Two remarks on my paper: "On the ideals' extension theorem and its equivalence to the axiom of choice"

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I. The proof, given in paper [1], stating that the following statement:

(T_1) *Let A be a subset of elements of a Boolean algebra B and let I be an ideal of B which is disjoint with A. Then there exists an ideal I^* including I which is disjoint from A and maximal with respect to this property.*

implies the axiom of choice is false, namely the systems F_τ mentioned on page 48_8 are not filters of the Boolean algebra of all subsets of the space P; they are filters of the lattice of all closed subsets of the space P (in fact, the union of a closed set and of an arbitrary set is not necessarily closed, whence it does not necessarily belong to F_τ). Thus, the reasoning of paper [1] shows only that the following statement implies the axiom of choice:

(T_1) *Let A be a non-empty subset of elements of a distributive lattice S with the maximal element 1 and let I be an ideal of S which is disjoint with A. Then there exists an ideal I^* including I which is disjoint from A and maximal with respect to this property.*

II. We give another result related to this topic. Consider the following statement:

(T_2) *Let S be a lattice with the maximal element 1 and let I be a proper ideal of S. Then there exists a maximal proper ideal I^* including I.*

We shall show that statement (T_2) implies the axiom of choice. First let us observe that (T_2) is equivalent to an analogous statement on filters:

(T_2') *Let S be a lattice with the minimal element 0 and let F be a proper filter of S. Then there exists a maximal proper filter F^* including F.*

To prove this, it suffices to consider the lattice ordered by the converse relation.

We shall show that statement (T_2') implies the following statement:

(T_3) *Let S be a lattice with 0 and 1 and let $\mathfrak{R} = \{F_\tau\}_{\tau \in T}$ be a family of proper filters of the lattice S. Then there exists a family $\mathfrak{R}^* = \{F_\tau^*\}_{\tau \in T}$ of maximal proper filters of S such that $F_\tau \subset R_\tau^*$ for each τ in T.*

Proof. Let $\bar{S}$ be the set composed of all functions defined on T with values from S and such that $f(\tau)\neq 0$ for each τ in T and of the function which is identically equal to 0 (the last will be denoted in the sequel as $\bar{0}$). The relation $\leqslant$ defined by the condition "$f\leqslant g$ *if and only if* $f(\tau)\leqslant g(\tau)$ *for each* τ *in* T" partly orders the set $\bar{S}$. Let us observe that each two elements of S have the least upper bound and the greatest lower bound; indeed, the least upper bound $f\vee g$ of functions $f, g\in\bar{S}$ is the function h defined by the equality

$$h(\tau)=f(\tau)\vee g(\tau)$$

and the greatest lower bound $f\wedge g$ of functions $f, g\in\bar{S}$ is the function h defined by the equality

$$h(\tau)=f(\tau)\wedge g(\tau) \quad \text{if} \quad f(\tau)\wedge g(\tau)\neq 0 \quad \text{for each} \quad \tau \text{ in } T;$$

$$h=\bar{0} \quad \text{in the opposite case.}$$

Hence the set $\bar{S}$ constitues a lattice and clearly $\bar{0}$ is the minimal element of $\bar{S}$. Let $\bar{F}$ be the set of all $f\in\bar{S}$ such that $f(\tau)\in F_\tau$ for each t in T. Let us observe that the set $\bar{F}$ is non-empty; indeed, the function identically equal to 1 belongs to $\bar{F}$. It can easily be proved that $\bar{F}$ is a proper filter of $\bar{S}$. In virtue of (T_2) there exists a maximal proper filter $\bar{F}^*$ including $\bar{F}$. For each $\tau_0\in T$ we define $F^*_{\tau_0}$ as the set of all $a\in S$ for which there is $f\in\bar{F}^*$ with $f(\tau_0)=a$. Clearly, $F^*_{\tau_0}$ is a maximal proper filter of S which contains F_{τ_0} and therefore the family $\mathfrak{N}^*$ of all F^*_τ $(t\in T)$ is the required family.

The axiom of choice can be deduced from (T_3) in an analogous manner to that used in [1] in the proof stating that lemma 2 implies the axiom of choice. An exact proof may be omitted here; one must only observe that the sets F_τ mentioned in the proof are filters of the lattice of all closed sets of the space P.

Reference

[1] S. Mrówka, *On the ideals' extension theorem and its equivalence to the axiom of choice*, Fund. Math. 43 (1955), p. 46-49.

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Reçu par la Rédaction le 22. 11. 1957

Note on ordered groups and rings

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Several criteria are known which ensure the existence of a linear order in groups and rings, in particular in abelian groups, in skewfields and fields (1). It is also known when a partial order of a group can be represented as conjunction of linear orders. But neither the same question for rings has as yet been answered nor a general criterion seems to have been stated explicitly under which a given partial order of a group or a ring can be extended to a linear one. Of course, it is not hard to formulate such conditions by standard methods of the theory of ordered groups and rings. The method which we follow below — although it contains no essentially new ideas — is simpler than the known ones, runs entirely parallel in groups and rings, and has the advantage that a great variety of previously known results may be derived from it in a quite simple manner (2).

§ 1. Groups. Let G be a group whose operation is written as multiplication. A *partial order* $\geqslant$ in G is a reflexive, antisymmetric and transitive relation defined for certain pairs a, b of elements of G such that $a\geqslant b$ implies $cad\geqslant cbd$ for all $c, d\in G$. a $(\in G)$ is called *positive* if $a\geqslant e$ for the group identity e, is strictly positive if $a>e$. The set of all positive elements of G is an invariant semigroup P in G (i. e. $a\in P$ and $b\in P$ imply $ab\in P$, and $a\in P$, $g\in G$ imply $g^{-1}ag\in P$) which contains e but no other element along with its inverse. The set P completely determines the partial order, for $a\geqslant b$ if and only if $ab^{-1}\in P$; thus we may denote a partial order $\geqslant$ and the set of positive elements under $\geqslant$ by the same symbol P. A partial order Q is said to be an *extension* of P if $Q\supseteq P$ (3). If Q_ν is a set of extensions of P such that their meet is P, then we call P the *conjunction* of the Q_ν. If for every $a\in G$ either a or a^{-1} belongs to P, then P defines a *linear order* in G in the sense that for any two elements

(1) See the References at the end of this note. (Numbers in brackets refer to it.)

(2) It is not difficult to see that our method can be applied to other algebraic systems too.

(3) The sign $\supseteq$ denotes inclusion, while $\supset$ is used to mean proper inclusion.