On the divisors of zero of the group algebra

by

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In this paper it will be shown that for every non-trivial locally compact group the group algebra has divisors of zero.

Let G be a locally compact multiplicative group. Its elements will be denoted by the letters a, b, t, τ, u, v; the unit element by e. The letters U, V, W will denote the neighbourhoods of the unit e (open sets with compact closure containing e). If A, B are the sets contained in G, then AB is defined as the set of points $u=ab$, where $a \in A$, and $b \in B$. The set A^{-1} is the set of all t such, that $t^{-1} \in A$. It is proved that for every neighbourhood U of the unit e there exists a symmetric neighbourhood V contained in U, *i. e.*, such a V that $V=V^{-1}$ and $V\subset U$. It is well known that for every locally compact group there exists a unique (to within a multiplicative constant) left invariant Haar measure μ (*i. e.*, $\mu(aA) =\mu(A)$ for every $a \in G$). Generally speaking the left invariant measure is not the right invariant one, but there exists such a continuous function $\Delta(a)$ called the *modular function* that $\mu(Aa)=\mu(A)\Delta(a)$ for every $a \in G$.

It is proved that Δ is a homomorphism into the multiplicative group of positive reals, *i. e.*, $\Delta(a)>0$, and $\Delta(ab)=\Delta(a)\Delta(b)$ for every $a,b \in G$. By the definition of the Haar measure if A is any non-void open set then $\mu(A)>0$, and if B is a compact one then $\mu(B)<\infty$.

We shall consider the Banach space $L_1(G)$ of all complex μ-integrable functions defined over G, with the customary norm

$$|f|_1=\int|f(\tau)|\,d\mu\,. \tag{1}$$

It is proved that $L_1(G)$ is a Banach algebra with multiplication as the convolution:

$$f*g=\int f(t\tau^{-1})g(\tau)\,d\mu(\tau)\,. \tag{2}$$

The algebra $L_1(G)$ is called the *group algebra* when the group G is discrete (in this case, and only in this case, the algebra has the unit element). The group algebra for a non-discrete group is the algebra obtained from $L_1(G)$ by joining the unit element. We shall prove that if G is non-trivial, then $L_1(G)$, and so the group algebra has divisors of zero. In the proof we shall give a construction of such divisors.

LEMMA 1. *If the group G contains a non-trivial compact subgroup G_0, then the group algebra has divisors of zero.*

Proof. We shall construct such functions f and g that

(3) $$0<|f|_1<\infty, \quad 0<|g|_1<\infty,$$

and

(4) $$f*g=0.$$

Let $a\in G_0$, and $a\neq e$. There exists such a symmetric neighbourhood V of e that $aV\cap V=\emptyset$.

We put

(5) $$f(t)=\chi_{VG_0}(t)$$

where $\chi_A(t)$ is the characteristic function of the set A, and

(6) $$g(t)=\chi_{aV}(t)-\chi_V(t).$$

We have

(7) $$|f|_1=\mu(VG_0), \quad \text{and} \quad |g|_1=2\mu(V).$$

Hence (3) holds.

Now we see that

$$f*g=\int\chi_{VG_0}(t\tau^{-1})[\chi_{aV}(\tau)-\chi_V(\tau)]d\mu(\tau)$$
$$=\mu(G_0Vt\cap aV)-\mu(G_0Vt\cap V)$$

and (4) holds by the left invariance of μ, q. e. d.

LEMMA 2. *The algebra l_1 (the group algebra of integers; its elements are the two-sided sequences $\{a_n\}_{n=-\infty}^{\infty}$ with absolute convergent series) has divisors of zero.*

Proof. It is well known that the Fourier transform T

$$T(\{a_n\})=\sum_{n=-\infty}^{\infty}a_ne^{inx}$$

is an isomorphism of l_1 into the algebra of all complex-valued continuous functions defined over $(-\pi,\pi)$ having absolutely convergent Fourier series. Then if $F(x)$, and $G(x)$ are two non-zero functions with absolutely convergent Fourier series and disjoint supports [1], then the convolution of the sequences of their Fourier coefficients is zero, q. e. d.

[1] *E. g.*,

$$F(x)=\begin{cases}\sin x & \text{for} \quad x\in(-\pi,0),\\ 0 & \text{for} \quad x\in(0,\pi),\end{cases}$$

$$G(x)=\begin{cases}0 & \text{for} \quad x\in(-\pi,0),\\ \sin x & \text{for} \quad x\in(0,\pi).\end{cases}$$

The following lemma is known:

LEMMA 3 (see [1]). *If a is any element of G and G_0 is the closure of the subgroup generated by a, then G_0 is either discrete or compact.*

It is obvious that if G_0 is discrete, then there exists such a U that

(8) $$a^kU\cap a^pU=\emptyset \quad \text{and} \quad Ua^k\cap Ua^p=\emptyset \quad \text{for} \quad k\neq p.$$

LEMMA 4 [2]. *For every non-trivial locally compact group there exists such an element $a\neq e$ that $\Delta(a)=1$.*

Proof. If G is an abelian group, then $\Delta(a)=1$ for every $a\in G$. If G is not abelian, then there exist such $x,y\in G$ that $xy\neq yx$. Then $(xy)(yx)^{-1}\neq e$ and

$$\Delta[(xy)(yx)^{-1}]=\Delta(x)\Delta(y)[\Delta(x)]^{-1}[\Delta(y)]^{-1}=1, \quad \text{q. e. d.}$$

Now we pass to the proof of the

THEOREM. *For every non-trivial locally compact group the group algebra has divisors of zero.*

Proof. By the lemma 4 there exists such an element a that $a\neq e$, and $\Delta(a)=1$. Let G_0 be the closure of the subgroup generated by this element. By lemma 3 G_0 is either compact or discrete. In the first case there are divisors of zero by lemma 1, in the second there exists such a symmetric neighbourhood U of the unit e that (8) holds.

We put

(9) $$f(t)=\sum_{n=-\infty}^{\infty}a_n\chi_{Ua^{-n}}(t)$$

and

(10) $$g(t)=\sum_{n=-\infty}^{\infty}\beta_{-n}\chi_{a^nU}(t)$$

where $\{a_n\}$ and $\{\beta_n\}$ are divisors of zero in l_1 (their existence proved in lemma 2).

We have

(11) $$|f|_1=\sum_{n=-\infty}^{\infty}|a_n|\mu(Ua^{-n})=\sum_{n=-\infty}^{\infty}|a_n|\mu(U)\Delta^{-n}(a)$$
$$=\mu(U)\sum_{n=-\infty}^{\infty}|a_n|$$

and

(12) $$|g|_1=\mu(U)\sum_{n=-\infty}^{\infty}|\beta_n|.$$

Hence we have (3).

[2] This lemma was suggested to me by C. Ryll-Nardzewski.

We have also

$$f*g=\int\sum_n \alpha_n\chi_{Ua^{-n}}(t\tau^{-1})\sum_k \beta_{-k}\chi_{a^kU}(\tau)\,d\mu(\tau)$$
$$=\sum_n\sum_k \alpha_n\beta_{-k}\int\chi_{Ua^{-n}}(t\tau^{-1})\chi_{a^kU}(\tau)\,d\mu(\tau)$$
$$=\sum_n\sum_k \alpha_n\beta_{-k}\mu(a^nUt\cap a^kU)$$
$$=\sum_n\sum_k \alpha_n\beta_{-k}A_{n-k},$$

where

$$A_s=\mu(a^sUt\cap U);$$

then in the sum $\sum_n\alpha_n\sum_k\beta_{-k}A_{n-k}$ we put $n-k=s$, and

$$f*g=\sum_n\alpha_n\sum_s\beta_{s-n}A_s=\sum_s A_s\sum_n\alpha_n\beta_{s-n}=0, \qquad \text{q. e. d.}$$

Reference

[1] B. Eckmann, *Über monothetische Gruppen*, Comment. Math. Helv. 16 (1943/44), p. 249-263.

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Completely regular mappings

by

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The class of continuous mappings f of a topological space X onto a topological space Y can be ordered roughly by the amount of information about the local and in the large properties of X which can be learned from information about those properties of Y and the inverses under f of points of Y. Some of the groupings in this ordering are open mappings, homologically regular mappings [5], mappings which have the covering homotopy property, homotopically regular mappings, and projection mappings of direct products.

In this paper a new type of mapping is defined — *the completely regular mapping*. It will be shown that this type of mapping occupies a position in the ordering mentioned above just before that of the projection mapping, that under certain additional hypotheses such mappings become projection mappings and that if the inverses are certain low dimensional spaces, 0-regular maps are completely regular. Thus we will be able to show in some cases that spaces on which certain 0-regular maps are defined are direct products. Some of our results of this sort are related to results of B. J. Ball [2] and others nicely complement a result of R. H. Bing [3]. Part of Theorem 7 is a special case of a theorem of Whitney [15] proved by quite different methods.

In particular, we show that if f is a 0-regular mapping of a compact metric space onto an arc such that each inverse under f is a 2-cell then X is a 3-cell. This answers a question raised in [7], p. 84. We also show that if f is a closed mapping of E^3 onto a metric space X such that each inverse under f of a point of X is a compact continuum lying in a horizontal plane and not separating that plane, then X is homeomorphic to E^3. For information on related problems the reader is referred to [3] and [4]. The lemma of Alexander which proves so important in the argument for our principal theorem was called to our attention by

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