Note on ordered groups and rings

by

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Several criteria are known which ensure the existence of a linear order in groups and rings, in particular in abelian groups, in skewfields and fields [1]. It is also known when a partial order of a group can be represented as conjunction of linear orders. But neither the same question for rings has as yet been answered nor a general criterion seems to have been stated explicitly under which a given partial order of a group or a ring can be extended to a linear one. Of course, it is not hard to formulate such conditions by standard methods of the theory of ordered groups and rings. The method which we follow below — although it contains no essentially new ideas — is simpler than the known ones, runs entirely parallel in groups and rings, and has the advantage that a great variety of previously known results may be derived from it in a quite simple manner [2].

§ 1. Groups. Let G be a group whose operation is written as multiplication. A *partial order* $\geqslant$ in G is a reflexive, antisymmetric and transitive relation defined for certain pairs a, b of elements of G such that $a \geqslant b$ implies $cad \geqslant cbd$ for all $c, d \in G$. a ($\in G$) is called *positive* if $a \geqslant e$ for the group identity e, is strictly positive if $a > e$. The set of all positive elements of G is an invariant semigroup P in G (i. e. $a \in P$ and $b \in P$ imply $ab \in P$, and $a \in P$, $g \in G$ imply $g^{-1}ag \in P$) which contains e but no other element along with its inverse. The set P completely determines the partial order, for $a \geqslant b$ if and only if $ab^{-1} \in P$; thus we may denote a partial order $\geqslant$ and the set of positive elements under $\geqslant$ by the same symbol P. A partial order Q is said to be an *extension* of P if $Q \supseteq P$ [3]. If Q_ν is a set of extensions of P such that their meet is P, then we call P the *conjunction* of the Q_ν. If for every $a \in G$ either a or a^{-1} belongs to P, then P defines a *linear order* in G in the sense that for any two elements

(1) See the References at the end of this note. (Numbers in brackets refer to it.)

(2) It is not difficult to see that our method can be applied to other algebraic systems too.

(3) The sign $\supseteq$ denotes inclusion, while $\supset$ is used to mean proper inclusion.

a, b in G either $a \geqslant b$ or $b \geqslant a$ holds. Finally, we introduce the notation $S(a_1, \ldots, a_n)$ $(a_1, \ldots, a_n \in G)$ for the invariant semigroup in G generated by the elements $a_1, \ldots, a_n$, and we put $S'(a_1, \ldots, a_n) = S(a_1, \ldots, a_n) \cup e$. Clearly, $S'(a_1, \ldots, a_n) = S'(a_1) \ldots S'(a_n)$.

Our main result is the following theorem.

THEOREM 1. *A partial order P of a group G can be extended to a linear order of G if and only if it has the following property:*

(*) *For any finite set of elements $a_1, \ldots, a_n$ in G $(a_i \neq e)$, one may choose $\varepsilon_1, \ldots, \varepsilon_n$ $(\varepsilon_i = 1$ or $-1)$ such that* (4)

$$P \cap S(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) = \wedge .$$

If P can be extended to a linear order L, then choose ε_i such that $a_i^{\varepsilon_i} < e$ in L. In this case all the elements of $S(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$ are strictly negative in L, and therefore the meet under consideration is indeed void.

For the proof of the converse we need the important

LEMMA 1. *If P has property (*), then for any $a \in G$ one of $PS'(a)$ and $PS'(a^{-1})$ defines a partial order P' in G, again satisfying (*).*

Suppose that there are elements $a_1, \ldots, a_n, b_1, \ldots, b_m$ $(\neq e)$ such that for every choice of signs ε_i, η_j one has

$$P \cap S(a, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) \neq \wedge \quad \text{and} \quad P \cap S(a^{-1}, b_1^{\eta_1}, \ldots, b_m^{\eta_m}) \neq \wedge .$$

Then the meet of P and $S(a^{\varepsilon}, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}, b_1^{\eta_1}, \ldots, b_m^{\eta_m})$ is never void, contrary to (*). Thus either $x = a$ or a^{-1} (or both) must be such that for any $a_1, \ldots, a_n$ in G $(a_i \neq e)$ the meet $P \cap S(x, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) = \wedge$ for suitably chosen signs $\varepsilon_i = \pm 1$. If a happens to have this property then put $P' = PS'(a^{-1})$, while if a^{-1} has it, then put $P' = PS'(a)$, and if both have it, put either one. For example, considering the first possibility, it is evident that P' is an invariant semigroup with e, which moreover satisfies (*), for if py $(p \in P,\ y \in S'(a^{-1}))$ belongs to $S(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$, then $p \in S(a, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$, which is impossible. Property (*) implies that P' does not contain any element $b \neq e$ along with its inverse $(P' \cap S(b^{\varepsilon}) = \wedge$ for $\varepsilon = 1$ or -1.)

Resuming the proof of the theorem, let L be a maximal element in the set of all partial orders of G which are extensions of P and satisfy (*); such an L exists, for (*) is satisfied by the union of an ascending chain of partial orders provided it is satisfied by all members of the chain. This L defines a linear order in G, for otherwise there would exist an $a \in G$ such that $a, a^{-1} \notin L$, but then by Lemma 1 L could further be extended. Q. e. d.

(4) By the symbol $\wedge$ we denote the void set.

If P consists of the group identity alone, from Theorem 1 we obtain the well known

COROLLARY 1. (Lorenzen [9] (5).) *A group G admits a linear order if and only if, given $a_1, \ldots, a_n$ in G with $a_i \neq e$, one has $e \notin S(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$ for at least one choice of signs $\varepsilon_i = \pm 1$.*

The following result has been obtained by Ohnishi [12].

COROLLARY 2. *All partial orders of a group can be extended to linear ones if and only if the group satisfies the following conditions:*

(i) *if $b, c \in S(a)$ then $S(b) \cap S(c)$ is not void;*

(ii) *$a \neq e$ implies $e \notin S(a)$.*

For the necessity take into account that (ii) is a special case of the condition in Corollary 1, while if $b, c \in S(a)$ but $S(b) \cap S(c)$ is void then $P = S'(b) S'(c^{-1})$ is an invariant semigroup containing no element other than e together with its inverse, so that P defines a partial order; this however cannot be extended to a linear one L, for $b > e$ and $c < e$ in L and these imply $a > e$ and $a < e$, respectively.

In order to establish the sufficiency, let P be any partial order and suppose that it fails to satisfy (*), i. e. that there are elements $a_1, \ldots, a_n$ in G $(a_i \neq e)$ such that $P \cap S(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n})$ is never empty. We show that the same holds for $a_1, \ldots, a_{n-1}$ too. For, if not, then for any fixed $\varepsilon_1, \ldots, \varepsilon_{n-1}$ there are elements $p_1, p_2 \in P$ such that $p_1 = t_1 s_1$, $p_2 = t_2 s_2$ with $t_i \in S'(a_1^{\varepsilon_1}, \ldots, a_{n-1}^{\varepsilon_{n-1}})$, $s_1 \in S(a_n)$, $s_2 \in S(a_n^{-1})$. Clearly, by (ii), $s_i \neq e$; further s_1, s_2^{-1} lie in $S(a_n)$ and therefore by (i) there is an element b common to $S(s_1)$ and $S(s_2^{-1})$,

$$b = x_1^{-1} s_1 x_1 \ldots x_k^{-1} s_1 x_k = y_1^{-1} s_2^{-1} y_1 \ldots y_l^{-1} s_2^{-1} y_l \quad (x_i, y_j \in G).$$

Either $t_1 \neq e$ or $t_2 \neq e$, for otherwise we should get $b \in P$ and $b^{-1} \in P$, whence $b = e$ contrary to (ii). Now an easy calculation leads us to the conclusion that

$$x_1^{-1} p_1 x_1 \ldots x_k^{-1} p_1 x_k y_l^{-1} p_2 y_l \ldots y_l^{-1} p_2 y_1 \in P \cap S(a_1^{\varepsilon_1}, \ldots, a_{n-1}^{\varepsilon_{n-1}}).$$

Consequently, we may assume $n = 0$, which is impossible.

COROLLARY 3. (Lorenzen [8]; see also Everett [3].) *All partial orders of an abelian group can be extended to linear orders if and only if the group is torsion free.*

Considering that $e \notin S(a)$ if and only if a is of infinite order whenever the group is commutative, the necessity is obvious. In an abelian

(5) Cf. also Łoś [10] and Ohnishi [13]. Lorenzen's original theorem has also a somewhat stronger form.

group, condition (i) of Corollary 2 is satisfied, for if $b, c \in S(a)$ then $b = a^n$, $c = a^m$ $(n, m > 0)$ and $a^{mn} \in S(b) \cap S(c)$. Corollary 2 completes the proof.

COROLLARY 4. (Levi [7].) *An abelian group admits a linear order if and only if it is torsion free.*

This is evident in view of Corollaries 1 and 3.

COROLLARY 5. (Lorenzen [9].) *A partial order P of a group G is a conjunction of linear orders if and only if $P \cap S(a, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) \neq \wedge$ for some $a_1, \ldots, a_n \in G$ $(a_i \neq e)$ and for all choices of signs $\varepsilon_i = \pm 1$ implies $a \in P$.*

This condition is necessary, for if $a \notin P$ and P is a conjunction of linear orders L_ν, then $a \notin L_\nu$ for some ν, i. e. $a^{-1} \in L_\nu$ and so $PS'(a^{-1})$ can be extended to a linear order L_ν. Lemma 1 implies that $PS'(a^{-1}) \cap S(a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) = \wedge$, that is to say, $P \cap S(a, a_1^{\varepsilon_1}, \ldots, a_n^{\varepsilon_n}) = \wedge$ for a suitable choice of the ε_i, whence the necessity follows. The sufficiency may be verified again by using Lemma 1: if the stated condition is fulfilled and $a \notin P$, then $PS'(a^{-1})$ possesses property (*), so that it has a linear extension L. Now $a^{-1} \in L$ implies $a \notin L$ and thus the meet of all linear extensions of P does not contain any element that is not in P.

COROLLARY 6. *A partial order P of an abelian group is a conjunction of linear orders if and only if $a^n \in P$ for some positive integer n implies $a \in P$.*

As in the proof of Corollary 2 it follows that under (i) the condition of Corollary 5 reduces to the condition: $P \cap S(a) \neq \wedge$ implies $a \in P$. In case when G is abelian, this is exactly the statement.

§ 2. Rings. Let R be an arbitrary ring [6]. A *partial order* $\geqslant$ in R is a reflexive, antisymmetric and transitive binary relation satisfying the condition: $a \geqslant b$ implies $a + c \geqslant b + c$ for all $c \in R$ and $ad \geqslant bd$, $da \geqslant db$ for all $d\ (\geqslant 0)$ in R. The set P of all positive elements $(x \geqslant 0)$ is a semiring (i. e. it is closed under addition and multiplication) containing 0 but no other element along with its negative. Again, the partial order is completely determined by P, because $a \geqslant b$ if and only if $a - b \in P$. We shall use the same notations as in case of groups. $H(a_1, \ldots, a_n)$ will denote the semiring generated by the elements $a_1, \ldots, a_n$ and 0.

THEOREM 2. *A partial order P of a ring R can be extended to a linear order of R if and only if P possesses the following property:*

(**) *For any two finite sets of elements $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ in R one may choose $\varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_m$ $(\varepsilon_i, \eta_j = \pm 1)$ such that*

$$H(P, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n) \cap -H(\eta_1 b_1, \ldots, \eta_m b_m) = 0\ .$$

(6) Thus R need be neither associative nor commutative.

The condition is necessary, for if L is a linear extension of P, then choosing ε_i, η_j such that $\varepsilon_i a_i \geqslant 0$ and $\eta_j b_j \geqslant 0$ in L, we see that the meet in question consists of 0 alone.

We begin the proof of the converse with

LEMMA 2. *If a partial order P of R has property (**), then for any $a \in R$ either $H(P, a)$ or $H(P, -a)$ defines a partial order P', which again satisfies (**).*

If both $H(P, a)$ and $H(P, -a)$ violate (**), then there are elements $a_1, \ldots, a_n$, $b_1, \ldots, b_m$, $a_1', \ldots, a_k'$, $b_1', \ldots, b_l'$ such that for every choice of signs ε_i, η_j, ε_r', η_t' we have

$$H(P, a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n) \cap -H(\eta_1 b_1, \ldots, \eta_m b_m) \supset 0$$

and

$$H(P, -a, \varepsilon_1' a_1', \ldots, \varepsilon_k' a_k') \cap -H(\eta_1' b_1', \ldots, \eta_l' b_l') \supset 0\ .$$

But then always

$$H(P, \varepsilon a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n, \varepsilon_1' a_1', \ldots, \varepsilon_k' a_k') \cap -H(\eta_1 b_1, \ldots, \eta_m b_m, \eta_1' b_1', \ldots, \eta_l' b_l') \supset 0 \qquad (\varepsilon = \pm 1),$$

contrary to the fact that P satisfies (**). Now put $P' = H(P, a)$ or $H(P, -a)$ according as $H(P, a)$ or $H(P, -a)$ possesses (**). Then P' is a semiring in R which contains 0 and (**) guarantees that P' contains no other element along with its negative ($H(P') \cap -H(\varepsilon b) = 0$ implies $\varepsilon b \notin P'$).

Now taking a maximal partial order L in the set of all extensions of P satisfying (**), we infer that, for any $a \in R$, L contains either a or $-a$. This concludes the proof of Theorem 2.

We turn our attention to the corollaries of this result.

COROLLARY 7. (Podderyugin [14].) *An arbitrary ring R can be linearly ordered if and only if for any $a_1, \ldots, a_n$ in R it is possible to select signs $\varepsilon_i = 1$ or -1 such that no sum of non-zero products of $\varepsilon_1 a_1, \ldots, \varepsilon_n a_n$ may vanish.*

In fact, if we set $P = 0$ in Theorem 2, then (**) expresses the same thing as is stated in this corollary.

COROLLARY 8. *A necessary and sufficient condition for a partial order P of a ring R without divisors of zero to be extensible to a linear order of R is the following: for any finite set of elements $a_1, \ldots, a_n$ of R, $a_i \neq 0$, no sum of products containing each a_i an even number of times and eventually (an arbitrary number of times) elements $\neq 0$ of P as factors may vanish.*

If $x_1, \ldots, x_s$ are products of the stated kind, built up from elements $a_1, \ldots, a_n$ of R, and $x_1 + \ldots + x_s = 0$, then $H(P, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n)$ contains for any choice of ε_i the elements $x_1\ (\neq 0)$ and $x_2 + \ldots + x_s = -x_1$, so that the meet of $H(P, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n)$ with $H(\varepsilon x_1)$ is never 0. By virtue of Theorem 2, the condition is necessary.

Conversely, assume that the condition holds in the ring R containing no divisors of zero. If for some sets $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ in R we always have

$$H(P, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n) \cap -H(\eta_1 b_1, \ldots, \eta_m b_m) \supset 0\,,$$

then take $h_1 = -h_2 \neq 0$ ($h_1 \in H(P, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n)$, $h_2 \in H(\eta_1 b_1, \ldots, \eta_m b_m)$) for every choice of ε_i and η_j. The product s of all these $h_1 + h_2$ is zero; it may be written in the form $s = s_0 + s_1$ where s_0 (s_1) is the sum of terms containing a_1 an even (odd) number of times; evidently s_0 or s_1 exists. Since s remains zero under any choice of signs ε_i, η_j of a_i, b_j, by the transition $a_1 \to -a_1$ we obtain $s_0 - s_1 = 0$, whence $2s_0 = 0$. Now starting with (7) $2s_0$ rather than s and following the same procedure with a_2, etc. we obtain successively non-zero terms with zero sum until we arrive at a sum where each term contains each of $a_1, \ldots, a_n, b_1, \ldots, b_m$ an even number of times. This contradicts the hypothesis, and therefore, owing to Theorem 2, P is extensible to a linear order.

From the statement just proved follows at once:

Corollary 9. (Johnson [6], Podderyugin [14].) *A ring without divisors of zero can be linearly ordered if and only if no sum of products containing each factor a_i ($\neq 0$) an even number of times vanishes.*

Corollary 10. *Let P be a partial order of a skewfield F. P has a linear extension in F if and only if no sum of terms of the form $b_1^2 \ldots b_n^2$ or $pb_1^2 \ldots b_n^2$ ($p \in P$, $b_i \in F$; $p, b_i \neq 0$) vanishes.*

By the associative law, the existence of inverses, and using the rule $aba = b(b^{-1})^2(ba)^2$ we may write each term indicated in Corollary 8 in the form $b_1^2 \ldots b_n^2$ or $pb_1^2 \ldots b_n^2$.

If F is commutative, i. e. if it is a field, then the terms may be brought to the form b^2, pb^2, and we obtain the theorem of Serre [16].

Corollary 11. (Szele [18].) *A skewfield F can be linearly ordered if and only if -1 cannot be represented as a sum of products $a_1^2 \ldots a_n^2$ ($a_i \in F$, $a_i \neq 0$).*

Take $P = 0$ in the preceding corollary and observe that if

$$a_1^2 \ldots a_n^2 + b_1^2 \ldots b_m^2 + \ldots = 0 \quad (a_i \neq 0,\ b_j \neq 0, \ldots)\,,$$

then

$$b_1^2 \ldots b_m^2 (a_n^{-1})^2 \ldots (a_1^{-1})^2 + \ldots = -1\,.$$

Corollary 12. (Artin-Schreier [1]). *A field admits a linear order if and only if it is formally real in the sense that -1 is no sum of squares.*

(7) If s_0 fails to exist, then we proceed with s_1^2.

Corollary 13. *For a partial order P of an arbitrary ring R to be a conjunction of linear orders it is necessary and sufficient that it satisfies the following condition: if for some $a_1, \ldots, a_n, b_1, \ldots, b_m$ in R*

(1) $$H(P, -a, \varepsilon_1 a_1, \ldots, \varepsilon_n a_n) \cap -H(\eta_1 b_1, \ldots, \eta_m b_m) \supset 0$$

holds for any choice of signs ε_i, η_j, then $a \in P$.

Assume that P is a conjunction of linear orders. Then $a \notin P$ implies the existence of a linear extension L of P such that $a \notin L$. Hence $-a \in L$ and, by Theorem 2, $H(P, -a)$ has property (**). Thus (1) does not always hold.

If the condition concerning P is fulfilled and $a \notin P$, then $H(P, -a)$ has property (**), so that it can be extended to a linear order L. This L contains $-a$, thus it does not contain a; consequently, the meet of all linear extensions of P contains no element that is not in P.

Note that if P satisfies the condition of Corollary 13, then it contains all sums of products whose factors occur an even number of times. In fact, if x is such a sum and $a_1, \ldots, a_n$ are the factors occurring in x, then the meet of $H(P, -x)$ with $-H(\varepsilon_1 a_1, \ldots, \varepsilon_n a_n)$ always contains $-x$, whence $x \in P$.

Corollary 14. *A partial order P of a ring R without divisors of zero is a conjunction of linear orders if and only if, for any $a \in R$, $a \notin P$, no sum of products whose factors are $-a$, non-zero elements of P and an even number of times non-zero elements of R may vanish.*

The proof is on the same lines as in Corollary 8, making use of Corollary 13 rather than Theorem 2.

Corollary 15. *A partial order P of a skewfield F is representable as a conjunction of linear orders if and only if*

(a) *the non-zero elements of P form a multiplicative group;*

(b) *P contains all sums of products of squares.*

If P is so representable, then $a = q^{-1}p$ ($p, q \in P$, $q \neq 0$) implies that $q(-a) \cdot 1^2 + p \cdot 1^2$ vanishes, and from the preceding corollary we get $a \in P$, whence (a) holds. (b) is an immediate consequence of the remark made after the proof of Corollary 13. Conversely, if (a) and (b) hold, then the sums in Corollary 14 may be brought to the form $q(-a) + p$ ($p, q \in P$); consequently the condition suffices, as asserted.

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Sur le prolongement des homéomorphies

par

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1. A. Kirkor a montré que A et B étant des ensembles compacts, homéomorphes, de dimension nulle et situés sur des sphères à $n>1$ dimensions, toute homéomorphie entre leurs compléments à ces sphères se laisse prolonger aux sphères tout entières. Sa démonstration (non publiée) fait un emploi essentiel des propriétés de polyèdres. Il a posé donc le problème de la validité du théorème en question pour les espaces plus généraux que les sphères.

Il sera démontré ici que ce théorème se laisse généraliser aux continus localement connexes arbitraires X et Y qui ne sont pas coupés localement (au sens qui sera précisé plus loin) par leurs sous-ensembles A et B compacts de dimension nulle (voir le théorème 3). L'hypothèse que A et B ne coupent localement X et Y respectivement est si forte que celle de l'homéomorphie entre A et B devient superflue: elle résulte de la démonstration. Cependant, cette hypothèse n'est qu'une condition suffisante pour l'existence du prolongement $h^*(X)=Y$ d'une homéomorphie $h(X-A)=Y-B$ quelconque. En effet, $A=B$ étant composé d'un seul point p qui coupe un segment rectiligne $X=Y$, l'identité (de même que toute homéomorphie) entre $X-A$ et $Y-B$ se laisse prolonger à celle entre X et Y.

Le problème d'une condition à la fois suffisante et nécessaire est donc ouvert, de même que celui des généralisations aux espaces qui ne sont pas des continus localement connexes.

Il est toutefois à observer que l'hypothèse de la dimension nulle de A et B est essentielle, même en admettant que ces ensembles sont homéomorphes et non-denses dans X et Y respectivement. Soient en effet X le solide de révolution du cercle $(x-1)^2+y^2\leqslant 1$ autour de l'axe des y, A le segment $0\leqslant x\leqslant 1$ de l'axe de x, Y la sphère massive $x^2+y^2+z^2\leqslant 1$ et B le segment $-1\leqslant y\leqslant 1$ de l'axe des y. Alors $X-A$ et $Y-B$ sont homéomorphes, de même que A et B, ni A n'est une coupure locale (voir plus loin, p. 179) de X, ni B de Y, et on a $\overline{X-A}=X$, de même que $\overline{Y-B}=Y$. Cependant, X n'est pas homéomorphe à Y. À plus forte rai-