shall prove $[n, n + a_0]$. Indeed, suppose that $[n, l, n + a_0]$ for a certain $l$. Thus $n < l$ implies $(l + a_0, a_0) < (a_0, 0)$ and $n < l$ implies $(n - l, 0) < (a_0, 0)$. These inequalities contradict the definition of $a_0$. Thus we have proved that if $a_0 < a_0$, then $[n, n + a_0]$ or $[n, n + a_0 - a_0]$ (only in the case $N < a_0 + a_0 - 1)$ for $n \leq N$. If $a_0 < a_0$, then we obtain by considerations quite similar to the above that $[n + a_0, n]$ or $[n + a_0 - a_0, n]$ or $[n - a_0, n]$ for $n \leq N$. Our theorem follows.

2. In order to prove Corollary 2 let us observe the formula

$$f_{n+1}x - f_n = (-1)^n x^{n+1}$$

(which can be obtained by induction). Since now $(0, x) = x (\text{mod } 1)$, the above formula implies

$$(*) \quad x^n = \begin{cases} (f_n, 0) & \text{if } n \text{ is even,} \\ (0, f_n) & \text{if } n \text{ is odd,} \end{cases}$$

Let us prove by induction that $a_n = f_{n+1}$. This equality can easily be verified for $n = 1, 2$. We assume $a_n = f_{n+1}$, $a_{n-1} = f_n$ for some $n$. Then from $(*)$ it follows that $a_{n-1}, a_n$, are of different kinds. Thus our Lemma (section 1) implies $a_{n+1} = a_n + a_{n-1}$. It remains to observe that $f_{n+2} = f_{n+1} + f_n$.

It is evident that $f_{n+2}$ is the greater of the numbers $a_n, a_{n-1}$ and the other is $f_{n+1}$. If $f_{n+1} = a_n$, then $m$ is odd by $(*)$ and thus $(0, a_n) = (0, f_{n+1}) = x^n$, $(a_{n-1}, 0) = (f_{n+1}, 0) = x^{n+1} = (a_n, a_{n-1}) = x^n$. If $f_n = a_n$, then $m$ is even and $(0, a_n) = (f_{n}, 0) = x^n$, $(0, a_{n-1}) = (f_{n-1}, 0) = x^n = (a_n, a_{n-1}) = x^{n+1}$. It remains to apply Corollary 1.