On covering theorems*

by

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1. Introduction. In his fundamental paper [2] (numbers in square brackets refer to the References at the end of this note) A. P. Morse derives various covering theorems of an extremely general character. The purpose of this note is to formulate an even more general covering theorem which is concerned with a pair of abstract binary relations σ, δ. This covering theorem (to be referred to as the (σ, δ)-covering theorem) is stated and proved in section 3. The relationship of this (σ, δ)-covering theorem to the corresponding result of A. P. Morse is explained in section 4. In applying such covering theorems in metric spaces, a conceptual ambiguity arises in the following manner. Let M be a metric space with distance function d. Given a point $a \in M$ and a finite, positive real number r, let us put

$$\gamma(a, r) = \{x \mid x \in M,\ d(x, a) \leqslant r\}.$$

A subset C of M is termed a closed sphere if it can be represented in the form $C = \gamma(a, r)$. Simple examples show that the center a and the radius r are generally not uniquely determined by C. This ambiguity necessitates a certain amount of care in formulating covering theorems in terms of closed spheres. Section 5 contains suggestions along these lines.

In the proof of the (σ, δ)-covering theorem we use the set inclusion form of Zorn's lemma (see section 2 for the statement of this lemma). In view of the generality of the (σ, δ)-covering theorem the question arises whether this theorem is sufficiently general to imply, conversely, the set inclusion form of Zorn's lemma. This is indeed the case. In fact a very special case of the (σ, δ)-covering theorem is already adequate to imply the general form of Zorn's lemma (see section 6 for the statement of the general form of Zorn's lemma and the proof of this fact).

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2. Notations and definitions. Let X be a non-empty set. Let there be given over X two binary relations σ and δ. For each element $x \in X$ we define the sets $N_\sigma(x) = \{y|\ y \in X, y\sigma x\}$, $N_\delta(x) = \{y|\ y \in X, y\delta x\}$, $N(x) = N_\sigma(x) \cap N_\delta(x)$. For each subset E of X we define (where $\emptyset$ is the empty set)

$$N_\sigma(E) = \bigcup_{x \in E} N_\sigma(x), \quad N_\delta(E) = \bigcup_{x \in E} N_\delta(x), \quad N(E) = \bigcup_{x \in E} N(x), \quad N(\emptyset) = \emptyset.$$

Concerning the binary relations σ and δ we make the following assumptions:

(i) σ and δ are both reflexive. That is, $x\sigma x$ and $x\delta x$ for every $x \in X$.

(ii) σ is symmetric. That is, $x\sigma y$ if and only if $y\sigma x$.

(iii) If E is any non-empty subset of X then there exists at least one element $x \in X$ such that $x \in E \subset N_\delta(x)$. Such an element x will be termed a *dominant element* of the set E.

A subset E of X will be termed *scattered* if E contains no pair of distinct elements x, y such that $x\sigma y$. Thus the empty set is scattered and every set consisting of a single element is scattered.

Let $\mathfrak{F}$ be a non-empty family of non-empty subsets of X. A subfamily $\mathfrak{C}$ of $\mathfrak{F}$ will be termed a chain in $\mathfrak{F}$ if $\mathfrak{C}$ is non-empty and for any two sets $F' \in \mathfrak{C}$, $F'' \in \mathfrak{C}$ at least one of the two relations $F' \subset F''$, $F'' \subset F'$ holds. $\mathfrak{F}$ will be termed *chain closed* if for any chain $\mathfrak{C}$ in $\mathfrak{F}$ the set $\bigcup_{F \in \mathfrak{C}} F$ is in $\mathfrak{F}$. A set $F \in \mathfrak{F}$ is termed *maximal* in $\mathfrak{F}$ if F is not a proper subset of any set in $\mathfrak{F}$. The set inclusion form of Zorn's lemma can be now stated as follows (see Zorn [3]). If a non-empty family $\mathfrak{F}$ of non-empty subsets of X is chain closed then $\mathfrak{F}$ contains a maximal set.

3. Covering theorems. (σ, δ)-COVERING THEOREM. *Under the circumstances described in section 2 there exists a scattered subset S of X such that $X = N(S)$.*

Proof. Let us denote by $\mathfrak{F}$ the family of all those non-empty, scattered subsets F of X which satisfy the condition that for each $x \in F$, $X - N(F-x) \subset N_\delta(x)$.

STATEMENT 1. $\mathfrak{F} \neq \emptyset$.

Indeed, since X is non-empty by assumption there exists a dominant element x_0 of X. It is obvious that the set F consisting of the single element x_0 belongs to $\mathfrak{F}$ and hence $\mathfrak{F} \neq \emptyset$.

STATEMENT 2. *$\mathfrak{F}$ is chain closed.*

Indeed, if $\mathfrak{C}$ is a chain in $\mathfrak{F}$, the set $F_0 = \bigcup_{F \in \mathfrak{C}} F$ is non-empty and it is easily verified that F_0 is a scattered set. For $x \in F_0$ there is an $F \in \mathfrak{C}$ such that $x \in F$ and hence, since $F \in \mathfrak{F}$, $X - N(F-x) \subset N_\delta(x)$. Since $F - x \subset F_0 - x$ it follows that $X - N(F_0 - x) \subset X - N(F-x) \subset N_\delta(x)$. Thus $F_0 \in \mathfrak{F}$ and $\mathfrak{F}$ is chain closed.

STATEMENT 3. *If the set $F \in \mathfrak{F}$ is such that $X - N(F) \neq \emptyset$ then F is not maximal in $\mathfrak{F}$.*

Indeed, let x^* be a dominant element of $X - N(F)$ and set $F^* = x^* \cup F$. Then F^* is a non-empty set. We now show that F^* is a scattered set. If there were an $x \in F$ such that $x^* \in N_\sigma(x)$ then, since $x^* \in X - N(F) \subset X - N(F-x) \subset N_\delta(x)$, it would follow that $x^* \in N_\sigma(x) \cap N_\delta(x) = N(x)$. This would contradict the fact that $x^* \in X - N(F) \subset X - N(x)$. Thus there is no $x \in F$ such that $x^* \in N_\sigma(x)$ and hence, since F is a scattered set, F^* is a scattered set. Finally, we show that for $x \in F^*$ we have $X - N(F^* - x) \subset N_\delta(x)$.

Case 1. $x \in F$. Then $X - N(F^* - x) \subset X - N(F - x) \subset N_\delta(x)$ since $F \in \mathfrak{F}$.

Case 2. $x \notin F$. Then $x = x^*$ and $X - N(F^* - x) = X - N(F^* - x^*) = X - N(F) \subset N_\delta(x^*)$, since x^* is a dominant element of $X - N(F)$. Therefore, $F^* \in \mathfrak{F}$. F is thus not maximal in $\mathfrak{F}$.

The proof of the (σ, δ)-covering theorem is now immediate. By Zorn's lemma we infer from Statements 1 and 2 that $\mathfrak{F}$ contains a maximal set F_0. By Statement 3 we must have then $X = N(F_0)$. Since $F_0 \in \mathfrak{F}$ this set F_0 is scattered and the (σ, δ)-covering theorem is thus proved.

Given any σ as described in section 2 and defining the binary relation δ to be such that $x\delta y$ hold for every pair of elements $x, y \in X$ we have $N_\delta(x) = X$ for $x \in X$ and hence $N(x) = N_\sigma(x) \cap N_\delta(x) = N_\sigma(x) \cap X = N_\sigma(x)$ for $x \in X$. As a corollary to the (σ, δ)-covering theorem we have thus the following covering theorem.

σ-COVERING THEOREM. *Under the circumstance described in section 2 there exists a scattered subset S of X such that $X = N_\sigma(S)$.*

It should be noted that a scattered set S for which $X = N_\sigma(S)$ is necessarily a *maximal scattered set.*

4. A special case. Let M be a non-empty set and let $\mathfrak{X}$ be a non-empty family of non-empty subsets of M. Let $f(E)$ be a real-valued, non-negative, bounded function defined for all sets $E \in \mathfrak{X}$. Let the binary relations σ and δ be defined over $\mathfrak{X}$ as follows. $E'\sigma E''$ if and only if $E' \cap E'' \neq \emptyset$ and $E'\delta E''$ if and only if $f(E') \leqslant 2f(E'')$. In applying the (σ, δ)-covering theorem to this situation the family $\mathfrak{X}$ plays the role of the set X in section 2 and accordingly the sets $E \in \mathfrak{X}$ and the sub-families of $\mathfrak{X}$ correspond to the elements and the subsets respectively of X. We shall also use $\mathfrak{N}$ instead of N. The conditions (i) and (ii) in section 2 are obviously satisfied for the given σ and δ. As regards condition (iii) in section 2 consider any non-empty sub-family $\mathfrak{E}$ of $\mathfrak{X}$ and put

$$\mu(\mathfrak{E}) = \operatorname*{lub}_{E\in\mathfrak{E}} f(E).$$

Since f is bounded we have then $0 \leqslant \mu(\mathfrak{E}) < \infty$. Hence we have a set $A \in \mathfrak{E}$ such that $\mu(\mathfrak{E}) \leqslant 2f(A)$. If E is any set in $\mathfrak{E}$ then $f(E) \leqslant \mu(\mathfrak{E}) \leqslant 2f(A)$ and hence $E\delta A$ for every set $E \in \mathfrak{E}$. Thus $A \in \mathfrak{E} \subset \mathfrak{N}_\delta(A)$ and condition (iii) in section 2 is verified.

Let us also note that under the present choice of the binary relation σ a sub-family $\mathfrak{E}$ of $\mathfrak{X}$ is scattered if and only if $\mathfrak{E}$ is disjoint, that is, if $E' \in \mathfrak{E}$, $E'' \in \mathfrak{E}$, $E' \neq E''$ implies that $E' \frown E'' = \emptyset$. Then the (σ, δ)-covering theorem of section 3 yields the existence of a disjoint sub-family $\mathfrak{S}$ of $\mathfrak{X}$ such that $X = \mathfrak{N}(\mathfrak{S})$. Explicitly (see the definitions in section 2): if E is any set of $\mathfrak{X}$ then there exists a set S in the disjoint family $\mathfrak{S}$ such that $f(E) \leqslant 2f(S)$ and $E \frown S \neq \emptyset$. This result is contained, essentially, in the important paper [2] of A. P. Morse. One sees that the comparative generality of the (σ, δ)-covering theorem in section 3 consists primarily in replacing the special binary relations considered by A. P. Morse by any two binary relations σ, δ satisfying conditions (i), (ii), (iii) in section 2.

5. Comments on metric spaces. Let M be a metric space with distance-function d. To simplify the presentation we assume that M is separable. Then every disjoint family of closed spheres (see section 1) is countable and hence consists of a (finite or countably infinite) sequence of pair-wise disjoint closed spheres C_n, $n = 1, 2, \ldots$

LEMMA 1. *Let A be a subset of M and let $\mathfrak{X}$ be a non-empty family of closed spheres in M which cover A. Assume further that the diameters of the closed spheres $C \in \mathfrak{X}$ are less than some finite positive constant K. Then there exists a sequence C_n, $n = 1, 2, \ldots$, of pair-wise disjoint closed spheres in $\mathfrak{X}$ such that if for each n an arbitrary representation $C_n = \gamma(a_n, r_n)$ is assigned (see section 1) then*

$$A \subset \bigcup_n \gamma(a_n, 5r_n). \tag{1}$$

Proof. To apply the covering theorem considered in section 4 we define $f(C) = \operatorname{diam} C$ for $C \in \mathfrak{X}$. By section 4 we infer (using the separability of M) the existence of a sequence of pair-wise disjoint closed spheres $C_n \in \mathfrak{X}$, $n = 1, 2, \ldots$, such that for every $C \in \mathfrak{X}$ there exists a C_n satisfying the relations

$$\operatorname{diam} C \leqslant 2 \operatorname{diam} C_n, \qquad C \frown C_n \neq \emptyset. \tag{2}$$

For each n, let there be assigned a representation

$$C_n = \gamma(a_n, r_n). \tag{3}$$

We assert that (2) and (3) imply that

$$C \subset \gamma(a_n, 5r_n). \tag{4}$$

Indeed, let x be any point of C. By (2) there is a point y such that $y \in C \frown C_n$.

Then, in view of (2) and (3),

$$d(x, y) \leqslant \operatorname{diam} C \leqslant 2 \operatorname{diam} C_n \leqslant 4r_n, \qquad d(y, a_n) \leqslant r_n,$$

and hence

$$d(x, a_n) \leqslant d(x, y) + d(y, a_n) \leqslant 5r_n.$$

Thus (4) is verified. Since by assumption

$$A \subset \bigcup_{C\in\mathfrak{X}} C,$$

the inclusion (1) follows.

DEFINITION. A family $\mathfrak{X}$ of closed spheres in M is said to cover a subset A of M in the Vitali sense if for each point $a \in A$ and for arbitrary $\varepsilon > 0$ there exists a closed sphere C (which depends upon both a and ε) such that $a \in C \in \mathfrak{X}$ and $\operatorname{diam} C < \varepsilon$.

LEMMA 2. *Let $\mathfrak{X}$ be a non-empty family of closed spheres in M which cover a subset A of M in the Vitali sense. Then there exists a sequence C_n, $n = 1, 2, \ldots$, of pair-wise disjoint closed spheres in $\mathfrak{X}$ such that if an arbitrary representation $C_n = \gamma(a_n, r_n)$ is assigned for $n = 1, 2, \ldots$, then*

$$A - \bigcup_n C_n \subset \bigcup_{n=N+1}^{\infty} \gamma(a_n, 5r_n) \tag{5}$$

for every choice of the positive integer N.

Proof. Clearly those closed spheres $C \in \mathfrak{X}$ for which $\operatorname{diam} C < 1$ cover A in the Vitali sense and hence we can assume without loss of generality that $\operatorname{diam} C < 1$ for $C \in \mathfrak{X}$. Applying again section 4 with $f(C) = \operatorname{diam} C$, there follows the existence of a sequence C_n, $n = 1, 2, \ldots$, of pair-wise disjoint closed spheres in $\mathfrak{X}$ such that for every $C \in \mathfrak{X}$ there exists a C_n satisfying the relations

$$\operatorname{diam} C \leqslant 2 \operatorname{diam} C_n, \qquad C \frown C_n \neq \emptyset. \tag{6}$$

Let the representation

$$C_n = \gamma(a_n, r_n) \tag{7}$$

and the positive integer N be arbitrarily assigned and consider any point $x \in A - \bigcup_n C_n$. We have *a fortiori*

$$x \in A - \bigcup_{n=1}^{N} C_n. \tag{8}$$

Since the union appearing in (8) is a closed set which does not contain the point x and since $\mathfrak{X}$ covers A in the Vitali sense, there follows the existence of a closed sphere C such that

$$x \in C \in \mathfrak{X}, \quad C \cap C_n = \emptyset \quad \text{for} \quad n = 1, 2, \dots, N. \tag{9}$$

On the other hand there exists a C_n satisfying (6). From (9) it follows that $n > N$ for this particular C_n. As in the proof of lemma 1, we conclude from (6) that $C \subset \gamma(a_n, 5r_n)$. Thus it follows that $x \in \gamma(a_n, 5r_n)$ for some $n > N$ and the inclusion (5) is proved.

Let us note that the essential content of the lemmas 1 and 2 is not new. Our sole purpose in discussing these lemmas was to suggest a way to deal with the issue arising from the fact, noted in the introduction, that in a metric space the center and the radius of a closed sphere are not uniquely determined in general.

6. Zorn's lemma. To state the general form of Zorn's lemma we need the following definitions (see Lefschetz [1], p. 4-5). In a non-empty set X let there be given a partial order in terms of an ordering relation to be denoted by $<$. The only assumption is that $<$ is transitive. That is to say, $x < y$, $y < z$ implies that $x < z$. We agree that we may write $y > x$ instead of $x < y$. We say that the elements x, y of X are comparable if one at least of the relations $x < y$, $y < x$ holds. A set E of X is simply ordered if any two *distinct* elements of E are comparable. An element $b \in X$ is an upper bound for a subset E of X if $x \in E$ implies that $x < b$. An element $m \in X$ is said to be maximal in X if every element $x \in X$ which is comparable with m satisfies the relation $x < m$. The general form of Zorn's lemma can now be stated as follows.

ZORN'S LEMMA. *If every simply ordered subset of X has an upper bound then for an arbitrary $x_0 \in X$, there is a maximal $m > x_0$.*

For the sake of completeness we shall show that this lemma follows from the σ-covering theorem of section 3. We define the binary relation σ in X as follows. For $x, y \in X$, $x\sigma y$ if and only if either $x = y$ or $x \neq y$ and x and y are not comparable. Then σ is reflexive and symmetric. A set E is scattered if and only if it is simply ordered. Let x_0 be an arbitrary element of X. By the σ-covering theorem (see the remark following it) there is a maximal scattered set S_0 in $X - N_\sigma(x_0)$ and hence $S = x_0 \cup S_0$ is a maximal scattered set in X. Since S is scattered it is simply ordered and by assumption there is an upper bound b for S. Then

$$x \in S \quad \text{implies that} \quad x < b. \tag{10}$$

We assert that $y > b$ implies that $y < b$. Indeed, $y > b$ implies that $y > x$ for $x \in S$ and hence $y \cup S$ is simply ordered. Since $y \cup S$ is thus a scattered set and S is a maximal scattered set it follows that $y \in S$. Hence by (10) $y < b$. Thus $y > b$ implies that $y < b$ and so b is maximal in X and, since $x_0 \in S$, by (10) $b > x_0$.

References

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[2] A. P. Morse, *A theory of covering and differentiation*, Transactions of the American Mathematical Society 55 (1944), p. 205-235.

[3] M. Zorn, *A remark on method in transfinite algebra*, Bulletin of the American Mathematical Society 41 (1935), p. 667-670.

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