Continuous associative multiplications in locally triangulable spaces

by

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A main problem on topological semi-groups is to answer the following question formulated by A. D. Wallace ([9], p. 96) (1): What compact connected Hausdorff spaces admit a continuous associative multiplication with two-sided unit? Since there have already been numerous results on the structure of topological groups, we may restrict our interest only to those continuous associative multiplications with two-sided unit which fail to be topological group operations. These multiplications will be conveniently called *essential multiplications*. Then we reformulate the problem as follows.

WALLACE'S PROBLEM. *Determine if a given compact connected Hausdorff space admits an essential multiplication.*

This problem is far from being solved. However, Wallace has settled a few interesting special cases stated in the following theorem, [9] and [10].

WALLACE'S THEOREM. *A compact connected Hausdorff space S admits no essential multiplication if one of the following conditions is satisfied:*

(1) *S is indecomposable.*

(2) *S is a manifold.*

(3) *S is homogeneous and* 1*-dimensional.*

The objective of the present paper is to investigate Wallace's problem for the class of locally triangulable spaces defined as follows.

Let S be a given topological space. A point $x \in S$ is called a *conic point* of S if there exists an open neighborhood U of x in S such that the closure $\mathrm{Cl}(U)$ is the join of the frontier $\mathrm{F}(U)$ to the point x; in other words, $\mathrm{Cl}(U)$ is topologically the cone over $\mathrm{F}(U)$ with x as vertex. Besides, if $\mathrm{F}(U)$ is a triangulable space, then S is said to be *locally triangulable at the point x*. We say that S is a *locally triangulable space* if it is locally triangulable at every point. In particular, every triangulable

(1) Numbers in brackets refer to the bibliography at the end of the paper.

space is locally triangulable. On the other hand, all manifolds are locally triangulable (since they are locally Euclidean), while it is still unknown if they are triangulable.

In a given compact connected locally triangulable space, the existence of an essential multiplication implies the existence of unstable points defined as follows.

A point x of a topological space S is said to be *unstable* if, for every open neighborhood U of x in S, there exists a homotopy

$$d_t\colon S\to S \qquad (0\leqslant t\leqslant 1),$$

satisfying the following conditions:

(D1) d_0 is the identity map.

(D2) $d_t(U)\subset U$ for every $t\in I$.

(D3) $d_t(x)=x$ for each $x\in S\setminus U$ and $t\in I$.

(D4) $d_1(S)\subset S\setminus x$.

Unstable points are called *labile points* by Hopf and Pannwitz [4]. See also Alexandroff and Hopf [1], p. 523, and Borsuk and Jaworowski [2]. A point $x\in S$ is said to be *stable* if it is not unstable. The set of all unstable points of S will be called the *boundary* of S, denoted by ∂S.

Now let x be a conic point of S and U an open neighborhood of x in S such that $\mathrm{Cl}(U)$ is the join of $\mathrm{F}(U)$ to the point x. Then the following lemma is obvious.

LEMMA. *The point x of S is unstable if, and only if, the frontier $\mathrm{F}(U)$ of U is contractible.*

Because of this lemma, it is very easy to determine the boundary of a given triangulable or locally triangulable space.

The main assertion of the paper is stated in the form of the following

THEOREM. *If a compact connected Hausdorff space S admits an essential multiplication with a conic point u as the two-sided unit, then u is an unstable point of S.*

In other words, if there is given in a compact connected Hausdorff space S a continuous associative multiplication with a two-sided unit u which is a stable conic point of S, then S becomes a topological group under this multiplication.

In particular, if the point u has a Euclidean neighborhood in S, this theorem reduces to a known result due to Mostert and Shields [5].

A proof of the theorem will be given at the end of the paper. Now let us deduce its immediate consequences.

COROLLARY 1. *If S is a compact connected locally triangulable space without unstable points, then S admits no essential multiplication.*

This follows from the fact that every point of a locally triangulable space is a conic point. A few special cases of this corollary will be listed as follows.

(1.1) *Every compact connected manifold admits no essential multiplication.* Hence, if the underlying space of a clan ([2]) is a manifold, then it must be a Lie group.

(1.2) *Every compact connected linear graph without end-points admits no essential multiplication.* Hence, if the underlying space of a clan is a linear graph without end-points, then it must be the circle group.

(1.3) *The one-point union of a finite number of compact connected manifolds admits no essential multiplication.* In fact, let $M_1, \ldots, M_n$ be compact connected manifolds. In each M_i, pick a point x_i. Then the one-point union $S=M_1\vee\ldots\vee M_n$ is defined to be the quotient space obtained from the disjoint union $M_1\cup\ldots\cup M_n$ by identifying the points $x_1, \ldots, x_n$ to a single point. This space S can be imbedded in the product space $M_1\times\ldots\times M_n$ in an obvious way. Our assertion follows from the fact that S has no unstable point. If $n>1$, then S is not homogeneous and hence cannot be the underlying space of a topological group. Therefore, the one-point union of two or more manifolds cannot be the underlying space of a clan.

(1.4) *The quotient space S obtained from a compact connected manifold M by identifying a finite number of finite sets into single points admits no essential multiplication.* This follows from the fact that S has no unstable point. Except the trivial case $S=M$, S is not homogeneous and hence it cannot be the underlying space of a clan.

The assertion (1.1) is a restatement of the case (2) of Wallace's theorem, and (1.2) gives partial generalization of the case (3) of Wallace's theorem. Some of these special cases may also be proved by a general theorem of Wallace, namely, if a clan S is a floor for some cohomology class of S, then S is a topological group, [8] and [9] (p. 106).

COROLLARY 2. *For a given essential multiplication in a compact connected locally triangulable space S, every point of S which has a two-sided inverse is an unstable point.*

This follows from the fact that, if $h\in S$ has a two-sided inverse, then the assignment $x\to hx$ defines a homeomorphism of S which carries

([2]) A clan is a compact connected Hausdorff space furnished with a continuous associative multiplication with two-sided unit. See [9].

the two-sided unit u into h. Let $I(S)$ denote the set of all points of S with two-sided inverse. According to Wallace ([7], p. 333), $I(S)$ is a compact topological group. Then Corollary 2 states that $I(S)$ is contained in the boundary $\partial(S)$ of S. A few special cases of this corollary will be listed as follows.

(2.1) *For a given essential multiplication in a compact connected linear graph S, $I(S)$ is a finite group which consists of some end-points of S.*

(2.2) *For a given essential multiplication in a compact connected manifold S with a regular boundary B, the group $I(S)$ is contained in B.* Furthermore, if B is connected and $BB \subset B$, then it follows from (1.1) that $I(S) = B$. This is a known result due to Mostert and Shields [5].

COROLLARY 3. *For any given essential multiplication in a compact connected subspace S of a locally triangulable space X with no unstable point, the group $I(S)$ is contained in the frontier $\mathrm{F}(S) = S \setminus \mathrm{int}(S)$ in X.*

This corollary can be deduced from the theorem as follows. Since every interior point of S is a stable conic point of S, it follows by the theorem that the two-sided unit u must be in $\mathrm{F}(S)$. Now let h be any point in $I(S)$. Define a new multiplication in S by taking $x \ast y = xh^{-1}y$, where h^{-1} denotes the two-sided inverse of h. Then it is easily verified that this new multiplication is an essential multiplication with h as the two-sided unit. Hence $h \in \mathrm{F}(S)$. This proves that $I(S) \subset \mathrm{F}(S)$. A few special cases of this corollary will be listed as follows.

(3.1) *If a clan S is topologically contained in the Euclidean n-space R^n, then $I(S)$ is contained in* $\mathrm{F}(S)$. In fact, if S is not a topological group, then this follows from the corollary by taking $X = R^n$. On the other hand, if S is a topological group, then it is easy to see that $I(S) = S = \mathrm{F}(S)$. For $n \geqslant 2$, this is a known theorem of Wallace ([9], p. 97).

(3.2) *If a clan S is topologically contained in a connected manifold M, then either $I(S) \subset \mathrm{F}(S)$ or $S = M$.* In fact, if S is not a topological group, then this follows from the corollary by taking $X = M$. On the other hand, if S is a topological group, then we have either $I(S) = S = \mathrm{F}(S)$ or $I(S) = M$. This is a known result due to Mostert and Shields [5].

Proof of the theorem. Let S be a clan with a two-sided unit u which is a stable conic point of the space S. We are going to prove that S is a topological group.

Since u is a conic point of S, there exists an open neighborhood W of u in S such that the closure $\mathrm{Cl}(W)$ is the join of the frontier $B = \mathrm{Cl}(W) \setminus W$ to the point u. Every point $w \in W \setminus u$ can be uniquely represented as

$$w = (b, t), \quad b \in B, \quad 0 < t < 1.$$

Let N denote the closed neighborhood of u which consists of the points $(b, t) \in W$ for all $b \in B$ and $0 < t \leqslant \frac{1}{2}$ together with the point u itself. Let $C \subset N$ denote the set of the points $(b, \frac{1}{2})$ for all $b \in B$. Then both N and C are compact. By continuity of the multiplication, there exists a neighborhood V of u such that

$$VN \subset W, \quad VC \subset W \setminus u.$$

We may assume that V is pathwise connected.

ASSERTION I. *Every element $v \in V$ has a right inverse in S.*

Proof. Assume that $v \in V$ has no right inverse in S. Then $vX \subset X \setminus u$ and hence

$$vN \subset W \setminus u.$$

Since V is pathwise connected, we may pick a path $\sigma\colon I \to V$ such that

$$\sigma(0) = u, \quad \sigma(1) = v.$$

Let $p\colon W \setminus u \to B$ and $q\colon B \to C$ denote the maps defined by

$$p(b, t) = b, \quad q(b) = (b, \tfrac{1}{2}) \quad (b \in B,\ 0 < t < 1).$$

Define a homotopy $h_t\colon B \to B$ $(0 \leqslant t \leqslant 1)$ by setting

$$h_t(b) = p[\sigma(t) \cdot q(b)] \quad (b \in B,\ 0 \leqslant t \leqslant 1).$$

Then h_0 is the identity map on B.

Since N is contractible to the point u, there exists a homotopy q_t: $B \to N$ $(0 \leqslant t \leqslant 1)$ such that $q_0 = q$ and $q_1(B) = u$.

Define a homotopy $k_t\colon B \to B$ $(0 \leqslant t \leqslant 1)$ by setting

$$k_t(b) = p[v \cdot q_t(b)] \quad (b \in B,\ 0 \leqslant t \leqslant 1).$$

Then $k_0 = h_1$ and $k_1(B) = p(v)$. Hence B is contractible. According to the lemma, this implies that u should be an unstable point of S. This contradiction proves the assertion.

ASSERTION II. *The set $R(S)$ of all elements of S which have right inverses is open.*

Proof. Let $a \in R(S)$. Then there exists an element $b \in S$ with $ab = u$. By continuity of the multiplication, there exists a neighborhood M of a such that $Mb \subset V$. Then it suffices to prove that every $x \in M$ has a right inverse. For this purpose, let $v = xb$. Since $v \in V$, it follows from Assertion I that there is an element $c \in S$ such that $vc = u$. Then we have

$$x(bc) = (xb)c = vc = u.$$

Hence x has a right inverse. This proves the assertion.

Assertion III. *The set $R(S)$ is closed.*

Proof. It suffices to prove that $S\backslash R(S)$ is open. Let $a \in S\backslash R(S)$. Then aS is contained in $S\backslash u$. By continuity of the multiplication, for each $x \in S$, there exists a neighborhood P_x of a and a neighborhood Q_x of x such that $P_xQ_x \subset X\backslash u$. Since S is compact, there exists a finite number of points $x_1, \ldots, x_n$ of S such that

$$S = Q_{x_1} \cup Q_{x_2} \cup \ldots \cup Q_{x_n}.$$

Let

$$P = P_{x_1} \cap P_{x_2} \cap \ldots \cap P_{x_n}.$$

Then P is a neighborhood of a and PS is contained in $S\backslash u$. This proves the assertion.

Since S is connected, the assertions II and III imply that $R(S) = S$. Therefore, S is algebraically a group under the given multiplication of the clan. Then, by a classical result (Gelbaum-Kalisch-Olmsted, Iwasawa, Peck), it follows that S is a topological group. For the sake of completeness, however, we shall establish this by the following

Assertion IV. *The function f: $S \to S$ defined by $f(x) = x^{-1}$ is continuous.*

Proof. Let A be any closed set in S. It suffices to prove that the inverse image $B = f^{-1}(A)$ is closed in S or, equivalently, that $S\backslash B$ is open. Let $x \in S\backslash B$. Then xA is contained in $S\backslash u$. Since A is compact, there exists a neighborhood U of x such that UA is also contained in $S\backslash u$. This implies that $U \subset S\backslash B$ and hence $S\backslash B$ is open. This proves the assertion.

This completes the proof of the theorem.

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