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The independence of various definitions of finiteness *

by

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In this paper we deal with concepts which are characteristic of a set theory without the axiom of choice. Since the independence of that axiom in a set theory which satisfies the axiom of foundation and permits no *Urelemente* has not yet been proved, we deal with a set theory which admits the existence of *Urelemente*. Recently Mendelson [4] has proved that the axiom of choice is also independent in a system which admits no *Urelemente* but, instead of satisfying the axiom of foundation, satisfies only weaker forms of that axiom. Mendelson has observed that Mostowski's results, which will be mentioned later, are valid also in his system. The same can also be said about the results of this paper. Those theorems in this paper which are true also for systems of axioms which include the axiom of foundation and admit no *Urelemente* will be denoted by an asterisk.

Almost all the results of the present paper regarding the independence of definitions of finiteness were announced by Mostowski and Lindenbaum in [3] and [5]. Some of them are proved in Mostowski [6]. All this was done for systems in which the ordering principle (the statement that every set can be ordered) is not provable. In the present paper the independence of the definitions of finiteness is also examined when the ordering axiom is assumed. One result in this direction was obtained by Doss [1].

The proofs in this paper are based on the models constructed by Mostowski in [7]. $\mathfrak{M}$ will denote the general model of Mostowski constructed in [7], § 3 and $\mathfrak{M}^+$ will denote the special model defined in [7], § 4 which satisfies the ordering axiom. The same letters will also denote, respectively, the universal classes of the models, without causing any confusion. The knowledge of the general features of these models will

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be assumed throughout the paper. Our notations are generally adapted to those of Mostowski in [7]. By „set theory" we shall mean the set theory of von Neumann in which the model $\mathfrak{M}$ is constructed. The consistency of the set theory will be assumed tacitly in all the metamathematical theorems in this paper. We shall not use any special terminology for the relativised concepts of the model. In the cases where the concepts of the model and the set theory coincide no confusion can arise; in other cases the phrase "in the model" or "of the model" will be added to denote the relativised concepts wherever an ambiguity may arise. For example, by "$A \sim B$ in the model" we mean that between the sets A and B of $\mathfrak{M}$ holds the relation of the equivalence of sets relativised to the model. $\mathfrak{S}$ will denote the axiomatic system of Mostowski ([7], § 1 — axioms 1-18), which is based on the axiomatic system of Bernays. The main difference between the systems of Bernays and Mostowski is that the latter permits *Urelemente*. $\mathfrak{S}$ is satisfied by $\mathfrak{M}$. $\mathfrak{S}^+$ will denote the system which consists of the axioms of $\mathfrak{S}$ plus the ordering axiom. $\mathfrak{S}^+$ is satisfied by $\mathfrak{M}^+$. Mostowski has proved in [7] that the axiom of choice does not hold in $\mathfrak{M}^+$; therefore the axiom of choice is not provable in $\mathfrak{S}$ or $\mathfrak{S}^+$.

By $\overline{\overline{A}}$ we shall mean the cardinal of the set A. We shall generally use this notation to denote cardinals of sets in the models with which we shall deal. We do not need any formal definition of the concept of the cardinal since any statement about cardinals can be understood as an abbreviation of a statement about sets.

DEFINITIONS OF FINITENESS.

I [1]. A is finite if every non-void family [2] of subsets of A has a maximal element (an element which is not a proper subset of any element of A).

We get an equivalent definition if we replace "maximal" by "minimal" (cf. Tarski [8]).

Ia. A is finite if it is not the union of two disjoint sets neither of which is finite according to definition I.

II [1]. A is finite if every non-void monotonic family (a family which is completely ordered by inclusion) of subsets of A has a maximal element.

We get an equivalent definition if we replace "maximal" by "minimal" (the proof is the same as in the case of definition I).

III [1]. A is finite if the power-set of A is irreflexive [3].

IV [1]. A is finite if it is irreflexive.

V [1]. A is finite if $\overline{\overline{A}} = 0$ or $2\overline{\overline{A}} > \overline{\overline{A}}$.

VI [4]. A is finite if $\overline{\overline{A}} = 0, 1$ or $\overline{\overline{A}}^2 > \overline{\overline{A}}$.

VII. A is finite if $\overline{\overline{A}}$ is not an aleph greater than $\aleph_0$ or equal to it.

All these definitions are equivalent if one assumes the axiom of choice. In fact, definition VII is the most inclusive definition that becomes equivalent to definition I if the choice axiom is assumed. What we usually mean by "finite" is finite according to I; therefore sets finite according to I will be simply called finite. The general properties of finite sets, which are proved in [8], are assumed to be known.

*THEOREM 1. *If a set is finite according to any of the above definitions it is finite also according to any definition which follows it.*

Proof. It is of course sufficient to prove that if a set A is finite according to any of the definitions then it is finite also according to the definition immediately following it. We shall prove that every set finite according to Ia is also finite according to II. The proof that every set finite according to II is also finite according to III was obtained by Kuratowski and may be found in [8]. The proofs of the other implications are easy.

Let A be finite according to Ia. Let us assume that A is infinite according to II and arrive at a contradiction. By assumption there exists a monotonic family P of subsets of A which has no maximal element. Assume further that every set of P is finite. P, being monotonic, is completely ordered by inclusion, whence the last assumption implies that every section of P is finite (Q is a section of P if there is an element x of P such that $Q = \{y; y \in P, y < x\}$). In this case it follows easily that the order type of P is a finite ordinal or ω. The first possibility is excluded by the assumption that P has no maximal element, therefore $P = \{A_0, A_1, A_2, \ldots\}$, $A_i \subset A_{i+1}$ for each $i \in \omega$. Now let

$$B = \sum_{i=0}^{\infty} (A_{2i+2} - A_{2i+1}), \qquad C = A - B.$$

Of course

$$C \supseteq \sum_{i=0}^{\infty} (A_{2i+1} - A_{2i}).$$

Neither B nor C are finite since the families

$$\left\{\sum_{i=0}^{n} (A_{2i+2} - A_{2i+1});\ n \in \omega\right\} \quad \text{and} \quad \left\{\sum_{i=0}^{n} (A_{2i+1} - A_{2i});\ n \in \omega\right\}$$

[1] This is one of the definitions of Tarski in [8].

[2] By "family" we mean a set of sets.

[3] A set is reflexive if it is equivalent to one of its proper subsets.

[4] This is a definition of Tarski mentioned in [5].

do not satisfy the conditions of definition I for B and C respectively. This cannot hold since we have assumed that A is finite according to Ia. Therefore the assumption that all the sets of P are finite is not true. Now let B be an infinite element of P. Define $C = A - B$. C is infinite because if C were finite A could have only a finite number of subsets which include B and therefore P should have a maximal element. Now we have arrived at a contradiction because we have proved that A is the union of the disjoint infinite sets B and C, contrary to the assumption that A is finite according to Ia.

In this paper we shall prove the independence of definitions II-VII in the system $\mathfrak{S}^+$ and the independence of definitions I, Ia, II in $\mathfrak{S}$. The definitions I, Ia, II will be shown to be equivalent in $\mathfrak{S}^+$ (i. e. when the ordering axiom is assumed).

*THEOREM 2. *If the set A is finite according to* II *and can be ordered, then it is finite* (5).

Proof. Let A be ordered in any order. We shall prove that A is well-ordered. Let B be a non-void subset of A. We have to prove that B has a first element. Let P be the family of the subsets of B which are sections of B. P is a monotonic family of subsets of A; and since A is finite according to II, P has a minimal set C. C is a section of B; therefore B has an element c such that $C = \{x;\ x \in B,\ x < c\}$. We shall prove that c is the first element of B. If c is not the first element of B, then there is an element d of B such that $d < c$. Let $D = \{x;\ x \in B,\ x < d\}$; then $D \subset C$ (because $d \in C$, $d \notin D$) and $D \in P$, being a section of B. But $D \subset C$ leads to a contradiction since C is minimal in P. Thus we have proved that A is well-ordered by any order of A; and since there exists at least one such order, A is finite. (Cf. [8], Th. 45.)

From Theorems 1 and 2 follows:

*THEOREM 3. *The definitions of finiteness* I, Ia *and* II *are equivalent if the ordering axiom is assumed.*

The basis of all the examples which are constructed in this paper is the set K (cf. [7], 16) of all the *Urelemente* of the model $\mathfrak{W}$. K is denumerable in the set theory. We recall that $\mathfrak{G}$ is a group of biunique mappings of K on itself. $|\varphi, x|$, for $\varphi \in \mathfrak{G}$ and $x \in \mathfrak{M}$ (6) is defined as follows: $|\varphi, x| = \varphi(x)$ for $x \in K$, $|\varphi, A_0| = A_0$ (7) and $|\varphi, x| = \{|\varphi, y|;\ y \in x\}$ other-

(5) Tarski mentions in [8] that defs. II and III are equivalent to def. I in the Euclidean space. Theorem 2 is a generalization of Tarski's statement.

(6) $\mathfrak{M}$ is the following class (cf. [7], 20): Let $\mathfrak{P}(A)$ be the set of all the non-void subsets of A in the set theory. $K_0 = K + \{A_0\}$ (cf. footnote (7)), $K_\xi = \sum_{\eta<\xi} K_\eta + \mathfrak{P}(\sum_{\eta<\xi} K_\eta)$. $\mathfrak{M}$ is the union class of all the K_ξ. By definition of $\mathfrak{W}$, $\mathfrak{W} \subseteq \mathfrak{M}$.

(7) A_0 is the void set of the model. It is not the void set of the set theory.

wise. Since $\left|\varphi^{-1}, |\varphi, x|\right| = x$ and $\left|\varphi, |\psi, x|\right| = |\varphi\psi, x|$ for each $\varphi, \psi \in \mathfrak{G}$ and each $x \in \mathfrak{M}$ (cf. [7], 33 and 36), $\mathfrak{G}$ is a group of biunique transformations of $\mathfrak{M}$ on itself, and the multiplication in $\mathfrak{G}$ is the ordinary multiplication of transformations of $\mathfrak{M}$ on itself. $\mathfrak{G}(A)$, for $A \subset K$, is the subgroup of $\mathfrak{G}$ consisting of the mappings which do not move the elements of A. $\mathfrak{N}(A)$ is the subclass of $\mathfrak{M}$ consisting of all the elements of $\mathfrak{M}$ which are invariant under all the mappings of $\mathfrak{G}(A)$.

For the model $\mathfrak{W}^+$ K is ordered in the same order as the rational numbers in the set theory. We shall use for K in $\mathfrak{W}^+$ the terminology of the rational line. The elements of K will be called *points*, and subsets of K of the types $(a,b) = \{x;\ x \in K,\ a < x < b\}$, $(-\infty, a) = \{x;\ x \in K,\ x < a\}$, $(a, \infty) = \{x;\ x \in K,\ x > a\}$, $(-\infty, \infty) = K$ will be called *open intervals*. We must remember that whenever intervals are mentioned we mean only intervals with "rational" or "infinite" endpoints (excluding intervals like the rational interval $(0, \sqrt{2})$). For $\mathfrak{W}^+$, we take for $\mathfrak{G}$ the group of all the order-preserving maps of K on itself, and denote it by $\mathfrak{G}^+$. $\mathfrak{G}^+(A)$ and $\mathfrak{N}^+(A)$ will denote with respect to $\mathfrak{G}^+$ what $\mathfrak{G}(A)$ and $\mathfrak{N}(A)$ denote with respect to $\mathfrak{G}$.

LEMMA 1. *A set A is a set of the model $\mathfrak{W}$ and is finite in the model if and only if either A is a finite non-void set of elements of $\mathfrak{W}$ in the set theory, or $A = A_0$.*

Proof. If $A = A_0$ then it is the void set of the model and therefore it is finite in the model.

Let A be a finite non-void set of elements of $\mathfrak{W}$. We shall use the following facts about the model $\mathfrak{W}$ which can easily be proved from the definition of $\mathfrak{W}$:

1. If $a \in \mathfrak{W}$ then $\{a\} \in \mathfrak{W}$ and $\{a\}$ is the set of the model containing the single element a in the model.

2. If $B, C \in \mathfrak{W}$, $B, C \subseteq \mathfrak{W}$ then $B + C \in \mathfrak{W}$ and $B + C$ is the union set of B and C in the model.

We shall prove that A is a finite set of the model by induction with respect to the number n of the elements of A. If $n = 1$ then $A = \{a\}$, $a \in \mathfrak{W}$, whence $A = \{a\} \in \mathfrak{W}$. A is finite in the model because it has only one element in the model. For $n > 1$, let a be an element of A; we have $A = (A - \{a\}) + \{a\}$. $A - \{a\}$ is a finite set of the model by the hypothesis of the induction. We have proved that $\{a\}$ is also a finite set of the model; therefore their union A is a set of the model, and A is finite in the model, being the union in the model of two sets finite in the model.

Let A be a set finite in the model, $A \neq A_0$. Let $\mathfrak{U}$ be the class of all the non-void finite subsets of $\mathfrak{W}$ in the set theory. According to the first part of this proof $\mathfrak{U} \subseteq \mathfrak{W}$. The class of the transforms of the elements of $\mathfrak{U}$ by a transformation $\varphi \in \mathfrak{G}$ is $\mathfrak{U}$ itself since the φ and the φ^{-1} transforms of elements of $\mathfrak{W}$ are elements of $\mathfrak{W}$ (cf. [7], 47), and the φ and φ^{-1} transforms of finite sets (in the set theory) are finite sets (the latter statement follows immediately from the definition of $|\varphi, x|$). Hence $\mathfrak{U}$ is a $M, \mathfrak{G}$-*ausgezeichneter Bereich* (cf. [7], 43), i. e., $\mathfrak{U}$ is a class of the model. Let P be the set of the model consisting of all the subsets of A which are elements of $\mathfrak{U}$. The existence of P is guaranteed by the axioms of $\mathfrak{S}$, which are satisfied by the model. A is a set of $\mathfrak{W}$, $A \neq \Lambda_0$, whence $A \subset \mathfrak{W}$ (cf. [7], 52 and 46), and there exists an element a of A such that $a \in \mathfrak{W}$; therefore $\{a\} \in \mathfrak{W}$, $\{a\} \in P$ and $P \neq \Lambda_0$. Since A is finite in the model, definition I and $P \neq \Lambda_0$ imply the existence of a maximal set B of P. We shall prove that $B = A$. If $B \subset A$, then there exists an element a of A such that $a \notin B$. $B \in P$ implies $B + \{a\} \in P$, which cannot hold since B is maximal in P. Hence $B = A$, and therefore $A \in P$ and $A \in \mathfrak{U}$, i. e., A is finite in the set theory.

LEMMA 2. *Let L be a non-void subset of K. $L \in \mathfrak{W}^+$ if and only if L is the non-void union of a finite number of open intervals and a finite number of points of K.*

Proof. Let $L \subseteq K$, $L \in \mathfrak{W}^+$. $L \in \mathfrak{W}^+$ implies the existence of a finite subset A of K such that $L \in \mathfrak{R}^+(A)$ (cf. [7], 43). The points of A divide K into a finite number of open intervals. Let (p, q) (8) be such an interval and let a, b be points in it. There exists a $\varphi \in \mathfrak{G}^+$ which is the identity on the whole of K except the interval (p, q) and maps a on b. $\varphi \in \mathfrak{G}^+(A)$ because φ does not move the points of A. Therefore $|\varphi, L| = L$ and if $a \in L$, then $b = \varphi(a)$ implies $b \in |\varphi, L| = L$. Thus we have proved that if L includes one point of an open interval generated by A it includes the whole interval. Since K is divided by A into a finite number of open intervals and the finite set of the points of A, L must consist of a finite number of open intervals and a finite number of points.

Let $L \subseteq K$, while L consists of a finite number of open intervals and a finite number of points. Let A be the set of the boundary points of L in K. A is finite, and we can easily see that $L \in \mathfrak{R}^+(A)$ (9). Therefore, since L is non-void, it is enough to show that $L \subseteq \mathfrak{W}^+$ in order to have $L \in \mathfrak{W}^+$; but $L \subseteq \mathfrak{W}^+$ follows from $L \subseteq K$ and $K \subseteq \mathfrak{W}^+$.

LEMMA 3. *Let $L \subseteq K$, $L \in \mathfrak{W}^+$. If L is an open (non-void) interval of K then L cannot be well-ordered in the model.*

(8) p is a point or $-\infty$, q a point or ∞.

(9) We recall our convention that every interval has endpoints in K at its finite ends (if any).

Proof. The proof that K cannot be well-ordered in the model, carried out in [7], 111, becomes a proof of this lemma if K is replaced by L throughout the proof.

LEMMA 4. *Let $L \subseteq K$, $L \in \mathfrak{W}^+$, $L \neq \Lambda_0$. If L is not finite then it cannot be well-ordered in the model.*

Proof. According to Lemma 2 L is the union of a finite number of open intervals and a finite number of points. Since L is not finite it must contain at least one open interval M. M is a subset of L in the model according to Lemma 2. If L can be well-ordered in the model then M, too, can be well-ordered in the model, in contradiction to Lemma 3.

LEMMA 5. *Let P be an infinite family of subsets of K in the model $\mathfrak{W}^+$. There exists a family Q and a function χ such that $Q \in \mathfrak{W}^+$, $\chi \in \mathfrak{W}^+$, $Q \subseteq P$, and χ maps Q onto an open interval of K.*

Proof. $P \in \mathfrak{W}^+$ implies the existence of a finite subset A of K such that $P \in \mathfrak{R}^+(A)$. Since $P \neq \Lambda_0$ it follows that $P \subseteq \mathfrak{W}^+$. There is only a finite number of subsets of K the boundaries of which are subsets of A; and since P is not finite there exists an element of P having at least one boundary point which is not in A. Let L be such an element of P; let a be one of its boundary points which is not in A; and let (p, q) be the open interval generated by A which contains a. Let $Q = \{|\varphi, L|; \varphi \in \mathfrak{G}^+(A)\}$. Since $\mathfrak{G}^+(A)$ is a group of transformations of $\mathfrak{M}$, for each $\varphi \in \mathfrak{G}^+(A)$, $|\varphi, M| \in Q$ if and only if $M \in Q$. Therefore $Q \in \mathfrak{R}^+(A)$. $L \in P$ and $P \in \mathfrak{R}^+(A)$ imply that $|\varphi, L| \in P$ for each $\varphi \in \mathfrak{G}^+(A)$, whence $Q \subseteq P$. $Q \subseteq P \subseteq \mathfrak{W}^+$ and $Q \in \mathfrak{R}^+(A)$ imply that $Q \in \mathfrak{W}^+$.

Let $M \in Q$. There exists a $\varphi \in \mathfrak{G}^+(A)$ such that $M = |\varphi, L|$. φ maps a on $\varphi(a)$. We shall show that $\varphi(a)$ is independent of the special mapping chosen from the mappings of $\mathfrak{G}^+(A)$ which map L on M. L and M have the structure mentioned in Lemma 2, whence their boundaries are finite sets. All the order-preserving biunique functions which map L on M map the boundary of L onto the boundary of M in the same manner, since the boundaries are finite. Therefore a is mapped on the same point by all the transformations of $\mathfrak{G}^+$ which map L onto M. Now, since $\varphi(a)$ depends only on M and not on φ, let us write $\varphi(a) = \chi(M)$. $\chi = \{\langle M, \chi(M)\rangle; M \in Q\}$ (10). By the pairing axiom, which the model satisfies, $\langle M, \chi(M)\rangle \in \mathfrak{W}^+$ for each $M \in Q$, and therefore $\chi \subseteq \mathfrak{W}^+$. In order to prove $\chi \in \mathfrak{W}^+$ it is now sufficient to prove $\chi \in \mathfrak{R}^+(A)$, i. e., $|\psi, \chi| = \chi$ for each $\psi \in \mathfrak{G}^+(A)$; but since $\mathfrak{G}^+(A)$ is a group this will follow from $|\psi, \chi| \subseteq \chi$. Let x be an element of χ. By the definition of χ, $x = \langle M, \chi(M)\rangle$ for some $M \in Q$, i. e., $x = \langle|\varphi, L|, \chi(|\varphi, L|)\rangle$ for some $\varphi \in \mathfrak{G}^+(A)$.

(10) $\langle a, b\rangle$ is the symbol for an ordered pair.

$$|\psi, x| = \left|\psi, \langle|\varphi, L|, \chi(|\varphi, L|)\rangle\right| = \left|\psi, \langle|\varphi, L|, \varphi(a)\rangle\right|$$
$$= \langle|\psi, |\varphi, L||, \psi\varphi(a)\rangle = \langle|\psi\varphi, L|, \chi(|\psi\varphi, L|)\rangle .$$

$|\psi\varphi, L| \in Q$ by the definition of Q, whence $|\psi, x| = \langle|\psi\varphi, L|, \chi(|\psi\varphi, L|)\rangle \in \chi$. Thus we have proved $|\psi, \chi| \subseteq \chi$, whence $\chi \in \mathfrak{R}^+(A)$.

We shall prove that the set of the images of the elements of Q by the function χ is the open interval (p, q). For every point b in the interval there exists a mapping $\varphi \in \mathfrak{G}^+(A)$ such that $\varphi(a) = b$, whence $b = \chi(|\varphi, L|)$, $|\varphi, L| \in Q$. On the other hand, each image by χ of an element of Q is of the type $\varphi(a)$, where $\varphi \in \mathfrak{G}^+(A)$; $\varphi(a)$ cannot be outside (p, q) since $p, q \in A$, and φ, being an element of $\mathfrak{G}^+(A)$, cannot carry a across p or q.

The sets Q and χ fulfil the requirements of the lemma.

LEMMA 6. *Let P be a set as required in Lemma 5. P cannot be well-ordered in the model $\mathfrak{W}^+$.*

Proof. By Lemma 5 there exist sets of the model Q and χ such that $Q \subseteq P$ and χ maps Q onto an open interval (p, q) of K. If P can be well-ordered in the model then the same applies to Q. The existence of χ implies that (p, q), too, can be well-ordered in the model, in contradiction to Lemma 3.

LEMMA 7. *The set K and the power-set of K in the model are irreflexive in the model $\mathfrak{W}^+$.*

Proof. If any of the two sets is reflexive in the model it must have a subset which is denumerable in the model (cf. [2], p. 57-58); but Lemmata 4 and 6 imply that neither set can have such a subset.

THEOREM 4. *The definitions of finiteness* I *and* III *are not equivalent in $\mathfrak{S}^+$.*

Proof. We shall prove that the set K is finite in the model $\mathfrak{W}^+$ according to III but is not finite in the same model according to I.

By Lemma 1, K is not finite in the model according to I. K is finite in the model according to III since Lemma 7 states that the power-set of K in the model is irreflexive in the model.

THEOREM 5. *The definitions of finiteness* III *and* IV *are not equivalent in $\mathfrak{S}^+$.*

Proof. Let P be the power-set of K in $\mathfrak{W}^+$. We shall prove that the set P is not finite in the model $\mathfrak{W}^+$ according to III but is finite in the same model according to IV.

K is infinite; therefore the power-set of P in the model is reflexive (because a set is finite if and only if the power-set of its power-set is irreflexive) (cf. [8], p. 74), whence P is not finite in the model according to III. P is finite in the model according to IV since Lemma 7 states that P is irreflexive in the model.

LEMMA 8. *Let $L, M \in \mathfrak{W}^+$, $L, M \subseteq K$. L and M are equivalent in the model if and only if both $L-M$ and $M-L$ are finite in the model and $\overline{\overline{L-M}} = \overline{\overline{M-L}}$.*

Proof. It obviously follows from the axioms of $\mathfrak{S}$, which are true in $\mathfrak{W}^+$, that if L and M are two sets for which $L-M$ and $M-L$ are equivalent in $\mathfrak{W}^+$, then L and M are equivalent in $\mathfrak{W}^+$. If, conversely, L and M are equivalent in the model, let φ be the biunique mapping of the model which maps L onto M. $L-M, M-L \in \mathfrak{W}^+$ by the axioms which are satisfied by the model. We shall prove that $L-M$ and $M-L$ are finite. Assume that one of them is infinite, say $L-M$. Since $L-M \subseteq K$ and $L-M$ is infinite, by Lemma 2 it must contain at least one open interval N of K. We have $N \subseteq L-M$, whence $N \subseteq L$, $N \cap M = \Lambda_0$. $\varphi \in \mathfrak{W}^+$ and therefore K has a finite subset A such that $\varphi \in \mathfrak{R}^+(A)$. A divides the open interval N into one or more open intervals. Let (p, q) be such an interval and let a, b be two different points in it. Let $\psi \in \mathfrak{G}^+$ be the biunique mapping which is the identity on K outside (p, q) and maps (p, q) on itself in such a way that $\psi(a) = b$. By the construction of ψ we have $\psi \in \mathfrak{G}^+(A)$. $a \in N \subseteq L$, whence a is an element of the domain of φ, $\langle a, \varphi(a)\rangle \in \varphi$. From $\varphi \in \mathfrak{R}^+(A)$ and $\psi \in \mathfrak{G}^+(A)$ follows $|\psi, \langle a, \varphi(a)\rangle| \in \varphi$, i. e., $\langle\psi(a), \psi\varphi(a)\rangle \in \varphi$. $\psi(a) = b$. $\varphi(a) \in M$, $N \cap M = \Lambda_0$ and $(p, q) \subseteq N$ imply that $\varphi(a) \notin (p, q)$, and therefore $\psi\varphi(a) = \varphi(a)$, since ψ is the identical map outside (p, q). Thus we have proved $\langle b, \varphi(a)\rangle = \langle\psi(a), \psi\varphi(a)\rangle \in \varphi$. But φ, being biunique, cannot map both a and b on $\varphi(a)$; therefore, by the contradiction obtained, we have proved that $L-M$ and $M-L$ are finite. We have still to prove that they are equivalent. We have in the model $L = L \cap M + (L-M)$, $M = L \cap M + (M-L)$. If $L-M$ and $M-L$ are not equivalent, we can assume, without loss of generality, that $\overline{\overline{L-M}} < \overline{\overline{M-L}}$. Let B be a proper subset of $M-L$ which is equivalent to $L-M$. We have $M \sim L = L \cap M + (L-M) \sim L \cap M + B$, whence $M \sim L \cap M + B$. But B is a proper subset of $M-L$; therefore $L \cap M + B$ is a proper subset of M. This proves that M is equivalent in the model to a proper subset of itself. This result cannot hold since M is irreflexive in the model, being a subset of K, which is irreflexive in the model (Lemma 7). Hence the assumption $\overline{\overline{L-M}} \neq \overline{\overline{M-L}}$ leads to a contradiction and $\overline{\overline{L-M}} = \overline{\overline{M-L}}$ is proved.

LEMMA 9. *Let U be an infinite well-ordered set in the model $\mathfrak{W}^+$ which has no common elements with K* [11]. *For any set L in the model we have $\overline{\overline{U+K}} \neq 2\overline{\overline{L}}$.*

[11] The latter requirement is not essential.

Proof. Assume $2\overline{\overline{L}} = \overline{\overline{U+K}}$. Let $L_1 = \{1\}\times L$, $L_2 = \{2\}\times L$. By assumption there is a biunique function $\varphi \in \mathfrak{W}^+$ which maps L_1+L_2 onto $U+K$.

Let $K_1 = K \cap \varphi(L_1)$, $K_2 = K \cap \varphi(L_2)$, $U_1 = U \cap \varphi(L_1)$, $U_2 = U \cap \varphi(L_2)$. We have $K = K_1+K_2$, $U = U_1+U_2$, $\overline{\overline{\varphi(L_1)}} = \overline{\overline{L_1}} = \overline{\overline{L_2}} = \overline{\overline{\varphi(L_2)}}$, and since $\varphi(L_1) = U_1+K_1$, $\varphi(L_2) = U_2+K_2$ we have $\overline{\overline{U_1+K_1}} = \overline{\overline{U_2+K_2}}$. Hence there exists a biunique function $\psi \in \mathfrak{W}^+$ mapping U_1+K_1 on U_2+K_2. $U_1 \subseteq U$, therefore U_1 can be well-ordered in the model. The same applies also to the set $\psi(U_1)$, which is equivalent to U_1 in the model, and to its subset $K_2 \cap \psi(U_1)$. $K_2 \cap \psi(U_1)$ is finite since it is a subset of K and K has no infinite subsets which can be well-ordered (Lemma 4). In the same way we can prove that $U_2 \cap \psi(K_1)$ is finite.

Let $N_1 = K_2 \cap \psi(U_1)$, $N_2 = U_2 \cap \psi(K_1)$; then N_1 and N_2 are finite.

$$\begin{aligned} \psi(K_1)+N_1 &= K_2 \cap \psi(K_1) + U_2 \cap \psi(K_1) + K_2 \cap \psi(U_1) \\ &= K_2 + U_2 \cap \psi(K_1) = K_2+N_2, \end{aligned}$$

whence $\overline{\overline{K_1}}+\overline{\overline{N_1}} = \overline{\overline{\psi(K_1)}}+\overline{\overline{N_1}} = \overline{\overline{K_2}}+\overline{\overline{N_2}}$. Assume, without loss of generality, that $\overline{\overline{N_1}} \geqslant \overline{\overline{N_2}}$ and let $n = \overline{\overline{N_1}}-\overline{\overline{N_2}}$. From $\overline{\overline{K_1}}+\overline{\overline{N_1}} = \overline{\overline{K_2}}+\overline{\overline{N_2}}$ we get $\overline{\overline{K_1}} = \overline{\overline{K_2}}-n$. Now let K_2^* be a subset of K_2 of the cardinal $\overline{\overline{K_2}}-n$. We have $\overline{\overline{K_1}} = \overline{\overline{K_2^*}}$, $K_1, K_2^* \subseteq K$. Hence, by Lemma 8, $K_1-K_2^*$ and $K_2^*-K_1$ are finite. But K_1 and K_2 are mutually exclusive by definition, and $K_2^* \subseteq K_2$, therefore $K_1 = K_1-K_2^*$, $K_2^* = K_2^*-K_1$. Hence K_1 and K_2^* are finite. $\overline{\overline{K}} = \overline{\overline{K_1}}+\overline{\overline{K_2}} = \overline{\overline{K_1}}+\overline{\overline{K_2^*}}+n$, which proves a contradiction, because the infinite set K cannot be the union of three finite sets.

THEOREM 6. *The definitions of finiteness* IV *and* V *are not equivalent in* $\mathfrak{S}^+$.

Proof. Let U be an infinite well-ordered set in $\mathfrak{W}^+$ which has no common elements with K (e. g. $\omega = \{0, 1, \ldots\}$ where $0, 1, \ldots$ are the natural numbers of the model). We shall prove that the set $U+K$ is not finite in the model $\mathfrak{W}^+$ according to IV but is finite in the same model according to V.

$U+K$ is not finite in the model according to IV since U has a subset which is denumerable in the model. $U+K$ is finite in the model according to V because by Lemma 9 $2\overline{\overline{L}} \neq \overline{\overline{U+K}}$ for any set L in the model, and hence $2(\overline{\overline{U+K}}) \neq \overline{\overline{U+K}}$, i. e. $2(\overline{\overline{U+K}}) > \overline{\overline{U+K}}$ in the model.

COROLLARY. *The following statement is consistent with the axioms of* $\mathfrak{S}^+$: *For any given aleph there exists a set which is finite according to* V *and has a cardinal greater than that aleph.*

THEOREM 7. *The definitions of finiteness* V *and* VI *are not equivalent in* $\mathfrak{S}^+$.

Proof. We shall prove that the set $\omega\times K$ is not finite in the model $\mathfrak{W}^+$ according to V but is finite in the same model according to VI.

$\omega\times K$ is not finite in the model according to V since we have in the model $2\overline{\overline{\omega\times K}} = 2\overline{\overline{\omega}}\,\overline{\overline{K}} = \overline{\overline{\omega}}\,\overline{\overline{K}} = \overline{\overline{\omega\times K}}$. In order to prove that $\omega\times K$ is finite according to VI we have to prove $\overline{\overline{\omega\times K}}\,\overline{\overline{\omega\times K}} > \overline{\overline{\omega\times K}}$. We have of course $\overline{\overline{\omega\times K}}\,\overline{\overline{\omega\times K}} \geqslant \overline{\overline{\omega\times K}}$. Assume $\overline{\overline{\omega\times K}}\,\overline{\overline{\omega\times K}} = \overline{\overline{\omega\times K}}$. Then we have $\overline{\overline{\omega\times K}} = \overline{\overline{\omega\times K}}\,\overline{\overline{\omega\times K}} = \overline{\overline{\omega}}\,\overline{\overline{K}}\,\overline{\overline{\omega}}\,\overline{\overline{K}} = \overline{\overline{\omega}}^2\overline{\overline{K}}^2 = \overline{\overline{\omega}}\,\overline{\overline{K^2}}$, whence $\overline{\overline{K^2}} \leqslant \overline{\overline{\omega\times K}}$. Now let φ be the biunique function of the model mapping K^2 into $\omega\times K$. $\varphi \in \mathfrak{W}^+$ implies the existence of a finite subset A of K such that $\varphi \in \mathfrak{R}^+(A)$. Let $a, b \in K$, $a \neq b$, $a, b \notin A$. We have $\langle a, b\rangle \in K^2$, $\varphi(\langle a, b\rangle) = \langle i, c\rangle$, $i \in \omega$, $c \in K$. Since $a \neq b$, $c \neq a$ or $c \neq b$. Without loss of generality we assume $c \neq a$. Let (p, q) be an interval which contains a but excludes c and all the points of A. There exists a biunique function $\psi \in \mathfrak{G}^+(A)$ which is the identity of K except the interval (p, q), and for which $\psi(a) \neq a$. We have $\langle\langle a, b\rangle, \langle i, c\rangle\rangle \in \varphi$, and since $\varphi \in \mathfrak{R}^+(A)$, $\psi \in \mathfrak{G}^+(A)$ it follows that $|\psi, \langle\langle a, b\rangle, \langle i, c\rangle\rangle| \in \varphi$, i. e., $\langle\langle\psi(a), \psi(b)\rangle, \langle|\psi, i|, \psi(c)\rangle\rangle \in \varphi$. The natural number i of the model is the set of all the natural numbers smaller then i. The zero of the model is Λ_0. Since $|\psi, \Lambda_0| = \Lambda_0$, it is easy to prove by induction that $|\psi, i| = i$. By the construction of ψ, $\psi(c) = c$. Therefore $\langle\langle\psi(a), \psi(b)\rangle, \langle i, c\rangle\rangle \in \varphi$, i. e., $\varphi(\langle\psi(a), \psi(b)\rangle) = \langle i, c\rangle$. Thus we have proved that φ maps both pairs $\langle a, b\rangle$ and $\langle\psi(a), \psi(b)\rangle$ on the same pair $\langle i, c\rangle$; the pairs $\langle a, b\rangle$ and $\langle\psi(a), \psi(b)\rangle$ are different because $\psi(a) \neq a$. Hence we have arrived at a contradiction by assuming φ to be a biunique function. Therefore must have $\overline{\overline{\omega\times K}}\,\overline{\overline{\omega\times K}} > \overline{\overline{\omega\times K}}$.

THEOREM 8. *The definitions of finiteness* VI *and* VII *are not equivalent in* $\mathfrak{S}^+$.

Proof. We shall prove that the set K^ω [12] is not finite in the model $\mathfrak{W}^+$ according to VI but is finite in the same model according to VII.

K^ω is not finite according to VI since $(\overline{\overline{K^\omega}})^2 = (\overline{\overline{K}}^{\overline{\overline{\omega}}})^2 = \overline{\overline{K}}^{\overline{\overline{\omega}}2} = \overline{\overline{K}}^{\overline{\overline{\omega}}} = \overline{\overline{K^\omega}}$. K^ω is finite according to VII, for if K^ω were an aleph, $\overline{\overline{K}} \leqslant \overline{\overline{K^\omega}}$ would imply that K is also an aleph, contradicting Lemma 3.

Now we are going to prove the independence of definitions I, Ia and II in $\mathfrak{S}$. For this we need another special model of the models of the type $\mathfrak{W}$ which will be named $\mathfrak{W}'$. For $\mathfrak{W}'$, we take for $\mathfrak{G}$ [13] the group consisting of all the biunique transformations of K on itself, and denote it by $\mathfrak{G}'$. The $\mathfrak{G}'$-ring (cf. [7], 42) will be again the set of all the finite subsets of K. $\mathfrak{G}'(A)$ and $\mathfrak{R}'(A)$ will denote with respect to $\mathfrak{G}'$ what $\mathfrak{G}(A)$ and $\mathfrak{R}(A)$ denote with respect to $\mathfrak{G}$.

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[12] By K^ω we mean the set of all functions of the model which map ω into K.
[13] Cf. the description of $\mathfrak{W}$ following Theorem 3.

Lemma 10. *Every subset L of K in the model $\mathfrak{W}'$ is finite in the model or is the complement in K of a set which is finite in the model.*

Proof. Since $L \in \mathfrak{W}'$ there exists a finite subset A of K such that $L \in \mathfrak{R}'(A)$. If $L \subseteq A$ does not hold in the model then there is an element b of L such that $b \notin A$. For each element c of $K-A$ there exists a function $\varphi \in \mathfrak{G}'(A)$ such that $\varphi(b)=c$. $b \in L$ and $L \in \mathfrak{R}'(A)$ imply that $c=\varphi(b) \in L$, whence $K-A \subseteq L$. Thus we have proved that either $L \subseteq A$ or $L \supseteq K-A$. From this and Lemma 1 follows our lemma.

Theorem 9. *The definitions of finiteness* I *and* Ia *are not equivalent in* $\mathfrak{S}$.

Proof. We shall prove that the set K is not finite in the model $\mathfrak{W}'$ according to I but is finite in the model according to Ia.

By Lemma 1 K is not finite in the model according to I. By Lemma 10 K is finite in the model according to Ia.

*Theorem 10. *The equivalence of definitions* Ia *and* II *implies the equivalence of definitions* I *and* Ia.

Proof. Assume that Ia and II are equivalent, I and Ia are not equivalent. Then there is a set L which is finite according to Ia but is not finite according to I. $2\times L$ is not finite according to Ia because $2\times L=\{0\}\times L+\{1\}\times L$ and both $\{0\}\times L$ and $\{1\}\times L$ are infinite. L is finite according to II (by Theorem 1) and it can easily be seen that the same applies also to $\{0\}\times L$ and $\{1\}\times L$. We can prove that the union of two sets which are finite according to II is also finite according to II ([14]). Therefore $2\times L$ is finite according to II. But $2\times L$ is not finite according to Ia, contrary to the assumption that definitions Ia and II are equivalent.

Theorem 11. *The definitions of finiteness* Ia *and* II *are not equivalent in* $\mathfrak{S}$.

Proof. Assume that Ia and II were equivalent in $\mathfrak{S}$. Then Theorem 10 would contradict Theorem 9.

References

[1] R. Doss, *Note on two theorems of Mostowski*, J. Symbolic Logic 10 (1945), p. 13-15.

[2] A. Fraenkel, *Abstract set theory*, Amsterdam 1953.

[3] A. Lindenbaum and A. Mostowski, *Über die Unabhängigkeit des Auswahlaxioms und einiger seiner Folgerungen*, C. R. des Séances de la Société des Sciences et des Lettres de Varsovie, Cl. III, 31 (1938), p. 27-32.

[4] E. Mendelson, *The independence of a weak axiom of choice*, J. Symbolic Logic 21 (1956), p. 350-366.

([14]) The proof is carried out in the same way as for Th. 6 in [8], using the second version of def. II.

[5] A. Mostowski, *Über den Begriff einer endlichen Menge*, C. R. des Séances de la Société des Sciences et des Lettres de Varsovie, Cl. III, 31 (1938), p. 13-20.

[6] — *On the independence of the definitions of finiteness in a system of logic* (Polish), Ann. Soc. Polon. Math. 11 (1938), p. 1-54. (Cf. the review of A. Tarski in the Journal of Symbolic Logic 3 (1938), p. 115).

[7] — *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, Fund. Math. 32 (1939), p. 201-252.

[8] A. Tarski, *Sur les ensembles finis*, Fund. Math. 6 (1924), p. 45-95.

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