On a class of continuous mappings

by

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1. Simple and elementary mappings. By a *mapping of order* $\leqslant k$ of a space X we understand a continuous mapping f defined on X and such that for every point $y \in f(X)$ the set $f^{-1}(y)$ contains at most k points (comp. [10], p. 52). A continuous mapping f is said to be *elementary* if its range X is metric and there exists a positive ε such that for every two different points $x, x' \in X$ from $f(x)=f(x')$ follows $\varrho(x,x') \geqslant \varepsilon$. The mappings of order $\leqslant 2$ are said to be *simple*.

The rather special class of all simple mappings constitutes a natural and intuitive generalization of the class of all homeomorphisms. In this note we study some elementary properties of these mappings, give some examples and formulate some problems.

The theory of mappings of order $\leqslant k$ is intimately related to the theory of upper semicontinuous decompositions (see [10], p. 42). A collection $\mathfrak{F}$ of subsets of a space X constitutes a *decomposition* of X if the sets of $\mathfrak{F}$ are disjoint, not empty and they fill up X. The decomposition $\mathfrak{F}$ is said to be *upper semicontinuous* if for every closed subset A of X the union of all sets of $\mathfrak{F}$ intersecting A is closed (in X). If, moreover, for every open subset B of X the union of all sets of $\mathfrak{F}$ intersecting B is open (in X), then the decomposition $\mathfrak{F}$ is said to be *continuous*.

P. Alexandroff ([1] and [2]; see also [10], p. 42) has proved the following theorem:

In order that a decomposition $\mathfrak{F}$ of a compact space X be upper semicontinuous it is sufficient and necessary that there exists a continuous mapping $f\colon X \to Y$ such that the sets belonging to $\mathfrak{F}$ are the same as the sets $f^{-1}(y)$ with $y \in Y$.

The continuous decompositions are characterised as the decompositions corresponding to the so called "interior mappings" (¹) (see, for instance [10], p. 48).

(¹) A continuous mapping $f\colon X \to Y$ is said to be *interior* if for every set G open in X the set $f(G)$ is open in Y.

If f denotes a continuous mapping of X, the topological properties of the space $Y=f(X)$ depend only on the upper semicontinuous decomposition $\mathfrak{F}$ of X into sets $f^{-1}(y)$, with $y \in Y$. In particular the study of simple mappings of compacta is thus reduced to the study of upper semicontinuous decompositions into sets containing at most two points.

Let $f\colon X \to Y$ be a continuous mapping. By the *seam* of f we mean the union of all sets $f^{-1}(y)$ containing at least two different points. The basic problem of the theory of simple mappings can be formulated as follows:

What are the topological properties of a space Y obtained from a given space X by a simple mapping having a given seam X_0?

In this note we shall give some rather simple remarks concerning this problem.

2. Examples. We begin with some examples.

Example 1. Consider the decomposition of the n-dimensional Euclidean sphere S_n onto pairs of antipodal points. This decomposition is continuous and its hyperspace is the n-dimensional projective space P_n. The corresponding mapping is simple and elementary. Its seam is the whole sphere S_n.

Example 2. Let Q denote the disk defined in the Euclidean plane E_2 by the inequality $x^2+y^2 \leqslant 1$. If we identify every point (x,y) lying on the boundary B of Q with the point $(x,-y)$, we obtain from Q a space homeomorphic with the 2-dimensional sphere S_2. The corresponding mapping is simple, but not elementary. Its seam is the set obtained from B by removing two points, $(1,0)$ and $(-1,0)$.

Example 3. Besides the identification made on the disk Q as in the example 2, let us identify two points, $(1,0)$ and $(-1,0)$. We obtain a space homeomorphic with the surface obtained by the rotation of the circle round one of its tangents. The corresponding mapping is simple but not elementary. Its seam is the boundary B of Q.

Example 4. Let X be the Cantor discontinuum (²) lying on the straight line E_1. If we identify every pair of endpoints of every open segment-component of $E_1 - X$, we obtain a simple (but not elementary) mapping of X onto a set homeomorphic with a segment. The seam is the enumerable set of all endpoints of the bounded components of $E_1 - X$. It is dense in X.

(²) *Cantor discontinuum* = the set of all real numbers expressible in the form $\sum_{n=1}^{\infty} 2 \cdot a_n \cdot 3^{-n}$, where $a_n = 0$ or 1.

Example 5. Consider in the Euclidean plane E_2 the dendrite (³) X defined as the union of the segment L_0 with the endpoints $(0,0)$ and $(1,0)$ and of the segments $L_1, L_2, \ldots$, where L_n has as endpoints the points $(2^{1-n},0)$ and $(2^{-n},2^{-n})$. Consider the mapping f defined as the orthogonal projection on the axis x. Hence $f(X)=L_0$. This mapping is simple but not elementary, its seam is the set obtained from X by removing two points, $(0,0)$ and $(1,0)$.

Example 6. Besides the identifications made on the dendrite X as in the example 5, let us identify two points, $(0,0)$ and $(1,0)$. We obtain a simple mapping of the dendrite X onto a simple closed curve. Its seam is the whole set X.

Example 7. We consider every point (x,y) of the plane E_2 as identical with the point $(x,y,0)$ of the 3-dimensional Euclidean space E_3. Let X_n $(n=1,2,\ldots)$ be the projection of the dendrite X (of the example 5) from the centre $(0,0,1)$ on to the plane $P(1-2^{-n})$ consisting of the points $(x,y,1-2^{-n})$. Moreover, let L denote the segment with endpoints $(0,0,0)$ and $(0,0,1)$ and L_{nk} $(n,k=1,2,\ldots)$, the segment with endpoints $\big(0,0,1-2^{1-n}+2^{-n}(1-2^{1-k})\big)$ and $\big(-2^{-k-n},0,1-2^{1-n}+2^{-n}(1-2^{-k})\big)$. Setting

$$Z=L\cup\bigcup_{n=1}^{\infty}[X_n\cup\bigcup_{k=1}^{\infty}L_{nk}]$$

we obtain a dendrite. Consider the decomposition of Z into the following sets: 1) The projections from the centre $(0,0,1)$ onto the plane $P(1-2^{-n})$ of the sets constituting the decomposition of X considered in the example 6; 2) the pairs of the points at which Z intersects the planes $P(c)$ with the equations $z=c$, where $0<c<1$ and $c\neq 1-2^{-n}$; 3) the pair of the points $(0,0,0)$ and $(0,0,1)$.

One easily sees that this decomposition is upper semicontinuous and the corresponding mapping is simple, not elementary and has Z as its seam. The hyperspace of this decomposition is homeomorphic with the union of the segment with endpoints $(-2,0)$ and $(2,0)$, of the half-circle with the equation $x^2+y^2=4$, $y\geqslant 0$ and of the circles with the equations $(x-2^{-n})^2+(y-2^{-n-3})^2=2^{-2n-6}$. It is clear that the space Y is not locally contractible (⁴).

Example 8. Consider in the plane E_2 the rectangle Q with the vertices $(-2,0)$, $(-2,1)$, $(2,0)$, $(2,1)$. Let us identify every point of the

(³) *Dendrite* = a locally connected continuum which contains no simple closed curve.

(⁴) A space X is said to be *locally contractible* if for every point $x\in X$ and every neighbourhood $U(x)$ of x there exists a neighbourhood $V(x)$ of x contained in $U(x)$ and a continuous deformation of $V(x)$ in $U(x)$ to a point. It is known that for every locally contractible continuum $Y\subset E_2$, the complement E_2-Y contains only finite number of components. See, for instance, [3], p. 230.

form $(x,0)$, where $0<x\leqslant 1$, with the point $\big(x-2,\,x-x\sin(1/x)\big)$ and also the point $(0,0)$ with the point $(-2,0)$. In this manner one obtains a simple and elementary mapping f having as its seam the union of two disjoint simple arcs. The space $f(Q)$ is a 2-dimensional compactum not homeomorphic with a polytope, because the set in which it is not locally homeomorphic with the plane has an infinite 1-dimensional Betti number.

Example 9. Consider in the space E_3 the square Q with the vertices $(0,0,0)$, $(0,1,0)$, $(1,0,0)$, $(1,1,0)$. Let $\{w_n\}$ denote the sequence of all rational numbers of the interval $0\leqslant x\leqslant 1$ such that $w_n\neq w_m$ for $n\neq m$. Let us denote by Y_n the union of the segment with endpoints $(w_n,0,0)$, $(w_n,1,0)$ and of n segments L_{nk} $(k=1,2,\ldots,n)$ with endpoints $\big(w_n,(k-1)/n,0\big)$, $(w_n,k/n,1/n)$. Setting

$$X=Q\cup\bigcup_{n=1}^{\infty}Y_n$$

we obtain a 2-dimensional compactum (which is an absolute retract (⁵)). The orthogonal projection f of X on the square Q is a simple (but not elementary) mapping of X onto Q. Its seam is the 1-dimensional set

$$\bigcup_{n=1}^{\infty}Y_n-\underset{(x,0,0)}{E}[0\leqslant x\leqslant 1]$$

dense in X.

Example 10. Let X_0 be an arbitrarily given subset of a space X. Setting $X^*=X\times(0)\cup X_0\times(1)$ and

$$f(x,t)=x\quad\text{for every}\quad(x,t)\in X^*$$

we obtain a simple mapping (elementary if X is a metric space) with the seam $X_0\times\big((0)\cup(1)\big)$.

Example 11. Let C denote the Cantor discontinuum lying on the segment $X=\underset{t}{E}[0\leqslant t\leqslant 1]$. The complement $X-C$ contains 2^{n-1} components of the length 3^{-n}:

$$L_{nk}=\underset{t}{E}[a_{nk}<t<a_{nk}+3^{-n}],\qquad n=1,2,\ldots;\ k=1,2,\ldots,2^{n-1},$$

where $a_{nk}<a_{n,k+1}$. If we identify in X every point of the form a_{nk} with the point $a_{nk}+3^{-n}$, for $n=1,2,\ldots$ and $k=1,2,\ldots,2^{n-1}$ and, moreover, if we identity every point of the form $a_{nk}+2\cdot 3^{-n-1}$ with the point $a_{n,k+1}+3^{-n-1}$, for every $n=2,3,\ldots$ and $k=1,2,\ldots,2^{n-1}-1$, then

(⁵) A space X is said to be an *absolute retract* (or an AR-set) if there exists a continuous mapping f of the Hilbert cube Q_ω onto X and a continuous mapping g of X into Q_ω such that $fg(x)=x$ for every $x\in X$. See, for instance, [10], p. 259.

we obtain a simple (but not elementary) mapping f. Let $Y=f(X)$. Evidently the set $f(C_n)$, $n=2,3,\ldots$, where

$$C_n=\bigcup_{k=1}^{2^{n-1}}\mathop{E}_{t}[a_{nk}+3^{-n-1}\leqslant t\leqslant a_{nk}+2\cdot 3^{-n-1}]$$

is a simple arc. The arcs $f(C_2), f(C_3), \ldots$ are disjoint and they converge to the simple arc $f(C)$. It follows that Y has infinite order at every point of the arc $f(C)$. In particular Y is not regular (in the sense of Menger, see for instance, [10], p. 201).

3. Involution assigned to a simple mapping. Let f be a simple mapping defined on a space X and let X_0 denote the seam of f. We assign to f the function i_f defined on X in the following manner:

If $x\in X_0$, then there exists exactly one point $x^*\in X_0$ different from x and such that $f(x)=f(x^*)$. We set $i_f(x)=x^*$.

If $x\in X-X_0$, then we set $i_f(x)=x$.

Evidently i_f maps X onto itself and

$$i_f[i_f(x)]=x \quad \text{for every} \quad x\in X.$$

THEOREM. *Let f be a simple mapping defined on a compactum X and let $\{x_n\}\subset X$ be a convergent sequence such that for an $\varepsilon>0$ it is $\varrho(x_n, i_f(x_n))\geqslant\varepsilon$ for every $n=1,2,\ldots$ Then*

$$\lim_{n\to\infty} i_f(x_n)=i_f(\lim_{n\to\infty} x_n)\neq\lim_{n\to\infty} x_n. \tag{1}$$

Proof. Let $x_0=\lim_{n\to\infty} x_n$. Since X is compact, the sequence $\{i_f(x_n)\}$ contains convergent subsequences. It remains to show that every convergent subsequence $\{i_f(x_{n_k})\}$ converges to the point $i_f(x_0)$. Since f is continuous we have

$$f\big(\lim_{k\to\infty} i_f(x_{n_k})\big)=\lim_{k\to\infty} f[i_f(x_{n_k})]=\lim_{k\to\infty} f(x_{n_k})=f(\lim_{k\to\infty} x_{n_k})=f(x_0).$$

It follows that $\lim_{n\to\infty} i_f(x_{n_k})$ is either x_0 or $i_f(x_0)$. But the first eventuality does not hold, because $\{x_{n_k}\}$ converges to x_0 and $\varrho(i_f(x_{n_k}), x_{n_k})\geqslant\varepsilon$. Hence we get (1).

COROLLARY 1. *Let f be a simple elementary mapping defined on a compact space X. Then the seam X_0 of f is closed in X and the involution i_f maps X_0 topologically onto itself.*

The first part of the statement is an immediate consequence of (1). In order to prove the second part, consider an $\varepsilon>0$ such that $\varrho(x, i_f(x))\geqslant\varepsilon$ for every $x\in X_0$. It follows, by the last theorem, that i_f is continuous on X_0. But i_f is the inverse of itself, hence i_f is a homeomorphism.

By example 2, the hypothesis that f is elementary is essential in this corollary.

COROLLARY 2. *Let f be a simple elementary mapping with compact range X and with the seam X_0. Then f is a local homeomorphism on X_0.*

By corollary 1, the seam X_0 is compact. Since f is elementary, there exists an $\varepsilon>0$ such that $\varrho(x, i_f(x))\geqslant\varepsilon$, for every $x\in X_0$. Consider a point $x_0\in X_0$ and a closed neighbourhood U of x_0 in X_0 with the diameter $<\varepsilon$. The mapping f, being continuous and univalent on U, maps U topologically. It remains to show that $f(U)$ is a neighbourhood in $f(X_0)$ of the point $f(x_0)$. In order to prove it, let us observe that the decomposition of X_0 into sets $(x, i_f(x))$ with $x\in X_0$ is the same as the decomposition of X_0 into sets $f^{-1}(y)$ with $y\in f(X_0)$. But i_f is continuous on X_0, whence f is interior on X_0. Consequently f maps the neighbourhood U of x_0 (in X_0) onto a neighbourhood $f(U)$ of $f(x_0)$ (in $f(X_0)$).

COROLLARY 3. *Let f be a simple mapping defined on a compactum X. Then the seam X_0 of f is a F_σ-set.*

Let $X(1/n)$ denote the union of all sets $f^{-1}(y)$ with the diameter $\geqslant 1/n$. Then $X_0=\bigcup_{n=1}^{\infty} X(1/n)$ and, by the last theorem, each of the sets $X(1/n)$ is closed in X.

By example 10, the hypothesis that X is compact is essential, even if we suppose that f is elementary.

4. Simple mappings and dimension. Let f be a continuous mapping of finite order, defined on a compact space X. Since the inverse-images $f^{-1}(y)$ are finite, we infer by a known theorem of W. Hurewicz (see, for instance, [10], p. 67), that

$$\dim f(X)\geqslant\dim X.$$

By example 4, even a simple mapping can raise the dimension (not more, however, than by one unit, by a theorem of Hurewicz; see, for instance, [10], p. 52). Now we shall show that for a certain class of mappings, containing all elementary mappings, the dimension remains invariant.

Let f be a continuous mapping defined on a space X. By a *substratum* of f we understand a system $\{X_a\}$ of closed subsets of X (indexed by a set $\mathfrak{A}=\{a\}$) such that $X=\bigcup_{a\in\mathfrak{A}} X_a$ and that for every $a\in\mathfrak{A}$ and every $y\in f(X)$ the set $X_a\cap f^{-1}(y)$ contains at most one point. By the *index* of the mapping f we understand the smallest of cardinals $\mathfrak{m}$ such that there exists a substratum $\{X_a\}$, $a\in\mathfrak{A}$, of f with $\overline{\overline{\mathfrak{A}}}=\mathfrak{m}$. In particular *the index of every elementary mapping over a compactum X is finite.* In fact, if $\varepsilon>0$ satisfies the condition that $\varrho(x,x')\geqslant\varepsilon$ for $x,x'\in X$, $x\neq x'$ and

$f(x)=f(x')$, then it suffices to split X into a finite number of compact sets with diameters $<\varepsilon$ in order to obtain a finite substratum of f.

Example 1 shows that there exist simple elementary mappings with arbitrary finite indices. In fact, for the simple elementary mapping f defined on the sphere S_n considered in example 1, the sets constituting a substratum have diameters <2. But it is known (see [12], p. 26 also [4], p. 178), that S_n can be split into $n+2$ closed sets with diameters <2, but can not be split into $<n+2$ such sets. Hence the index of f is equal to $n+2$.

The simple mapping (defined on the disk Q) considered in example 2 is not elementary, but its index is finite. In order to obtain a finite substratum of this mapping it suffices to split Q into 4 quarters (closed) by the straight lines with equations $x=0$ and $y=0$.

Example 9 gives a simple mapping f over a compactum X with an infinite index. In fact, if $X=X_1\cup X_2\cup\ldots\cup X_n$, where the sets X_i are closed, then the set $\bigcup_{i=1}^{n}\overline{X_i-Q}$ is closed in X and contains $X-Q$. Since $X-Q$ is dense in X we infer that $\bigcup_{i=1}^{n}\overline{X_i-Q}=X$. It follows that for an i_0 the set $\overline{X_{i_0}-Q}\cap Q$ contains some set G open in Q. Since $G\subset\overline{X_{i_0}-Q}$ we infer that there exists a sequence $\{p_k\}\subset X_{i_0}-Q$ which converges to a point $p_0\in G\subset X_{i_0}$. Then $\lim_{k\to\infty}f(p_k)=f(p_0)=p_0$ and $f(p_k)\in Q$. Since G is open in Q, we infer that $f(p_k)\in G$ for almost all k. For those k the points $p_k\in X_{i_0}-Q$ and $f(p_k)\in G\subset Q$ are distinct, both belong to X_{i_0} and satisfy the condition $f(p_k)=f[f(p_k)]$. It follows that the sets $X_1,X_2,\ldots,X_n$ do not constitute a substratum for the mapping f.

THEOREM. *If f is a continuous mapping defined on a compactum X and the index of f is finite, then $\dim X=\dim f(X)$.*

Proof. Let $\{X_1,X_2,\ldots,X_n\}$ be a finite substratum of f. Then f is univalent and continuous on every set X_i. Since X_i is compact, we infer that f is topological on X_i. Hence

$$\dim f(X)=\dim[\bigcup_{i=1}^{n}f(X_i)]=\max_{1\leqslant i\leqslant n}\dim f(X_i)=\sup_{1\leqslant i\leqslant n}\dim X_i=\dim X\,.$$

PROBLEM 1. *Does there exist a simple mapping defined on an absolute neighbourhood retract* [6] *which raises the dimension?*

[6] A space X is said to be an *absolute neighbourhood retract* (or an ANR-set) provided that it is compact and there exists a continuous mapping f of an open subset G of the Hilbert cube Q_ω onto X and a continuous mapping g of X into Q_ω such that $fg(x)=x$ for every $x\in X$. The finite-dimensional ANR-sets are the same as locally contractible compacta. See [3], p. 240.

PROBLEM 2. *Let X be a rational curve* [7] *and f a simple mapping defined on X. Is it true that $f(X)$ is necessarily also a rational curve?*

Let us observe that, by example 11, the image of a segment by a simple mapping is not necessarily regular.

5. Superpositions of simple mappings. We shall prove the following

THEOREM. *Let f be a continuous mapping defined on a compact space X and having a finite index n. Then f is a finite superposition of simple mappings with indices $\leqslant n$.*

Proof. Let $\mathfrak{X}=\{X_1,X_2,\ldots,X_n\}$ be a substratum for f. For every pair of indices (i,j) with $1\leqslant i<j\leqslant n$ and for every point $y\in f(X)$ the set $(X_i\cup X_j)\cap f^{-1}(y)$ contains at most 2 points. Let $N(\mathfrak{X},f)$ denote the number of all pairs (i,j) such that for some $y\in f(X)$ the set $(X_i\cup X_j)\cap f^{-1}(y)$ contains two distinct points. Evidently if $N(\mathfrak{X},f)=1$ then the mapping f is simple. Proceeding by induction let us assume that the statement holds for all compact spaces X and all continuous mappings f for which there exists a substratum $\mathfrak{X}$ such that $N(\mathfrak{X},f)\leqslant m$ and let us assume that in our case $N(\mathfrak{X},f)=m+1$.

Let (i_0,j_0) be a pair of indices such that $1\leqslant i_0<j_0\leqslant n$ and that for some $y\in f(X)$ the set $(X_{i_0}\cup X_{j_0})\cap f^{-1}(y)$ contains two distinct points. Consider the decomposition $\mathfrak{F}$ of the space X into the sets of the form $(X_{i_0}\cup X_{j_0})\cap f^{-1}(y)$ and into the individual points of $X-(X_{i_0}\cup X_{j_0})$. Evidently every set belonging to $\mathfrak{F}$ contains at most two points. Moreover the decomposition $\mathfrak{F}$ is upper semicontinuous, because for every closed subset A of X the union of all sets of $\mathfrak{F}$ not disjoint with A is identical with the set

$$A\cup f^{-1}[f(A\cap(X_{i_0}\cup X_{j_0}))]\cap(X_{i_0}\cup X_{j_0})\,,$$

whence it is closed in X. It follows that $\mathfrak{F}$ induces a simple mapping φ defined on X which maps X onto a compactum X^*. Evidently one obtains X^* from X by the identification of points belonging to every set $f^{-1}(y)\cap(X_{i_0}\cup X_{j_0})$.

Now let us observe that setting

$$f^*(x^*)=f[\varphi^{-1}(x^*)]\quad\text{for every}\quad x^*\in X^*$$

we obtain a continuous mapping f^* of X^* onto $Y=f(X)$. The closed sets $\varphi(X_1),\ldots,\varphi(X_n)$ constitute a decomposition of X^* and it is clear that none of them contains two distinct points belonging to one of the

[7] A curve X (*i. e.*, a 1-dimensional continuum) is said to be *rational* if each point of X has arbitrarily small neighbourhoods (in X) whose boundaries are finite or countable.

sets $f^{*-1}(y)$. Hence the system $\mathfrak{X}^*=\{\varphi(X_1),\ldots,\varphi(X_n)\}$ is a substratum of f^*. Hence f^* is a continuous mapping with index $\leqslant n$.

If there exist in the set $\varphi(X_i)\cup\varphi(X_j)$, where $1\leqslant i<j\leqslant n$, two points x^*, x'^* such that

$$f^*(x^*)=f^*(x'^*)$$

then one of these points belongs to $\varphi(X_i)$ and the other to $\varphi(X_j)$. For instance $x^*=\varphi(x)$, where $x\in X_i$ and $x'^*=\varphi(x')$, where $x'\in X_j$. It follows that $x\neq x'$ and $f(x)=f(x')$. Thus we have shown that if the set $\varphi(X_i)\cup\varphi(X_j)$ contains two distinct points belonging to one of the sets $f^{*-1}(y)$, then the set $X_i\cup X_j$ contains two points belonging to one of the sets $f^{-1}(y)$. But the converse is not true, because $X_{i_0}\cup X_{i_0}$ contains two distinct points belonging to one of the sets $f^{-1}(y)$, but for every two points $x^*=\varphi(x)$, $x'^*=\varphi(x')$ with $x\in X_{i_0}$, $x'\in X_{j_0}$ satisfying the condition $f^*(x^*)=f^*(x'^*)$ we have

$$f(x)=f^*\varphi(x)=f^*(x^*)=f^*(x'^*)=f^*\varphi(x')=f(x'),$$

which implies, by the definition of the set X^*, that $x^*=x'^*$.

Thus we have shown that $N(\mathfrak{X}^*,f^*)<N(\mathfrak{X},f)$, *i. e.*, $N(\mathfrak{X}^*,f^*)\leqslant m$. By the induction hypothesis we infer that f^* is a superposition of a finite number of simple mappings with indices $\leqslant n$. To finish the proof it remains to observe that $f=f^*\varphi$, where φ is a simple mapping with index $\leqslant n$.

COROLLARY. *Every elementary mapping over a compactum is a superposition of a finite number of simple mappings.*

PROBLEM 3. *Is it true that every elementary mapping is a superposition of a finite number of elementary simple mappings?*

PROBLEM 4. *Does there exist a continuous mapping of finite order which is not a superposition of a finite number of simple mappings?*

6. Simple mappings on the ANR-sets. It follows by example 8 that a simple elementary mapping with the seam being a (curvilinear) polytope can transform a polytope onto a compactum which is not a curvilinear polytope. However, the question whether the image of a polytope X by a simple elementary mapping with the seam being a rectilinear polytope (*i. e.*, a subcomplex of a triangulation of X) is necessarily a polytope remains open.

In example 7 we have constructed a simple mapping f of a dendrite X (hence of an AR-set: see, for instance, [10], p. 290) with the seam being also an AR-set, such that the image $f(X)$ is not an ANR-set. Now we shall prove that for simple elementary mappings an analogous phenomenon is not possible. More exactly, we shall prove the following

THEOREM. *Let f be a simple elementary mapping, defined on a locally contractible compactum X of a finite dimension. If the seam of f is locally contractible then $Y=f(X)$ is an ANR-set.*

First we establish the following

LEMMA. *Let X and $X_0\subset X$ be two ANR-sets. For every point $a\in X_0$ and every neighbourhood G of a in X there exists a compact neighbourhood A of a in X and a continuous function $\psi(x,t)$ defined for $(x,t)\in A\times\langle 0,1\rangle$ and satisfying the following conditions:*

(1) $$\psi(x,t)\in G \quad \text{for every} \quad (x,t)\in A\times\langle 0,1\rangle,$$

(2) $$\psi(x,0)=x;\quad \psi(x,1)\in X_0 \quad \text{for every} \quad x\in A,$$

(3) $$\psi(x,t)=x \quad \text{for every} \quad (x,t)\in(A\cap X_0)\times\langle 0,1\rangle.$$

Proof. We may assume that X is a subset of the Hilbert cube Q_ω. Let r be a retraction of a neighbourhood U of X (in Q_ω) to X and r_0 a retraction of a neighbourhood U_0 of X_0 (in Q_ω) to X_0. Setting

$$\varphi(x,t)=tr_0(x)+(1-t)x \quad \text{for every} \quad (x,t)\in U_0\times\langle 0,1\rangle,$$

we obtain a continuous deformation of the set U_0 in Q_ω to the set X_0 such that all points of X_0 remain fixed during this deformation. It follows that there exists a neighbourhood $U_0^*\subset U_0$ of X_0 in Q_ω such that all values of $\varphi(x,t)$ for $(x,t)\in U_0^*\times\langle 0,1\rangle$ belong to U. Hence setting

$$\psi(x,t)=r\varphi(x,t) \quad \text{for every} \quad (x,t)\in(U_0^*\cap X)\times\langle 0,1\rangle,$$

we obtain a continuous deformation of $U_0^*\cap X$ in the space X to X_0, and all points of X_0 remain fixed during this deformation. It follows that for every point $a\in X_0$ and every neighbourhood G of a in X there exists a compact neighbourhood A of a in X such that ψ satisfies conditions (1), (2) and (3).

Proof of the theorem. It suffices to show that Y is locally contractible. By corollary 1 of section 3, the seam X_0 of f is closed. We infer that the set $X-X_0$ is open in X and f maps it topologically onto the set $Y-f(X_0)$ open in Y. Since X is locally contractible, we conclude that Y is locally contractible at each point $y_0\in Y-f(X_0)$.

Consider now a point $y_0\in f(X_0)$ and an arbitrarily given neighbourhood V of y_0 in the space Y. It remains to show that there exists a neighbourhood W of y_0 in Y contractible in V to a point.

Let $y_0=f(x_1)=f(x_2)$ where $x_1\neq x_2$. Since f is elementary, there exists for $\nu=1,2$ a compact neighbourhood U_ν of x_ν in X such that

(4) $$U_1\cap U_2=0,$$

(5) $$f \text{ is univalent on } U_\nu,$$

(6) $$f(U_\nu)\subset V \quad \text{for} \quad \nu=1,2.$$

Since U_ν is compact we infer by (5) that f maps U_ν onto a set $f(U_\nu)\subset X$ topologically. Consequently there exists a homeomorphism g_ν defined in the set $f(U_\nu)$ such that

$$(7) \qquad g_\nu(y) \in U_\nu \text{ and } fg_\nu(y)=y \quad \text{for every} \quad y \in f(U_\nu),\ \nu=1,2 .$$

Moreover, let us observe that

$$(8) \qquad f(U_1)\cap f(U_2)\subset f(X_0) ,$$

$$(9) \qquad f(U_1)\cup f(U_2) \text{ is a neighbourhood of } y_0 \text{ in } Y.$$

By corollary 1 of section 3 the sets $f^{-1}(y)$, where $y \in f(X_0)$, constitute a continuous decomposition of the set X_0. It follows that the mapping f is interior on X_0 and consequently each of the sets $f(U_\nu\cap X_0)$, $\nu=1,2$, hence also the set $f(U_1\cap X_0)\cap f(U_2\cap X_0)$ constitutes a neighbourhood of y_0 in $f(X_0)$. Since f is topological on the set U_ν and the set X_0 is locally contractible, the set $f(X_0)$ is also locally contractible. Hence there exists a compact neighbourhood B of y_0 in $f(X_0)$ such that

$$(10) \qquad B\subset f(U_1\cap X_0)\cap f(U_2\cap X_0) ,$$

$$(11) \qquad B \text{ is contractible to a point in the set } V\cap f(X_0) .$$

We infer by (7) and (10) that the set $g_\nu(B)$ is a neighbourhood of x_ν in X_0.

Let G_ν be a neighbourhood of x_ν in X such that

$$(12) \qquad G_\nu\subset U_\nu \text{ and } X_0\cap G_\nu\subset g_\nu(B) \quad \text{for} \quad \nu=1,2 .$$

Applying our lemma we infer that there exists in X a compact neighbourhood $A_\nu\subset G_\nu$ of the point x_ν and a continuous function $\psi_\nu(x,t)$ defined for $(x,t) \in A_\nu \times\langle 0,1\rangle$ and satisfying the following conditions:

$$(13) \qquad \psi_\nu(x,t) \in G_\nu \quad \text{for every} \quad (x,t) \in A_\nu \times\langle 0,1\rangle ,$$

$$(14) \qquad \psi_\nu(x,0) = x, \quad \psi_\nu(x,1) \in X_0 \quad \text{for every} \quad x \in A_\nu ,$$

$$(15) \qquad \psi_\nu(x,t) = x \quad \text{for every} \quad (x,t) \in (A_\nu\cap X_0) \times\langle 0,1\rangle .$$

Since f maps A_ν topologically onto $f(A_\nu)\subset f(U_\nu)$, we infer by (9) that the set

$$(16) \qquad W=f(A_1)\cup f(A_2)$$

is a neighbourhood of y_0 in Y. It remains to show that W is contractible in V to a point. To prove it, let us set

$$(17) \qquad \varphi(y,t)=f\psi_\nu[g_\nu(y),t] \quad \text{for every} \quad (y,t) \in f(A_\nu) \times\langle 0,1\rangle,\ \nu=1,2 .$$

By (13), (12) and (6), the values of φ lie in V. Moreover, by (14) and (7) we have

$$f\psi_\nu[g_\nu(y),0]=f(g_\nu(y))=y \quad \text{for every} \quad y \in f(A_\nu),\ \nu=1,2$$

and, by (14), (13) and (12)

$$f\psi_\nu[g_\nu(y),1] \in f(X_0\cap G_\nu)\subset fg_\nu(B)=B .$$

Since $A_\nu\subset U_\nu$, we infer by (8) that for $y \in f(A_1)\cap f(A_2)\subset f(U_1)\cap f(U_2)\subset f(X_0)$ it is $g_\nu(y) \in A_\nu\cap X_0$. Hence, by (15)

$$f\psi_\nu[g_\nu(y),t]=fg_\nu(y)=y \quad \text{for} \quad \nu=1,2 .$$

It follows that formula (17) defines a continuous deformation of the set W in the set V to a subset of B. By (11) we infer that the set W is contractible to a point in the set V. Thus our proof is finished.

7. Simple mappings and retractions. Now we shall prove the following

THEOREM. *Let f be a simple mapping defined on a compact space X and let A be a retract* (8) *of X containing the seam X_0. Then $f(A)$ is a retract of $f(X)$.*

Proof. Let r denote a retraction of X to A. For every $y \in f(X_0)$ the set $f^{-1}(y)$ lies in the seam $X_0\subset A$. Consequently $frf^{-1}(y)=ff^{-1}(y)=y$. It follows that setting

$$s(y)=frf^{-1}(y) \quad \text{for every} \quad y \in f(X)$$

we obtain a single-valued function mapping $f(X)$ onto $f(A)$. Let us show that s is a retraction of $f(X)$ to $f(A)$.

If $y \in f(A)$ then $f^{-1}(y)\subset A$ and consequently

$$s(y)=frf^{-1}(y)=ff^{-1}(y)=y .$$

Moreover, f maps $X-A$ homeomorphically onto $f(X)-f(A)$ (see [8], p. 266). It follows that the mapping s is continuous on the set $f(X)-f(A)$. It remains to show that for every sequence $\{y_n\}\subset f(X)-f(A)$ convergent to a point $y_0 \in f(A)$ we have $\lim_{n\to\infty} s(y_n)=y_0$. In order to show it, let us observe that the decomposition of X into sets $f^{-1}(y)$ is upper semi-continuous. Consequently $y_n\to y_0$ implies that for each neighbourhood V of $f^{-1}(y_0)$ (in the space X)

$$(1) \qquad f^{-1}(y_n)\subset V \quad \text{for almost all } n .$$

(8) A subset A of a space X is said to be a *retract* of X if there exists a continuous mapping r of X onto A such that $r(x)=x$ for every $x \in A$.

But f and r are continuous. Hence for each neighbourhood U of y_0 (in $f(X)$) there exists a neighbourhood V of $f^{-1}(y_0)$ such that $fr(V)\subset U$. We infer by (1) that

$$frf^{-1}(y_n)\subset U \quad \text{for almost all } n\,,$$

i. e., $\lim frf^{-1}(y_n)=y_0$.

8. Simple mappings and homology groups. Let X, Y be compact spaces and let $A\subset X$, $B\subset Y$ be closed. A continuous mapping

$$\varphi\colon\ (X,A)\to(Y,B)\,,$$

that is a continuous function mapping X onto Y and such that $f(A)\subset B$ is said to be a *relative homeomorphism* (comp. [8], p. 266) if φ maps $X-A$ homeomorphically onto $Y-B$.

By $H_n(X)$ we denote the nth Čech homology group of X and by $H_n(X,A)$ — the nth Čech relative homology group of the pair (X,A) over the group of rational numbers (see [8], p. 237). If $\varphi\colon (X,A)\to(Y,B)$ is a relative homeomorphism then φ induces an isomorphism φ_{n*} mapping the group $H_n(X,A)$ onto the group $H_n(Y,B)$ (see [8], p. 266).

Let f be a simple mapping of a compactum X onto Y and let X_0 denote the seam of f. Then $f\colon (X,\bar X_0)\to\big(Y,f(\bar X_0)\big)$ is a relative homeomorphism inducing an isomorphism f_{n*} of $H_n(X,\bar X_0)$ onto $H_n\big(Y,f(\bar X_0)\big)$. Moreover, f induces a homomorphism f^1_{n*} of $H_n(X)$ into $H_n(Y)$ and a homomorphism f^2_{n*} of $H_n(\bar X_0)$ into $H_n\big(f(\bar X_0)\big)$. The inclusions $\bar X_0\subset X$ and $f(\bar X_0)\subset Y$ induce the homomorphisms i_{n*} and i'_{n*} of $H_n(\bar X_0)$ into $H_n(X)$ and of $H_n\big(f(\bar X_0)\big)$ into $H_n(Y)$. The inclusions $X=(X,0)\subset(X,\bar X_0)$ and $Y=(Y,0)\subset\big(Y,f(\bar X_0)\big)$ induce the homomorphisms j_{n*} and j'_{n*} of $H_n(X)$ into $H_n(X,\bar X_0)$ and of $H_n(Y)$ into $H_n\big(Y,f(\bar X_0)\big)$. Finally we have the boundary homomorphism ∂_n of $H_n(X,\bar X_0)$ into $H_{n-1}(\bar X_0)$ and ∂'_n of $H_n\big(Y,f(\bar X_0)\big)$ into $H_{n-1}\big(f(\bar X_0)\big)$. It is known (see [8], p. 15) that the diagram

$$\begin{array}{ccccccccc}
\ldots\to & H_n(\bar X_0) & \xrightarrow{i_{n*}} & H_n(X) & \xrightarrow{j_{n*}} & H_n(X,\bar X_0) & \xrightarrow{\partial_n} & H_{n-1}(\bar X_0) & \xrightarrow{i_{n-1*}}\ \ldots \\
 & \downarrow f^2_{n*} & & \downarrow f^1_{n*} & & \downarrow f_{n*} & & \downarrow f^2_{n-1*} & \\
\ldots\to & H_n\big(f(\bar X_0)\big) & \xrightarrow{i'_{n*}} & H_n(Y) & \xrightarrow{j'_{n*}} & H_n\big(Y,f(\bar X_0)\big) & \xrightarrow{\partial'_n} & H_{n-1}\big(f(\bar X_0)\big) & \xrightarrow{i'_{n-1*}}\ \ldots
\end{array}$$

is exact in rows and commutative. If the image of i_{n*} is zero then j_{n*} is an isomorphism into and consequently also $j'_{n*}f^1_{n*}=f_{n*}j_{n*}$ is an isomorphism into. Hence

(I) *If the image of i_{n*} is zero then f^1_{n*} is an isomorphism into.*

Moreover,

(II) *If the kernel of f^2_{n-1*} is zero and if f^2_{n*} is onto then f^1_{n*} is onto.*

Proof. It is known (see [8], p. 16) that the hypothesis of (II) implies that $f^1_{n*}[H_n(X)]=j'^{-1}_{n*}\big[f_{n*}\big(H_n(X,\bar X_0)\big)\big]$. But f_{n*} is an isomorphism onto, whence $f_{n*}\big(H_n(X,\bar X_0)\big)=H_n[Y,f(\bar X_0)]$ and thus $f^1_{n*}[H_n(X)]=H_n(Y)$.

(III) *If the image of i_{n*} is zero and the kernel of i_{n-1*} is zero then $H_n(Y)\simeq H_n(X)+i'_{n*}\big[H_n\big(f(\bar X_0)\big)\big]$.*

Proof. In this case j_{n*} maps $H_n(X)$ isomorphically onto $H_n(X,\bar X_0)$. Moreover f_{n*} maps $H_n(X,\bar X_0)$ isomorphically onto $H_n\big(Y,f(\bar X_0)\big)$. Since $f_{n*}j_{n*}=j'_{n*}f^1_{n*}$ we infer that $\chi=j'_{n*}f^1_{n*}$ is an isomorphism onto and the inverse isomorphism χ^{-1} maps $H_n\big(Y,f(\bar X_0)\big)$ onto $H_n(X)$. Setting $\vartheta=f^1_{n*}\chi^{-1}$ we obtain a homomorphism of $H_n\big(Y,f(\bar X_0)\big)$ into $H_n(Y)$. Moreover the homomorphism $j'_{n*}\vartheta=j'_{n*}f^1_{n*}\chi_n^{-1}=\chi\chi^{-1}$ is the identity $\big(\text{on } H_n\big(Y,f(\bar X_0)\big)\big)$. Thus we see that for the homomorphism j'_{n*} there exists a homomorphism ϑ satisfying the condition $j'_n\vartheta=$ identity on $H_n\big(Y,f(\bar X_0)\big)$, *i. e.*, j'_{n*} is an r-homomorphism (in the sense of [6]). It follows (see [6], p. 331) that $H_n(Y)$ is isomorphic with the direct sum of $H_n(X)$ and of the kernel of j'_{n*}, and consequently also with the direct sum of $H_n(X)$ and of the group $i'_{n*}\big[H_n\big(Y,f(\bar X_0)\big)\big]$.

THEOREM (9). *If f is an elementary simple mapping defined on a compact space X and if X_0 denotes the seam of f then the relations*

a) $H_n(X)=0$ *or* $H_n(X_0)=0$,

b) $H_{n-1}(X_0)=0$

imply the isomorphism $H_n(X)\simeq H(Y)$.

Proof. By corollaries 1 and 2 of section 3, f is a local homeomorphism on $X_0=\bar X_0$. It is known (see [7], p. 40) that in this case the homeomorphism f^2_{n*} maps the group $H_n(\bar X_0)$ onto the group $H_n\big(f(\bar X_0)\big)$. It follows by (I) and (II) that f^1_{n*} maps the group $H_n(X)$ onto the group $H_n(Y)$.

COROLLARY 1. *Let f be an elementary simple mapping of a compact space X such that the dimension of the seam X_0 is $\leqslant n-2$. Then $H_n(X)\simeq H_n(Y)$.*

Since for local connected compacta X the condition $H_1(X)=0$ (see [11], p. 273) means that X is unicoherent (see [5], p. 230), we infer

COROLLARY 2. *Let f be an elementary simple mapping of a compact, locally connected, unicoherent space X. If the seam X_0 of f is connected then the space $f(X)$ is unicoherent.*

(9) Similar results, but concerning some other subclass of simple mappings that is a little more general than the class of elementary simple mappings, is obtained in a different way, by J. Jaworowski (see [9], theorem 4 and corollary 3). Let us observe that in our proof the hypothesis that f is elementar is used only to show that the homomorphism f^2_n induced by f, maps $H_n(\bar X_0)$ onto $H_n\big(f(\bar X_0)\big)$.

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On the divisors of zero of the group algebra

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In this paper it will be shown that for every non-trivial locally compact group the group algebra has divisors of zero.

Let G be a locally compact multiplicative group. Its elements will be denoted by the letters a, b, t, τ, u, v; the unit element by e. The letters U, V, W will denote the neighbourhoods of the unit e (open sets with compact closure containing e). If A, B are the sets contained in G, then AB is defined as the set of points $u=ab$, where $a \in A$, and $b \in B$. The set A^{-1} is the set of all t such, that $t^{-1} \in A$. It is proved that for every neighbourhood U of the unit e there exists a symmetric neighbourhood V contained in U, *i. e.*, such a V that $V=V^{-1}$ and $V \subset U$. It is well known that for every locally compact group there exists a unique (to within a multiplicative constant) left invariant Haar measure μ (*i. e.*, $\mu(aA) = \mu(A)$ for every $a \in G$). Generally speaking the left invariant measure is not the right invariant one, but there exists such a continuous function $\Delta(a)$ called the *modular function* that $\mu(Aa)=\mu(A)\Delta(a)$ for every $a \in G$.

It is proved that Δ is a homomorphism into the multiplicative group of positive reals, *i. e.*, $\Delta(a)>0$, and $\Delta(ab)=\Delta(a)\Delta(b)$ for every $a,b \in G$. By the definition of the Haar measure if A is any non-void open set then $\mu(A)>0$, and if B is a compact one then $\mu(B)<\infty$.

We shall consider the Banach space $L_1(G)$ of all complex μ-integrable functions defined over G, with the customary norm

$$|f|_1 = \int |f(\tau)|\,d\mu\,. \tag{1}$$

It is proved that $L_1(G)$ is a Banach algebra with multiplication as the convolution:

$$f * g = \int f(t\tau^{-1})\,g(\tau)\,d\mu(\tau)\,. \tag{2}$$

The algebra $L_1(G)$ is called the *group algebra* when the group G is discrete (in this case, and only in this case, the algebra has the unit element). The group algebra for a non-discrete group is the algebra obtained from $L_1(G)$ by joining the unit element. We shall prove that if G is non-