Applications of formalized consistency proofs II *

by

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R. Montague asked us in conversation to explain the exact relation between Theorem 5, p. 109, and the argument of [12], p. 48. Since the details may be of interest to other readers we think it useful to publish them here. Though it is necessary to use Herbrand's theorem to get an arithmetic formalization of the relative consistency proof, the intuitive idea of [12], p. 48, is simply this: we start with an arithmetic model of set theory (S_1) without class variables, and get a model of a set theory (S) with class variables as follows. The models of the sets of (S) and of the non-logical constants common to (S) and (S_1) are the same as in the given model for (S_1), and the models for other non-logical constants of (S) are defined arithmetically from the model for (S_1): this applies particularly to the functions introduced which make the axiom of domain existence (and extensionality) of (S) quantifier-free. The models of the classes of (S) are the first order predicates definable in terms of (the models of) these non-logical constants. At first sight it appears that this is not sufficient to deal with the axiom of replacement and substitution for the set theories considered in Theorem 5 because not all non-logical constants of (S) appear in (S_1).

Though argument (2) below takes care of all the predicative extensions of set theory considered in the present paper and in [12], it is more natural to deal separately with the cases where (S_1) is general set theory and where (S_1) is ZF.

Consider the axiom of substitution. Let F be a class constant which represents a function and let A be a class constant which represents a set (of the model of (S_1)). We wish to show that $F``A$ is also a set.

(1) In the case of general set theory (p. 48 of [12]), the model of (S_1) consists of finite sets. So since F represents a function, $F``A$ is a finite collection of finite sets, and so it is a set of the model of (S_1) too.

* This is an addition to our paper *Applications of formalized consistency proofs* which appeared in Fundamenta Mathematicae 42 (1955), p. 101-110. We use here the notations of that paper.

(2) By Gödel's theory of constructible sets [1] there is a formula $\mathfrak{A}(n,m)$ of ZF with the following property: in arithmetic (even primitive recursive arithmetic) one can prove that if ZF is consistent so is ZF^* obtained by adding to ZF the proposition: $\mathfrak{A}(n,m)$ defines a function from non-empty sets to a member of the set, *i. e.*

$$\mathfrak{A}(n,m)\rightarrow m\in n$$

$$(n)(p)(Em)[p\in n\rightarrow\mathfrak{A}(n,m)]\ \&\ (n)(m)(p)[\mathfrak{A}(n,m)\ \&\ \mathfrak{A}(n,p)\rightarrow m=p]\,.$$

So, on the assumption of $\mathrm{Con}(ZF)$ we get a model in arithmetic of ZF^*. With the help of the choice predicate there are explicit definitions of the function symbols introduced to make the axioms of NB quantifier-free, in terms of the notation of ZF, and so we get a consistency proof for NB on the assumption of $\mathrm{Con}(ZF)$ by the method of [12].

We take this opportunity to draw the reader's attention to the following convention: when counting quantifiers, we count the number of distinct symbols and not the number of occurrences; also we do not use distinct symbols unless one quantifier is within the scope of the other, *e. g.* we write $(x)[A(x)\rightarrow(Ex)B(x)]$ instead of $(x)[A(x)\rightarrow(Ey)B(y)]$. The truth definition applies under this convention which is used throughout the paper.

[1] This theory was developed by Gödel for NB. The modifications necessary for applying the theory to ZF are obvious.

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