Topological characterization of the Sierpiński curve *

by

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1. Introduction. The universal plane curve described by Sierpiński [4] in 1916 has proven highly useful in the developments of various phases of topology and analysis which have gone ahead at such a rapid pace in the intervening period of over forty years. Interest in this curve and its analog in 3-space is currently much alive and its rôle in mathematics is surely by no means finished. The curve is obtained very simply as the residual set remaining when one begins with a square and applies the operation of dividing it into nine equal squares and omitting the interior of the center one, then repeats this operation on each of the surviving 8 squares, then repeats again on the surviving 64 squares, and so on indefinitely. Sierpiński showed that this set contains a topological image of every plane continuum having no interior point and thus it has come to be known as the *Sierpiński plane universal curve.*

The main results in the present paper were first stated by the author in a talk before the Warsaw Mathematical Colloquium in the spring of 1930, on which occasion he was honored to be introduced to the group by Professor Sierpiński. An abstract was published at that time [5]; but since the treatment and proofs in hand then were long and complicated, no detailed publication has been made of the results until now, though they were utilised by other authors (cf. [1], p. 258). Recently the author received a letter from B. Knaster indicating that reference to these results would be appropriate in connection with recent work of his and A. Lelek's and urging that publication be made. As a result of this letter, the subject has been restudied by the author from a more modern viewpoint with the result that new and simpler proofs have been devised, based on markedly different techniques. This new treatment is presented in this paper.

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2. Preliminaries. By an *S-curve* will be meant a plane locally connected 1-dimensional continuum S such that the boundary of each complementary domain of S is a simple closed curve and no two of these complementary domain boundaries intersect. It will be noted that the Sierpiński curve is an S-curve; and one of our main results is that all S-curves are homeomorphic. The boundary of the unbounded complementary domain of a plane continuum is called its *outer boundary.*

Remark. *If C is any simple closed curve lying in an S-curve S, I is the interior of C, and if no complementary domain boundary of S lying in $I+C$ has a point in common with C, then $T=S\cdot(C+I)$ is an S-curve.*

For the complementary domain boundaries of T are C together with all complementary domain boundaries of S lying entirely in I, and no two of these intersect.

Definition. A *subdivision* σ of an S-curve S is a division of S into a finite number of S-curves effected by taking a simplicial or cellular subdivision σ' of the elementary closed region R obtained by adding to S all save a finite number $I_1, I_2, \ldots, I_n$ of its bounded complementary regions in such a way that the union K of all 1-cells of σ' (*i. e.*, the 1-dimensional structure of σ') lies entirely in S and contains the boundary of R but does not intersect the boundary of any bounded complementary region of S other than $I_1, I_2, \ldots, I_n$. The intersections of the 2-cells of σ' with S give a collection of S-curves (see Remark) constituting the "2-cells" of the subdivision σ of S.

Lemma 1. *If S and S' are S-curves with outer boundaries C_0 and C_0' respectively, ε is any positive number and h is any homeomorphism of C_0 onto C_0', there exist ε-subdivisions of S and S' whose* 1*-dimensional structures K and K' correspond under a homeomorphism which is an extension of h.*

Proof of Lemma 1. Let n be an integer such that there are not more than n bounded complementary domains of either S or S' of diameter $\geqslant\varepsilon$. Let $C_0, C_1, C_2, \ldots, C_n$ and $C_0', C_1', C_2', \ldots, C_n'$ be sets of distinct complementary domain boundaries of S and S' respectively including all those of diameter $\geqslant\varepsilon$ in each case. Decompose S into the complementary domain boundaries not included in $(C_n)_0^n$ together with the individual points of S not on such a boundary and let W be the hyperspace and φ the natural mapping. Decompose S' similarly, letting W' be the hyperspace and φ' the natural mapping. Then both W and W' are topologically equivalent to a closed plane elementary region with $n+1$ boundary curves. (By the theorem of R. L. Moore [3], since the decomposition extends to an upper semi-continuous decomposition of the whole plane into continua not separating the plane by adding its interior to each non-degenerate element and including as new elements all individual points in the other complementary regions of S or of S' respectively.) Thus the homeomorphism $\varphi' h \varphi^{-1}$ of the boundary curve $\varphi(C_0)$ of W onto the boundary curve $\varphi'(C_0')$ of W' can be extended to a homeomorphism

$$t(W) = W'.$$

Now let Q be the (countable) set in W' consisting of all points which are images of non-degenerate elements of the decomposition of S' under φ' together with images of non-degenerate elements of the decomposition of S under $t\varphi$. For a given $\delta > 0$ let us take a simplicial subdivision Σ of W' of mesh $< \delta$ effected by a 1-dimensional graph G in W' which does not intersect Q but which contains all the boundary curves of W'. Then on G both φ'^{-1} and $\varphi^{-1}t^{-1}$ are homeomorphisms. Thus the sets $K' = \varphi'^{-1}(G)$ and $K = \varphi^{-1}t^{-1}(G)$ effect subdivisions σ and σ' respectively of S and S' which correspond in 1-1 fashion with Σ. Also $\varphi'^{-1}t\varphi(K) = K'$ is a homeomorphism which on C_0 reduces to h, since t reduces to $\varphi' h \varphi^{-1}$ on $\varphi(C_0)$ so that we have

$$\varphi'^{-1}\varphi' h \varphi^{-1}\varphi(C_0) = h(C_0) = C_0'.$$

Thus h extends to a homeomorphism from K onto K'.

Finally, for δ sufficiently small, both σ and σ' will be of mesh $< \varepsilon$. This is a direct consequence of the fact that the mappings $t\varphi$ and φ' are ε-mappings, *i. e.*, point inverses are of diameter $< \varepsilon$ in each case, together with the easily verified

LEMMA 2. *If A and B are compact metric spaces and $f(A) = B$ is an ε-mapping, there exists a $\delta > 0$ such that if B_0 is any subset of B of diameter $< \delta$, the diameter of $f^{-1}(B_0)$ is $< \varepsilon$.*

For, if not, there exists a sequence of subsets $B_1, B_2, \ldots$ of B such that $\delta[B_n] < 1/n$ and $\delta[f^{-1}(B_n)] \geqslant \varepsilon$. We may suppose the B_n so chosen that these sets converge to a single point $b \in B$ and so that the sets $f^{-1}(B_n)$ converge to a set A_0 in A. We then have $\delta(A_0) \geqslant \varepsilon$ whereas $f(A_0) = b$ by continuity of f; and this contradicts the fact that f is an ε-mapping.

3. THEOREM. *Any two S-curves are homeomorphic.*

Proof. Let S and S' be S-curves with outer boundaries C_0 and C_0' respectively and let h_0 be any homeomorphism of C_0 onto C_0'. Taking $\varepsilon = 1$ in Lemma 1, there exist subdivisions σ_1 and σ_1' of S and S' respectively of mesh < 1 whose 1-dimensional structures K_1 and K_1' correspond under a homeomorphism $h_1(K_1) = K_1'$ which is an extension of h_0, (*i. e.*, $h_1(x) \equiv h_0(x)$ for $x \in K$) and whose "2-cells" correspond in 1-1 fashion under h_1.

Now taking $\varepsilon = \frac{1}{2}$, for each pair S_1, S_1' of corresponding "2-cells" of σ_1 and σ_1' Lemma 1 yields an extension of h_1 from the outer boundaries of S_1 and S_1' to the 1-dimensional structures of $\frac{1}{2}$-subdivisions of S_1 and S_1'. These separate extensions and subdivisions of the corresponding cells of σ_1 and σ_1' combine to yield $\frac{1}{2}$-subdivisions σ_2 and σ_2' which are refinements of σ_1 and σ_1', respectively, and an extension h_2 of h_1 which is a homeomorphism of the 1-dimensional structure K_2 of σ_2 onto the 1-dimensional structure K_2' of σ_2'.

We next take $\varepsilon = \frac{1}{3}$ and obtain $\frac{1}{3}$-refinements σ_3 and σ_3' of σ_2 and σ_2' similarly and a homeomorphism $h_3(K_3) = K_3'$ which is an extension of h_2 to the 1-dimensional structures K_3 and K_3' of σ_3 and σ_3' respectively. Continuing in this fashion, for each n we obtain $1/n$-refinements σ_n and σ_n' of σ_{n-1} and σ_{n-1}' respectively and a homeomorphism

$$h_n(K_n) = K_n'$$

which is an extension of h_{n-1} to the 1-dimensional structures K_n and K_n' of σ_n and σ_n' respectively.

For each n and each $x \in K_n$ we define $h(x) = h_n(x)$. Then since h_n is in every case an extension of h_{n-1} it follows that $h(x)$ is a 1-1 transformation of $\sum K_n = K$ onto $\sum K_n' = K'$.

Further, h is uniformly continuous. For let $\varepsilon > 0$ be given. Then if we choose a fixed $n > 2/\varepsilon$, we can take $\delta > 0$ so that any two different cells of σ_n either intersect or else they are at a distance $> \delta$ apart. Then if $x, y \in K$ and $\varrho(x, y) < \delta$, x and y must be in cells S_x and S_y of σ_n which are either the same or else they intersect. Thus the corresponding cells S_x' and S_y' of σ_n' must intersect so that their union $S_x' + S_y'$ is of diameter $< \varepsilon$, because each is of diameter $< 1/n < \varepsilon/2$. However, this gives

$$\varrho[h(x), h(y)] < \varepsilon,$$

because by the method of defining h we have

$$h(x) \in S_x', \quad h(y) \in S_y'.$$

Thus h is uniformly continuous. By exactly the same type of argument, h^{-1} is uniformly continuous.

Since K and K' are dense in S and S', there exist unique continuous extensions [6] of h and h^{-1} to S and S' respectively. The extension of h^{-1} thus obtained is the inverse of the extended h so that we have a homeomorphism of S onto S' as required.

COROLLARY. *Every S-curve is homeomorphic with the Sierpiński curve.*

4. THEOREM. *In order that a plane 1-dimensional locally connected continuum M be an S-curve it is necessary and sufficient that it have no local separating point.*

A point p is a local separating point of a locally connected continuum M provided it is a cut point of some connected open subset of M. Suppose, first, that M has no local separating point. Then M has no cut point and hence, by a theorem of R. L. Moore's [2], the boundary of every complementary domain of M is a simple closed curve. Further, no two of these complementary domain boundaries can intersect, because any point of intersection of two such boundaries would be accessible from both domains and hence, by a theorem of the author's [7], would be a local separating point of M. Thus M is an S-curve.

On the other hand, if M is an S-curve, it can have no local separating point because it is homeomorphic with the Sierpiński curve S and S has no such point. To see this latter, we have only to note that clearly each point p of S is interior relative to S to an arbitrarily small S-curve in S and thus no region in S can have a cut point, as no S-curve has a cut point.

Note. An elementary proof for the above theorem, independent of the theorem in § 3 is easily given based on the Plane Separation Theorem (see Chapter VI, § 3, of the author's book referred to in [4], p. 28, 38 and 56, for example).

References

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On covering theorems*

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1. Introduction. In his fundamental paper [2] (numbers in square brackets refer to the References at the end of this note) A. P. Morse derives various covering theorems of an extremely general character. The purpose of this note is to formulate an even more general covering theorem which is concerned with a pair of abstract binary relations σ, δ. This covering theorem (to be referred to as the (σ, δ)-covering theorem) is stated and proved in section 3. The relationship of this (σ, δ)-covering theorem to the corresponding result of A. P. Morse is explained in section 4. In applying such covering theorems in metric spaces, a conceptual ambiguity arises in the following manner. Let M be a metric space with distance function d. Given a point $a \in M$ and a finite, positive real number r, let us put

$$\gamma(a, r) = \{x \mid x \in M,\ d(x, a) \leqslant r\}.$$

A subset C of M is termed a closed sphere if it can be represented in the form $C = \gamma(a, r)$. Simple examples show that the center a and the radius r are generally not uniquely determined by C. This ambiguity necessitates a certain amount of care in formulating covering theorems in terms of closed spheres. Section 5 contains suggestions along these lines.

In the proof of the (σ, δ)-covering theorem we use the set inclusion form of Zorn's lemma (see section 2 for the statement of this lemma). In view of the generality of the (σ, δ)-covering theorem the question arises whether this theorem is sufficiently general to imply, conversely, the set inclusion form of Zorn's lemma. This is indeed the case. In fact a very special case of the (σ, δ)-covering theorem is already adequate to imply the general form of Zorn's lemma (see section 6 for the statement of the general form of Zorn's lemma and the proof of this fact).

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