Ideals in rings of continuous functions *

by

C. W. Kohls (Lafayette, Ind.)

1. Introduction. Let X be a completely regular Hausdorff space. In this paper, we study several problems about ideals in the ring $C(X)$ of all continuous real-valued functions on X, and in the ring $C^*(X)$ of all bounded continuous real-valued functions on X.

Familiarity with the main results of [3], [5] and [8] will be assumed. However, a brief review of some of the concepts needed is given in section 2. We also make use of the results of the preceding paper [13], which will be referred to throughout as [K].

In sections 3 and 4, we are concerned with certain ideals of $C^*(X)$, namely, the ring of functions "vanishing at ∞", the subring of functions with compact supports, and the ideals of $C^*(X)$ which are contained in the first ring and contain the second. In section 4, we obtain an algebraic characterization of a certain subclass of this collection. In these sections, we usually assume that X is a locally compact Hausdorff space.

Section 5 is devoted to an investigation of the ideals contained in a given maximal ideal, and the quotient rings obtained from some of these ideals, under the hypothesis that the prime ideals in this family intersect in a prime ideal. The study of rings of functions satisfying this requirement was initiated in [4].

The last section contains miscellaneous results connected with some algebraic questions raised in [K], and with the concept of P-space introduced in [3].

2. Preliminary remarks. Throughout this paper, X denotes a completely regular Hausdorff space. The letter R is reserved for the field of real numbers. We are primarily concerned with the following rings: $C(X)$, the ring of all continuous real-valued functions on X; $C^*(X)$, the subring of all bounded functions of $C(X)$; $C_s(X)$, the subring of all

* This paper, which was prepared while the author was a Predoctoral Fellow of the National Science Foundation, U. S. A., constitutes a section of a doctoral dissertation, written under the supervision of Prof. Leonard Gillman. The author wishes to express his gratitude to Prof. Gillman for the advice and encouragement given during the preparation of this paper.

functions of $C(X)$ with compact supports; and $C_\infty(X)$, the subring of all functions of $C(X)$ which "vanish at ∞": $f \in C_\infty(X)$ if and only if $f \in C(X)$ and for each $\varepsilon > 0$, the set $\{x \in X\colon |f(x)| \geqslant \varepsilon\}$ is compact. This last concept can be generalized. We define the subring of $C(X)$ of functions which "approach a limit at ∞" to be all functions of the form $f + r\cdot 1$, where $f \in C_\infty(X)$, $r \in R$ and 1 is the identity of $C(X)$.

As is well known, with each space X there is associated a compact Hausdorff space βX, the Čech compactification of X, having the properties: (1) X is (homeomorphic to) a dense subspace of βX; (2) every $f \in C^*(X)$ has a continuous extension f^β over βX. The space βX is unique (up to homeomorphism). The closure in X of any set $A \subseteq X$ will be written as $\bar{A}$; and in βX, as A^β. The space vX is the largest subspace of βX over which *every* function in $C(X)$ (whether bounded or not) has a continuous extension. Furthermore, if $f \in C(X)$ is regarded as a function from X to the one-point compactification of R, designated by $R \cup \{\infty\}$, then f may be extended to a continuous function $\hat{f}$ from βX to $R \cup \{\infty\}$. As observed in [5], this follows from a theorem of Stone. (See [5] for further discussion of the function $\hat{f}$).

For every $f \in C(X)$, the set $Z(f) = \{x \in X\colon f(x) = 0\}$ is called the zero-set of f. For any subset I of $C(X)$, we let $\mathcal{Z}(I) = \{Z(f)\colon f \in I\}$.

Let A be a commutative ring. The set of primitive (*i. e.*, prime maximal) ideals of A is denoted by $\mathfrak{M}(A)$. If this set is given the Stone topology (cf. [K], § 2), it will be written $\mathfrak{M}_s(A)$; and if it is given some other topology T, this will be indicated as $\mathfrak{M}_T(A)$. When A is a subring of B, and the mapping γ defined by $\gamma(M) = M \cap A$, $M \in \mathfrak{M}_s(B)$, is into $\mathfrak{M}_s(A)$, then it is continuous. This statement follows from the discussion in [9], § 3.

It is well known that $\mathfrak{M}_s(C^*(X))$ and $\mathfrak{M}_s(C(X))$ are both homeomorphic to βX. In the first case, $M \in \mathfrak{M}_s(C^*(X))$ if and only if $M = M^{*p} = \{f \in C^*(X)\colon f^\beta(p) = 0\}$ for some $p \in \beta X$. The correspondence in the second case is given explicitly in:

LEMMA 2.1 (Gelfand-Kolmogoroff). *For every point p in βX, the set $M^p = \{f \in C(X)\colon p \in Z(f)^\beta\}$ is a maximal ideal of $C(X)$. Conversely, for every maximal ideal M of $C(X)$ there is a unique $p \in \beta X$ such that $M = M^p$.*

For a proof, see [5].

Furthermore, in either ring, the subspace of all fixed ideals (*i. e.*, such that $p \in X$) is homeomorphic to X.

For any ideal I of $C(X)$, we define $\Delta(I) = \bigcap_{f \in I} Z(f)^\beta = \bigcap_{Z \in \mathcal{Z}(I)} Z^\beta$. Equivalently, $\Delta(I) = \{p \in \beta X\colon M^p \supseteq I\}$. As noted in [5], p. 453, the equivalence is a consequence of the Gelfand-Kolmogoroff lemma. It is evident that $\Delta(I)$ is a closed subset of βX.

An ideal of $C(X)$ of particular interest to us, which was introduced in [3], is N^p ($p \in \beta X$). This is defined to be all $f \in C(X)$ such that $Z(f)$ contains the intersection of X with a neighborhood of p in βX.

In the terminology of [K], for any commutative ring A, and any $a \in A$, the set $\mathfrak{M}(a) = \{M \in \mathfrak{M}(A)\colon a \in M\}$ is called the $\mathfrak{M}$-set of a. Let I be an ideal of A. We shall say that I is a $\mathfrak{Z}$*-ideal* if whenever $\mathfrak{M}(a) = \mathfrak{M}(b)$ and $b \in I$, then $a \in I$. It is useful to examine this definition for the rings $C^*(X)$ and $C(X)$. If $f \in C^*(X)$, then $\mathfrak{M}(f)$ is the zero-set of f^β, regarded as an element of $C(\beta X)$. Thus, an ideal I of $C^*(X)$ is a $\mathfrak{Z}$-ideal if whenever $Z(f^\beta) = Z(g^\beta)$ (in βX), and $g \in I$, then $f \in I$. On the other hand, if $f \in C(X)$, then $\mathfrak{M}(f)$ is the set $Z(f)^\beta$. Now it is easily shown that $Z(f)^\beta = Z(g)^\beta$ if and only if $Z(f) = Z(g)$. Thus an ideal I of $C(X)$ is a $\mathfrak{Z}$-ideal if whenever $Z(f) = Z(g)$ and $g \in I$, then $f \in I$.

LEMMA 2.2. *Every $\mathfrak{Z}$-ideal of $C(X)$ is an intersection of prime ideals.*

The proof is almost identical with the first part of [4], Theorem 1.4.

It was shown in [8], that if $C(X)/M^p$ is not isomorphic to R, then it is isomorphic to a non-Archimedean ordered field containing R. In section 5, we obtain a similar result for other quotient rings.

We conclude these remarks with a lemma of McKnight [16] about topologies on $\mathfrak{M}(A)$:

LEMMA 2.3. *If T is a topology on $\mathfrak{M}(A)$ such that each $a \in A$ is a continuous function from $\mathfrak{M}(A)$ to a (T_1) topological ring, then T is at least as strong as the Stone topology.*

Proof. The inverse images of zero by elements of A are closed; these are precisely the $\mathfrak{M}$-sets. But the $\mathfrak{M}$-sets form a base for the closed sets of the Stone topology on $\mathfrak{M}(A)$ (see [K], § 2).

3. $C_s(X)$, $C_\infty(X)$ **and related ideals of** $C^*(X)$. As is well known, if X is compact, the space $\mathfrak{M}_s(C(X))$ is homeomorphic to X. The first part of the section is devoted to a study of a generalization of this statement.

The following lemma and proof are taken from [16], with some minor expository modifications, and a slight generalization.

LEMMA 3.1 (McKnight). *Let X be a completely regular Hausdorff space. Let Δ be any closed subset of βX. Then the set I consisting of all $f \in C(X)$ for which $Z(f)^\beta$ contains a neighborhood of Δ, is the smallest ideal of $C(X)$ such that $\Delta(I) = \Delta$.*

Proof. If $f, g \in I$, then there exist open subsets V, W of βX such that $\Delta \subseteq V \subseteq Z(f)^\beta$, $\Delta \subseteq W \subseteq Z(g)^\beta$; thus

$$\Delta \subseteq V \cap W \subseteq Z(f)^\beta \cap Z(g)^\beta \subseteq Z(f-g)^\beta.$$

And for any $h \in C(X)$, we have $\Delta \subseteq V \subseteq Z(hf)^\beta$. Thus, I is an ideal.

Obviously, $\Delta(I) \supseteq \Delta$. For any $p \notin \Delta$, there is an $f \in C^*(X)$ such that $f^\beta(p) = 1$, $f^\beta(\Delta) = -1$. Let $g = \max\{f, 0\}$. Then $g \in I$, and $p \notin Z(g)^\beta$. It follows that $\Delta(I) = \Delta$.

Finally, let J be any ideal satisfying $\Delta(J) = \Delta$. Given $f \in I$, there is an open subset U of βX such that $\Delta \subseteq U \subseteq Z(f)^\beta$. For each point $p \in \beta X - U$, there is a non-negative function $g_p \in J \cap C^*(X)$ such that $g_p^\beta(p) > 1$. The open sets $U_p = \{q\colon g_p^\beta(q) > 1\}$ cover $\beta X - U$; by compactness, there is a finite subcover, say $\{U_1, \ldots, U_n\}$. The sum $g = g_1 + \ldots + g_n$, where g_i is the defining function of U_i, is in $J \cap C^*(X)$; and $g^\beta(p) > 1$ for all $p \in \beta X - U$. Since βX is normal, there is an $h \in C^*(X)$ such that $h^\beta(\Delta) = 0$, $h^\beta(\beta X - U) = 1$. Define $m \in C(X)$ by: $m(x) = h(x)/g(x)$ if $g(x) \geqslant 1$, and $m(x) = h(x)$ if $g(x) < 1$. Let $e = mg$. Then $e \in J$; and $e(x) = 1$ for $x \in X - Z(f)$; so $ef = f$, which implies that $f \in J$. Thus, $I \subseteq J$.

We recall that an ideal I of any subring of $C(X)$ is said to be *free* if for each $p \in X$, there is an $f \in I$ such that $p \notin Z(f)$.

LEMMA 3.2. *Let X be a completely regular Hausdorff space. The ring $C_\infty(X)$ is the intersection of the free maximal ideals of $C^*(X)$.*

Proof. The intersection of the free maximal ideals of $C^*(X)$ coincides with the set $I = \{f \in C^*(X)\colon f^\beta(\beta X - X) = 0\}$. Now it is easily seen that the following statements are equivalent: $f \in C_\infty(X)$, *i. e.*, for every $\varepsilon > 0$, the set $F_\varepsilon = \{x \in X\colon |f(x)| \geqslant \varepsilon\}$ is compact; for every $\varepsilon > 0$, $F_\varepsilon^\beta = F_\varepsilon$; for every $\varepsilon > 0$, $|f^\beta(p)| < \varepsilon$ for all $p \in \beta X - X$; and $f \in I$.

THEOREM 3.3 [1]. *Let X be a locally compact Hausdorff space, and let A be an ideal of $C^*(X)$. Then the following statements are equivalent.*

(a) $C_s(X) \subseteq A \subseteq C_\infty(X)$.

(b) *For all* $M \in \mathfrak{M}_s(C^*(X))$, $M \supseteq A$ *if and only if M is a free ideal.*

(c) *The mapping* $p \to M^{*p} \cap A$ ($p \in X$) *is a homeomorphism from X to* $\mathfrak{M}_s(A)$.

Proof. (b)$\leftrightarrow$(c). Since the mapping $p \to M^{*p}$ ($p \in X$) is a homeomorphism from X to the space of fixed maximal ideals of $C^*(X)$, this follows without difficulty from [K], Theorem 5.2 (for the spaces $\mathfrak{M}_s(C^*(X))$ and $\mathfrak{M}_s(A)$).

(a)$\to$(b). Let $M \in \mathfrak{M}_s(C^*(X))$ be a free ideal. Then by Lemma 3.2, $A \subseteq C_\infty(X) \subseteq M$.

[1] The statement (a)$\to$(c) has been obtained independently by J. G. Horne, using the concept of 0-ideal. For $A = C_s(X)$, this result was announced by M. E. Shanks in [17]. His proof (which has not been published) is also based on a viewpoint which is different from ours. For $A = C_\infty(X)$, (and Lemma 3.4), cf. Loomis [14], p. 60. (Added in proof: See also Théorèm 1 of K. Fujiwara, *Sur les anneaux des fonctions continues à support compact*, Math. J. Okayama Univ. 3 (1954)- p. 175-184.)

Now suppose $M \in \mathfrak{M}_s(C^*(X))$ is a fixed ideal, *i. e.* $M = M^p$ for some $p \in X$. Since X is locally compact, there is a neighborhood V of p with compact closure. And since X is completely regular, there is an $f \in C^*(X)$ such that $f(p)=1$, $f(X-V)=0$. Thus $f \in C_s(X)$, and $f \notin M^p$, so $C_s(X) \not\subset M^p$. Hence $A \not\subset M^p$.

Before concluding the proof of Theorem 3.3, we state and prove a lemma.

The one-point compactification $X \cup \{\infty\}$ of a locally compact Hausdorff space X will be denoted by X^*.

LEMMA 3.4. *Let X be a locally compact Hausdorff space. Then $(C_\infty(X);R)$ (notation as in [K], § 6) is isomorphic to $C(X^*)$.*

Proof. By Theorem 3.3, (a)→(c), whose proof has been completed, the mapping $p \to M^{*p} \cap C_\infty(X)$ $(p \in X)$ is a homeomorphism from X to $\mathfrak{M}_s(C_\infty(X))$. Thus, $C_\infty(X)$ may be regarded as the ring $C_\infty(\mathfrak{M}_s(C_\infty(X)))$. By [K], Theorem 6.3, $(C_\infty(X);R)$ is a subring of $C(X^*)$. But it is evident that in this case, all of $C(X^*)$ is obtained.

We return to the proof of 3.3.

(b)→(a). Since $A \subseteq M$ for every free ideal $M \in \mathfrak{M}_s(C^*(X))$, it follows from Lemma 3.2 that $A \subseteq C_\infty(X)$.

By 3.4, we imbed $C_\infty(X)$ in $C(X^*)$. Now $C(X^*)$ is isomorphic to the subring of $C^*(X)$ consisting of all functions which "approach a limit at ∞". Thus, A may be viewed as an ideal of $C(X^*)$ contained in M^∞. Now for each $p \in X$, there exists an $f \in A$ such that $f(p) \neq 0$. Hence $\Delta(A) = \bigcap_{f \in A} Z(f) = \{\infty\}$. By Lemma 3.1, the ideal of all functions vanishing in a neighborhood of ∞ is contained in A, *i. e.*, $C_s(X) \subseteq A$.

Before stating the next theorem, we point out that it was shown in [3], Theorem 3.3, that every prime ideal of $C(X)$ is contained in a unique maximal ideal.

THEOREM 3.5. *Let X be a compact Hausdorff space; P, a prime ideal of $C(X)$; and M^p, the unique maximal ideal containing P. Then every maximal ideal of P has the form $P \cap M^q$, where $q \neq p$, $M^q \in \mathfrak{M}(C(X))$.*

Proof. Suppose not, and let I be a maximal ideal of P which is not of the form indicated. There is a non-negative function $g \in P-I$. For let $f \in P-I$ be arbitrary. The relations $\max\{f,0\} \cdot \min\{f,0\} = 0 \in P$, $\max\{f,0\} + \min\{f,0\} = f \in P-I$ imply that both $\max\{f,0\}$, $\min\{f,0\}$ are in P but not both are in I; hence, either $\max\{f,0\}$ or $-\min\{f,0\}$ is in $P-I$. Now we have also $\sqrt{g} \in P-I$. Thus P/I is not a zero-ring; so I must be a prime ideal of P. It follows that P is isomorphic to an ideal P' of $C^*(X-\{p\})$ satisfying $C_s(X-\{p\}) \subseteq P' \subseteq C_\infty(X-\{p\})$ and having a free primitive ideal. This contradicts Theorem 3.3.

COROLLARY 3.6. *Let X be a locally compact Hausdorff space. Then $C_s(X)$ is an ideal of $C_\infty(X)$ which is contained in no maximal ideal.*

Proof. M^∞ is a prime ideal of $C(X^*)$; by Theorem 3.5, every maximal ideal of M^∞ has the form $M^\infty \cap M^q$, where $q \neq \infty$, $M^q \in \mathfrak{M}(C(X^*))$. But $C_s(X) \not\subset M^q$.

COROLLARY 3.7. *Let X be a compact Hausdorff space; F, a closed subset of X; and A, the subring of $C(X)$ consisting of all functions which vanish on F. Then every maximal ideal of A has the form $A \cap M^q$, where $q \notin F$, $M^q \in \mathfrak{M}(C(X))$.*

Proof. Form a quotient space by reducing F to a point.

EXAMPLE 3.8. The ring $C_\infty(R)$ is a ring in which to every maximal ideal there corresponds an element in its complement having a relative identity (so that every maximal ideal is primitive [K], § 4), but not every element has a relative identity. For, it is clear that any maximal ideal failing to have this property would necessarily contain $C_s(R)$. By 3.6, there are no such maximal ideals. And, the function f in $C_\infty(R)$ defined by $f(x) = 1/(x^2+1)$, for all $x \in R$, is an element with no relative identity.

It has been noted that we may view $C_\infty(X)$ as the ideal M^∞ in $C(X^*)$. It is easily seen that, similarly, $C_s(X)$ may be considered as the ideal N^∞ of $C(X^*)$ (see section 2 for the definition of N^p). In fact, $C_s(X)$ and $C_\infty(X)$ are the minimal and maximal ideals associated with the closed set $\{\infty\}$ in X^*, in the sense of Lemma 3.1. If I is a proper ideal of $C_\infty(X)$, it follows from Theorem 3.3 that I is contained in no primitive ideal if and only if $\Delta(I) \cap X$ is empty, or, regarding $\Delta(I)$ as a subset of X^*, if and only if $\Delta(I) = \{\infty\}$. Thus, by Lemma 3.1, $C_s(X)$ is the minimal ideal contained in no primitive ideal.

Now $N^\infty = M^\infty$ if and only if ∞ is a P-point of X^*; equivalently, every countable union of compact subsets of X is contained in a compact set (cf. [3], 4.2). It is easily seen that if X is a σ-compact non-compact space, or a non-countably compact space, then $N^\infty \neq M^\infty$, *i. e.*, $C_s(X) \neq C_\infty(X)$.

According to Lemma 3.2, $C_\infty(X)$ is the intersection of the free maximal ideals of $C^*(X)$. We consider now the intersection D of the free maximal ideals of $C(X)$. The ideal D must be a subring of $C^*(X)$; for if $f \in C(X)$ is unbounded, there is a $p \in \beta X - X$ such that $\hat{f}(p) = \infty$ (cf. section 2), so that $f \notin M^p$. But D is then a subring of $C_\infty(X)$, because $(M^p \cap C^*(X)) \subseteq M^{*p}$ for every p. Since D is an ideal of $C(X)$, it is an ideal of $C^*(X)$.

If X is a locally compact, σ-compact space, then D is contained properly in $C_\infty(X)$. For, $\beta X - X$ is clearly a G_δ-set of βX, and it is closed in βX (cf. [11], p. 163, Exer. G); hence there is an $f \in C^*(X)$ such that $Z(f^\beta) = \beta X - X$ (cf. [11], p. 134, Exer. J). Thus, $f \in C_\infty(X)$, but f is in *no* maximal ideal of $C(X)$, since it is a unit of $C(X)$. It follows that in this case, $C_\infty(X)$ is not an ideal of $C(X)$.

By Lemma 2.1, D coincides with $\{f \in C(X): Z(f)^\beta \supset \beta X - X\}$. Now if $f \in C_s(X)$, then $\beta X = X^\beta = (X - Z(f))^\beta \cup Z(f)^\beta = (\overline{X - Z(f)}) \cup Z(f)^\beta$; so $Z(f)^\beta \supset \beta X - X$, *i. e.*, $f \in D$. Thus $C_s(X) \subseteq D$. We next give two sufficient conditions that $C_s(X) = D$.

THEOREM 3.9. *If either* (a) X *is a* P*-space (not necessarily locally compact), or* (b) X *is a locally compact Hausdorff space and* ∞ *is a* P*-point of* X^*, *then* $C_s(X)$ *coincides with the intersection of the free maximal ideals of* $C(X)$.

Proof. (a) Let $f \in D$. Since X is a P-space, $Z(f)$ is open ([3], Theorem 5.3). Hence $X - Z(f)$ and $Z(f)$ are completely separated; so $(X - Z(f))^\beta$ and $Z(f)^\beta$ are disjoint subsets of βX. Now $Z(f)^\beta \supset \beta X - X$ implies that $(X - Z(f))^\beta \subseteq X$. Since $\overline{X - Z(f)} \subseteq (X - Z(f))^\beta$, and $(X - Z(f))^\beta$ is a compact subset of X, it follows that $\overline{X - Z(f)}$ is compact, *i. e.*, that $f \in C_s(X)$. Therefore $D \subseteq C_s(X)$. Combining with the remark preceding the theorem, we have $C_s(X) = D$.

(b) It has been shown that $C_s(X) \subseteq D \subseteq C_\infty(X)$. Thus, if ∞ is a P-point of X^*, we have $C_s(X) = C_\infty(X) = D$.

Note that the two cases considered in 3.9 are mutually exclusive in all spaces with an infinite number of points. For, if X is a locally compact P-space, and ∞ is a P-point of X^*, then X^* is a compact P-space; so X^*, and hence X, is finite ([3], Cor. 5.4).

We designate by $W(a)$ the space of all ordinals less than the ordinal a, with the interval topology. The space $X = W(\omega_1 + 1) \times W(\omega_0 + 1) - \{(\omega_1, \omega_0)\}$ shows that it need not be true that $C_s(X) = D$. Every continuous function of this space is bounded, so D is identical with the intersection of the free maximal ideals of $C^*(X)$, *i. e.*, $D = C_\infty(X)$. But $C_s(X) \neq C_\infty(X)$, in other words, ∞ is not a P-point of X^*, since X is evidently not countably compact.

A familiar question related to these matters is, for what spaces X is it true that $X^* = \beta X$ (cf. *e. g.*, [8], p. 62-63)? It is of course necessary that X be locally compact Hausdorff. And for this class, $X^* = \beta X$ if and only if every function in $C(X)$ has a continuous extension to X^*, which is clearly equivalent to the condition that $C(X)$ coincide with the set of functions which "approach a limit at ∞". Thus, it certainly suffices that every function in $C(X)$ be constant outside a compact set; this is equivalent to the statement that each $f \in C(X)$ has the form $g + r \cdot 1$, $g \in C_\infty(X)$, $r \in R$, and that ∞ is a P-point of X^*. That this is not necessary is shown by the space

$$W(\omega_1 + 1) \times W(\omega_0 + 1) - \{(\omega_1, \omega_0)\}.$$

The final result of this section indicates that an interesting condition on $C(X)$ (see [3], Theorem 5.3) is too restrictive when applied to $C_\infty(X)$.

THEOREM 3.10. *Let* X *be a locally compact Hausdorff space. Then* $C_\infty(X)$ *is a regular* (²) *ring if and only if* X *is finite.*

Proof. If X is finite, then X is discrete; so $C_\infty(X)$ is the direct sum of a finite number of fields (each isomorphic to R), and hence is regular.

Conversely, suppose that $C_\infty(X)$ is regular. Now $C_\infty(X)$ is an ideal of $(C_\infty(X); R)$, which by Lemma 3.4, is isomorphic to $C(X^*)$. It follows from [12], Theorem 1, and the fact that R is regular, that $C(X^*)$ is regular. Thus X^* is a P-space ([3], Theorem 5.3). But X^* is compact; so X^*, and hence X, is finite ([3], Cor. 5.4).

4. Algebraic characterizations. In this section we give ring characterizations of the $\mathfrak{Z}$-ideals of $C^*(X)$ (cf. section 2) containing $C_s(X)$ and contained in $C_\infty(X)$, where X is a locally compact Hausdorff space. Characterizing conditions are given separately for the ideals of greatest interest, $C_s(X)$ and $C_\infty(X)$.

Before giving these theorems, we state in 4.1 an essentially known result, which will be needed in the proofs. We include a proof of 4.1 based on a different viewpoint, which we believe to be interesting in itself; it is developed from the method used by Heider in his characterization of the lattice of all continuous real-valued functions on a compact Hausdorff space [6].

Let A be a subring of a commutative ring B. If the mapping a defined by $a(M) = M \cap A$ is one-to-one from $\mathfrak{M}(B)$ onto $\mathfrak{M}(A)$, we shall say that B is an $\mathfrak{M}$*-extension* of A. Throughout this section, the symbol a will denote a mapping defined as above for the pair of rings under discussion.

There are many examples of $\mathfrak{M}$-extensions which are *proper* extensions. The ring $C([0,1])$ is a proper $\mathfrak{M}$-extension of the ring of continuous rational functions over R defined on $[0,1]$, as well as of the ring of differentiable real-valued functions on $[0,1]$. More generally, $C([0,1])$ is a proper $\mathfrak{M}$-extension of any subring A of $C([0,1])$ having the properties: (1) For each $p \in [0,1]$, $\{r \in R: f(p) = r$ for some $f \in A\}$ is a field; (2) if $f \in A$ and $Z(f)$ is empty, then f is a unit of A. For, (1) evidently implies that for each $p \in [0,1]$, $\{f \in A: f(p) = 0\}$ is a primitive ideal of A; while it follows from (2), by a familiar argument of Gelfand and Kolmogoroff, that every primitive ideal of A is fixed. Many proper subrings of $C([0,1])$ satisfying (1) and (2) can be constructed merely by restricting the functions at a single point p. For instance, the collection of rings $A_1, A_2, A_3, A_4, \ldots$, consisting of all functions f in $C([0,1])$ such

(²) For the definition and simple properties, see [15], p. 147-149.

that $f(p)$ lies in Ra, Ra$(\sqrt{2})$, Ra$(\sqrt{2},\sqrt{3})$, Ra$(\sqrt{2},\sqrt{3},\sqrt{5})$, ..., respectively, (where Ra denotes the rationals) is a sequence of subrings such that A_{i+1} is a proper $\mathfrak{M}$-extension of A_i $(i=1,2,3,\ldots)$. Also, $C([0,1])$ is a proper $\mathfrak{M}$-extension of each A_i. In every example given above, the mapping α is actually a homeomorphism if the two spaces are given the Stone topology.

Any ring A satisfying $(\mathfrak{N})$ of 4.1, (1) possesses an induced partial order defined as follows: Given $a \in A$, we set $a \geqslant 0$ if and only if the image of a in A/M is non-negative for each $M \in \mathfrak{M}(A)$. The symbol $\leqslant$ used in the statement of condition $(\mathfrak{B})$ of 4.1, (1) and in condition $(\mathfrak{B}_s)$ of 4.6, signifies this partial order.

It would be incorrect to say that Theorem 4.1 is merely a translation of Heider's result on lattices into the terminology of rings. If the "maximality" condition is omitted from [6], Theorem 5.1, the analogue of our 4.1, (1) is not obtained. For example, the rings of rational functions and differentiable functions mentioned above satisfy all the hypotheses of 4.1, (1); but they do not satisfy the remaining conditions of [6], Theorem 5.1, since they are not lattices.

Whenever it is convenient, we shall identify a ring with any ring with which it is known to be isomorphic, without further notice.

THEOREM 4.1 [3]. (1) *Let A be a commutative ring satisfying*

$(\mathfrak{N})$ *A is a semi-simple algebra over R such that for each $M \in \mathfrak{M}(A)$, we have $A/M \cong R$.*

$(\mathfrak{J})$ *A has an identity (denoted by 1).*

$(\mathfrak{B})$ *For each $a \in A$, there exists an $r \in R$ such that $a \leqslant r\cdot 1$.*

Then A is isomorphic to a dense subring of $C(\mathfrak{M}_K(A))$, where K is a suitable compact Hausdorff topology.

(2) *If, in addition, A satisfies*

$(\mathfrak{E})$ *Any $\mathfrak{M}$-extension of A satisfying $(\mathfrak{N})$, $(\mathfrak{J})$ and $(\mathfrak{B})$ coincides with A,*

then A is isomorphic to $C(\mathfrak{M}_K(A))$.

(3) *Conversely, if X is a compact Hausdorff space, then $C(X)$ satisfies $(\mathfrak{N})$, $(\mathfrak{J})$, $(\mathfrak{B})$ and $(\mathfrak{E})$.*

[3] The author is indebted to J. E. Kist for pointing out the similarity of 4.1 to the ordered algebra theorem of Stone (see, e. g., [10], Theorem 3.1). (Added in proof: Stone's theorem is more general than our 4.1; but we have since obtained a substantial improvement in 4.1, which is almost the same as Stone's theorem. The rest of the section can be correspondingly improved. In effect, we assume that A is an "almost Archimedean" ordered algebra (rather than Archimedean); in compensation, only an algebra isomorphism can be obtained – the order is not necessarily preserved. The additions and changes in the proofs are too lengthy to be indicated here.)

Proof. (1) By $(\mathfrak{N})$, A is a ring of functions from $\mathfrak{M}(A)$ to R; and by $(\mathfrak{J})$ and $(\mathfrak{B})$, each function is bounded. Thus, A is a subring of $C^*(\mathfrak{M}(A))$ (where $\mathfrak{M}(A)$ has the discrete topology). We form $\beta\mathfrak{M}(A)$, and extend each $a \in A$ to a^β, an element of $C(\beta\mathfrak{M}(A))$. Now for each $x \in \beta\mathfrak{M}(A)-\mathfrak{M}(A)$, there is an $M \in \mathfrak{M}(A)$ such that $a^\beta(x)=a^\beta(M)$ for all $a \in A$, namely, the kernel of the homomorphism $\tau\colon A\to R$ defined by $\tau(a)=a^\beta(x)$. Furthermore, there is not more than one M corresponding to x. For, if $M_1, M_2 \in \mathfrak{M}(A)$, $M_1 \neq M_2$, then there is an $a \in A$ such that $a(M_1) \neq a(M_2)$; so if M_1 corresponds to x, $a^\beta(M_2) \neq a^\beta(x)$.

We partition $\beta\mathfrak{M}(A)$ by identifying all points which are not distinguished by elements of A, *i. e.*, we stipulate that for any $x, y \in \beta\mathfrak{M}(A)$, $x \equiv y$ if and only if $a^\beta(x)=a^\beta(y)$ for all $a \in A$. It has just been shown that the points of the resulting quotient space are in one-to-one correspondence with the points of $\mathfrak{M}(A)$. It can easily be shown that the quotient topology is compact Hausdorff. From this, a compact Hausdorff topology K may be given to $\mathfrak{M}(A)$ in the natural way. Thus, A is a subalgebra of $C(\mathfrak{M}_K(A))$. Furthermore, A separates points of $\mathfrak{M}(A)$. By the Stone-Weierstrass Theorem, A is dense in $C(\mathfrak{M}_K(A))$.

(2) Let X be a compact Hausdorff space. It is well known that $C(X)$ satisfies $(\mathfrak{N})$; and it is evident that $C(X)$ satisfies $(\mathfrak{J})$ and $(\mathfrak{B})$. Thus, $C(\mathfrak{M}_K(A))$ is an $\mathfrak{M}$-extension of A satisfying $(\mathfrak{N})$, $(\mathfrak{J})$ and $(\mathfrak{B})$. From $(\mathfrak{E})$, $A=C(\mathfrak{M}_K(A))$.

(3) Let X be a compact Hausdorff space. It remains only to show that $C(X)$ satisfies $(\mathfrak{E})$. Suppose B is an $\mathfrak{M}$-extension of $C(X)$ satisfying $(\mathfrak{N})$, $(\mathfrak{J})$ and $(\mathfrak{B})$. It follows from (1) that B is a subring of $C(\mathfrak{M}_{K'}(B))$, where K' is a compact Hausdorff topology. Since the elements of B are continuous, K' is at least as strong as the Stone topology (Lemma 2.3). Thus, α is continuous from $\mathfrak{M}_{K'}(B)$ to $\mathfrak{M}_s(C(X))$. Since α is also one-to-one and onto, it is a homeomorphism.

Now X is homeomorphic to $\mathfrak{M}_s(C(X))$ under the natural mapping $p \to M^p$ $(p \in X)$. Since $\alpha\colon \mathfrak{M}_{K'}(B)\to\mathfrak{M}_s(C(X))$ is defined by $\alpha(M)=M\cap A$, $M \in \mathfrak{M}_{K'}(B)$, it is clear that X and $\mathfrak{M}_{K'}(B)$ are homeomorphic under the natural mapping. Thus, B may be *identified* with a subring of $C(X)$. Since also $B \supseteq C(X)$, we have $B=C(X)$.

Let A be a commutative ring satisfying $(\mathfrak{N})$. As in [K], § 6, we may imbed A in the ring with identity $(A; R)$. From [K], Theorems 6.1 and 6.3, it follows that $(A; R)$ also satisfies $(\mathfrak{N})$. Thus, as above, $(A; R)$ possesses an induced partial order. The symbol $\leqslant$ used in the statement of $(\mathfrak{B}')$, 4.2, signifies this partial order.

THEOREM 4.2. *Let A be a commutative ring satisfying* $(\mathfrak{R})$ *and*

$(\mathfrak{B}')$ *For each $a \in A$, there exists an $s \in R$ such that* $(a,0) \leqslant s(0,1)$ *in* $(A;R)$.

Then A is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology.

Proof. Imbed A in $(A;R)$. As observed above, $(A;R)$ satisfies $(\mathfrak{R})$; and it is evident that $(A;R)$ satisfies $(\mathfrak{I})$. Finally, $(\mathfrak{B}')$ implies that $(A;R)$ satisfies $(\mathfrak{B})$. For let $(a,t) \in (A;R)$ be given, and let $s \in R$ be such that $(a,0) \leqslant s(0,1)$. Then $(a,t) \leqslant (s+t)(0,1)$.

We conclude that $(A;R)$ is isomorphic to a dense subring of $C(\mathfrak{M}_K(A;R))$, where K is some compact Hausdorff topology. Let L denote the relative topology on $\mathfrak{M}(A_0)$; then $\mathfrak{M}_L(A_0)$ is locally compact Hausdorff, and $\mathfrak{M}_K(A;R)$ is its one-point compactification, with $\infty = A_0$ (notation as in [K], § 6). Given $f \in C(\mathfrak{M}_K(A;R))$, the elements of $(A;R)$ which approximate f may always be chosen so as to coincide with f at ∞. (If $f(\infty)=r$, and $|(a,s)-f|<\varepsilon/2$, then $|(a,r)-f|<\varepsilon$, since $|(a,r)-(a,s)| = |r-s| < \varepsilon/2$). In particular, every element of $C_\infty(\mathfrak{M}_L(A_0))$ may be approximated by elements of $(A;R)$ that vanish at ∞, *i. e.*, by elements of A_0. Thus, A is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_L(A_0))$, or, of $C_\infty(\mathfrak{M}_L(A))$.

We shall now utilize the concept of $\mathfrak{Z}$-ideal which was introduced in section 2. It is clear that every intersection of $\mathfrak{Z}$-ideals is a $\mathfrak{Z}$-ideal.

LEMMA 4.3. *If A is a subring of $C_\infty(X)$, there is a smallest $\mathfrak{Z}$-ideal $\mathfrak{Z}(A,X)$ of $C^*(X)$ such that $A \subseteq \mathfrak{Z}(A,X) \subseteq C_\infty(X)$.*

Proof. $C_\infty(X)$ is a $\mathfrak{Z}$-ideal, since it is the intersection of the free maximal ideals of $C^*(X)$ (Lemma 3.2). Therefore the desired ideal $\mathfrak{Z}(A,X)$ is simply the intersection of all the $\mathfrak{Z}$-ideals containing A.

THEOREM 4.4. (1) *Let A be a commutative ring satisfying* $(\mathfrak{R})$, $(\mathfrak{B}')$ *and*

$(\mathfrak{E}')$ *Any $\mathfrak{M}$-extension of A satisfying $(\mathfrak{R})$ and $(\mathfrak{B}')$ such that $a(\{\mathfrak{M}(a)\colon a \in \mathfrak{Z}(B,\mathfrak{M}(B))\}) = \{\mathfrak{M}(a)\colon a \in \mathfrak{Z}(A,\mathfrak{M}(A))\}$ coincides with A.*

Then A is isomorphic to a $\mathfrak{Z}$-ideal of $C^*(\mathfrak{M}_L(A))$ containing $C_s(\mathfrak{M}_L(A))$ and contained in $C_\infty(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology.

(2) *Conversely, if X is a locally compact Hausdorff space, then every $\mathfrak{Z}$-ideal of $C^*(X)$ containing $C_s(X)$ and contained in $C_\infty(X)$ satisfies* $(\mathfrak{R})$, $(\mathfrak{B}')$ *and* $(\mathfrak{E}')$.

Proof. (1) Let X be a locally compact Hausdorff space, and let H be a $\mathfrak{Z}$-ideal of $C^*(X)$ such that $C_s(X) \subseteq H \subseteq C_\infty(X)$. Then H satisfies $(\mathfrak{R})$ and $(\mathfrak{B}')$. For, by Theorem 3.3, each ideal of $\mathfrak{M}_s(H)$ is the intersection of H with a fixed ideal of $C^*(X)$. Condition $(\mathfrak{R})$ then follows immediately.

Now imbed H in $(H;R)$. Since $\mathfrak{M}_s(H;R)$ is compact, every element of $(H;R)$ is bounded, so H satisfies $(\mathfrak{B}')$.

Suppose A satisfies $(\mathfrak{R})$, $(\mathfrak{B}')$ and $(\mathfrak{E}')$. By Theorem 4.2, A is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology. In view of Theorem 3.3, it follows that $\mathfrak{Z}(A,\mathfrak{M}_L(A))$ is an $\mathfrak{M}$-extension of A as in $(\mathfrak{E}')$. From $(\mathfrak{E}')$, we have $A = \mathfrak{Z}(A,\mathfrak{M}_L(A))$.

(2) Let X be a locally compact Hausdorff space, and let H be such that $C_s(X) \subseteq H \subseteq C_\infty(X)$. It remains only to show that H satisfies $(\mathfrak{E}')$. Suppose B is an $\mathfrak{M}$-extension of H as in $(\mathfrak{E}')$. It follows from 4.2 that B is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_{L'}(B))$, where L' is a locally compact Hausdorff topology. Since the elements of B are continuous, L' is at least as strong as the Stone topology (Lemma 2.3). Thus, a is continuous from $\mathfrak{M}_{L'}(B)$ to $\mathfrak{M}_s(H)$.

Next, let $\mathfrak{K} \subseteq \mathfrak{M}_s(H)$ be compact. By Theorem 3.3, the Stone topology on $\mathfrak{M}_s(H)$ is locally compact Hausdorff. Thus, using the same method as in 3.1, we find an $a \in C_s(\mathfrak{M}_s(H))$ whose support contains $\mathfrak{K}$, and repeating the argument, a non-negative $b \in C_s(\mathfrak{M}_s(H))$ exceeding 1 everywhere on the support of a. Let $e = \min\{b,1\}$. Then $e \in C_s(\mathfrak{M}_s(H))$, and e is a relative identity for a. By hypothesis, $H \supseteq C_s(\mathfrak{M}_s(H))$; so $a, e \in H$. We now view a and e as functions on $\mathfrak{M}_{L'}(B)$. Since $ae = a$ in B also, it is clear that $e(M) = 1$ for each $M \in \mathfrak{M}_{L'}(B)$ in the support of a. Thus, a has a support which is a closed subset of $\mathfrak{D} = \{M \in \mathfrak{M}_{L'}(B)\colon e(M) = 1\}$. The set $\{x \in \beta\mathfrak{M}(B;R)\colon (e,0)^\beta(x) = 1\}$ is compact, so its continuous image in the quotient space which yields the one-point compactification of $\mathfrak{M}_{L'}(B)$ is also compact. But this latter set, since it does not contain the point at infinity, coincides with $\mathfrak{D}$. It follows that $a \in C_s(\mathfrak{M}_{L'}(B))$. Since a is continuous and $\mathfrak{K}$ is closed, $a^{-1}(\mathfrak{K})$ is compact. Thus, we have shown that a^{-1} takes compact sets into compact sets.

Finally, we extend a to a mapping $a^*\colon \mathfrak{M}_*(B) \to \mathfrak{M}_*(H)$, where $\mathfrak{M}_*(B) = \mathfrak{M}_{L'}(B) \cup \{\infty_B\}$, $\mathfrak{M}_*(H) = \mathfrak{M}_s(H) \cup \{\infty_H\}$ are the one-point compactifications of $\mathfrak{M}_{L'}(B)$, $\mathfrak{M}_s(H)$, respectively, by stipulating that $a^*(\infty_B) = \infty_H$. If $\mathfrak{V} \subseteq \mathfrak{M}_*(H)$ is any open set containing ∞_H, then $\mathfrak{M}_*(H) - \mathfrak{V}$ is compact; so $a^{*-1}(\mathfrak{M}_*(H) - \mathfrak{V}) = a^{-1}(\mathfrak{M}_*(H) - \mathfrak{V})$ is compact, and hence is the complement of an open set containing ∞_B. It follows that a^* is continuous, and hence a homeomorphism. Thus, a is a homeomorphism from $\mathfrak{M}_{L'}(B)$ to $\mathfrak{M}_s(H)$.

By Theorem 3.3, X is homeomorphic to $\mathfrak{M}_s(H)$ under the mapping $p \to M^{*p} \cap H$, $(p \in X)$. Hence X and $\mathfrak{M}_{L'}(B)$ are homeomorphic under the natural mapping. Thus, $\mathfrak{Z}(B,\mathfrak{M}_{L'}(B))$ may be identified with $\mathfrak{Z}(H,X)$, *i. e.*, with H. Since $B \subseteq \mathfrak{Z}(B,\mathfrak{M}_{L'}(B))$, and $B \supseteq H$, we have $B = H$.

COROLLARY 4.5 (4). (1) *Let A be a commutative ring satisfying* ($\mathfrak{R}$), ($\mathfrak{B}'$) *and*

($\mathfrak{E}_\infty$) *Any $\mathfrak{M}$-extension of A satisfying* ($\mathfrak{R}$) *and* ($\mathfrak{B}'$) *coincides with A.*

Then A is isomorphic to $C_\infty(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology.

(2) *Conversely, if X is a locally compact Hausdorff space, then $C_\infty(X)$ satisfies* ($\mathfrak{R}$), ($\mathfrak{B}'$) *and* ($\mathfrak{E}_\infty$).

Proof. The only statement which might require a remark is that if X is a locally compact Hausdorff space, then $C_\infty(X)$ satisfies ($\mathfrak{E}_\infty$). A proof of this can be given by using the proof of 4.4, (2) in a manner similar to 4.6, (2) below.

The final theorem is a direct generalization of Theorem 4.1, (2) and (3).

THEOREM 4.6 (5). (1) *Let A be a commutative ring satisfying* ($\mathfrak{R}$) *and*

($\mathfrak{I}_s$) *Every element of A has a relative identity.*

($\mathfrak{B}_s$) *For each $a \in A$, there exists an $r \in R$ such that $a \leqslant re$, where e is a suitable relative identity for a.*

($\mathfrak{E}_s$) *Any $\mathfrak{M}$-extension of A satisfying* ($\mathfrak{R}$), ($\mathfrak{I}_s$) *and* ($\mathfrak{B}_s$) *coincides with A.*

Then A is isomorphic to $C_s(\mathfrak{M}_L(A))$, where L is a suitable locally compact Hausdorff topology.

(2) *Conversely, if X is a locally compact Hausdorff space, then $C_s(X)$ satisfies* ($\mathfrak{R}$), ($\mathfrak{I}_s$), ($\mathfrak{B}_s$) *and* ($\mathfrak{E}_s$).

Proof. (1) We modify the proof of Theorem 4.2 as follows: By ($\mathfrak{I}_s$) and ($\mathfrak{B}_s$), each element of A_0 in $(A; R)$ is a bounded function. Now [K], Theorem 6.3, shows that each element of $(A; R)$ is the sum of a function in A_0 and a constant function, and hence is bounded. Since condition ($\mathfrak{B}$) of Theorem 4.1, (1) is used only to obtain boundedness, we may again apply 4.1, (1) to $(A; R)$. Thus, from 4.2, we conclude that A is isomorphic to a dense subring of $C_\infty(\mathfrak{M}_L(A))$, where L is a locally compact Hausdorff topology. We now show that, in fact, $A \subseteq C_s(\mathfrak{M}_L(A))$. Let $a \in A$, and let $e \in A$ be a relative identity for a. Since $e \in C_\infty(\mathfrak{M}_L(A))$, $\{M \in \mathfrak{M}_L(A) \colon |e(M)| \geqslant 1\}$ is compact. The support of a is a closed subset of $\{M \in \mathfrak{M}_L(A) \colon |e(M)| \geqslant 1\}$. Hence $a \in C_s(\mathfrak{M}_L(A))$.

Now if X is a locally compact Hausdorff space, then $C_s(X)$ satisfies ($\mathfrak{R}$), ($\mathfrak{I}_s$) and ($\mathfrak{B}_s$). For, ($\mathfrak{R}$) was shown to hold in 4.4; ($\mathfrak{I}_s$) follows from a construction like that used in 4.4, (2); and ($\mathfrak{B}_s$) is evident. Thus, in view of Theorem 3.3, it follows that $C_s(\mathfrak{M}_L(A))$ is an $\mathfrak{M}$-extension of A satisfying ($\mathfrak{R}$), ($\mathfrak{I}_s$) and ($\mathfrak{B}_s$). From ($\mathfrak{E}_s$), we have $A = C_s(\mathfrak{M}_L(A))$.

(2) Let X be a locally compact Hausdorff space. It remains only to show that $C_s(X)$ satisfies ($\mathfrak{E}_s$). Suppose B is an $\mathfrak{M}$-extension of $C_s(X)$ satisfying ($\mathfrak{R}$), ($\mathfrak{I}_s$) and ($\mathfrak{B}_s$). It follows from the preceding part of the proof that B is a subring of $C_s(\mathfrak{M}_{L'}(B))$, where L' is a locally compact Hausdorff topology. As in the proof of 4.4, (2), X is homeomorphic to $\mathfrak{M}_{L'}(B)$. Thus $B = C_s(X)$.

(4) For a characterization of $C_\infty(X)$ as a ring and lattice, see [1], Theorem 5.

(5) A similar characterization of $C_s(X)$, using its vector lattice properties, has been obtained by J. E. Kist.

5. βF-points (6).

The set U is called an *X-neighborhood* of $p \in \beta X$ if U has the form $X \cap \Omega$, where Ω is a neighborhood of p in βX. Thus when $p \in X$, the set of X-neighborhoods of p coincides with the set of neighborhoods of p in X. For convenience, when U is an X-neighborhood of p, we shall refer to the set $U - \{p\}$ as a *deleted X-neighborhood of p* — even when $p \in \beta X - X$, so that $U - \{p\} = U$.

Let $f \in C(X)$, and Y be a subset of X. If a statement about $f(x)$ is true for each $x \in Y$, we shall say the statement is true for f *on* Y.

We recall that for any $f \in C(X)$, $\hat{f}$ denotes the extension to βX of f (as a function into the one-point compactification of R). For any maximal ideal M^p of $C(X)$, $f \in M^p$ implies $\hat{f}(p) = 0$. But the converse is false in general, as can be seen from Lemma 2.1.

In the following definition, we generalize several concepts discussed in [4].

DEFINITION 5.1. Let $p \in \beta X$. We define p to be a:

(1) *βF-point* (with respect to X), if for each $f \in C(X)$ such that $\hat{f}(p) = 0$, there is an X-neighborhood of p on which one of the relations $f \geqslant 0$, $f \leqslant 0$ holds;

(2) *$\beta P'$-point* (with respect to X), if for each $f \in C(X)$ such that $\hat{f}(p) = 0$, there is a deleted X-neighborhood of p on which one of the relations $f > 0$, $f < 0$, $f = 0$ holds;

(3) *βP-point* (with respect to X), if for each $f \in C(X)$ such that $\hat{f}(p) = 0$, there is an X-neighborhood of p on which $f = 0$.

We observe that if $\hat{f}(p) = 0$, then $\hat{f}$ is continuous and finite-valued in some neighborhood of p in βX. Thus, since X is dense in βX, each of the conclusions in 5.1, (1) and (3), is equivalent to that obtained by replacing "X-neighborhood" with "neighborhood in βX", and f with $\hat{f}$. However, this is not the case for (2).

(6) It is interesting to compare this section with the results obtained in [7] for the ring of entire functions. (Added in proof: We hawe since shown that the conclusions of 5.6 hold for any prime ideal of $C(X)$, and that all conclusions of 5.8, 5.11 and 5.13 hold when p is *any* βF-point. The proofs will be given elsewhere.)

A point $p \in X$ is a βP-point (resp. $\beta P'$-point) if and only if it is a P-point (resp. P'-point) as defined in [3], 4.1 (resp. [4], 8.1).

When $p \in vX - X$, the concept of βP-point coincides with that of "P-point with respect to X" given in [3], § 4; but when $p \notin vX$, the concept of βP-point is more restrictive. For, whenever $p \notin vX$, there is an $f \notin M^p$ such that $\hat{f}(p) = 0$ (cf. [8], Theorem 45). Now there is a restriction placed on f if p is a βP-point, but not if p is a P-point with respect to X. As an example, let X be the discrete countable space $\{e_1, e_2, \ldots, e_n, \ldots\}$; then $vX = X$. Let p be any point in $\beta X - X$. Since X is a P-space, p is a P-point with respect to X (cf. [3], Theorem 5.3, (4)). But p cannot be a βP-point, since the function f defined by $f(e_n) = 1/n$ $(n = 1, 2, \ldots)$, vanishes on no X-neighborhood of p, although $\hat{f}(p) = 0$.

If $p \notin X$ is a $\beta P'$-point, then p is a P-point with respect to X. For, if p is a $\beta P'$-point, every $f \in C(X)$ satisfying $\hat{f}(p) = 0$ and such that neither $f > 0$, $f < 0$ holds on any X-neighborhood of p, must vanish on some X-neighborhood of p. In particular, each $f \in M^p$ satisfies these conditions (cf. 2.1).

Now suppose $p \notin X$ is a βF-point, and a P-point with respect to X. Then every $f \in C(X)$ satisfying $\hat{f}(p) = 0$ either (1) belongs to M^p, so that, by the second condition, it vanishes on an X-neighborhood of p; or (2) is non-zero on some X-neighborhood of p, so that, by the first condition, it is positive or negative on some X-neighborhood of p. Thus, p is a $\beta P'$-point.

THEOREM 5.2. *Let $p \in \beta X$. Then p is a βF-point if and only if the ideal N^p is prime.*

The proof is almost identical with the last part of the proof of [4], Theorem 2.5.

Since N^p coincides with the intersection of the prime ideals contained in M^p ([4], Theorem 1.4), 5.2 justifies the formulation of the hypothesis that p is a βF-point which was given in the introduction. Theorem 5.2 and [4], Theorem 2.5 together show that every point of βX is a βF-point if and only if X is an F-space ([4], Definition 2.1).

We next relate the concept of βF-point to zero-sets of functions in $C(X)$.

THEOREM 5.3. *Let $p \in \beta X$. Then p is a βF-point if and only if for each pair of functions $f, g \in C(X)$ satisfying $\hat{f}(p) = \hat{g}(p) = 0$, there is an X-neighborhood U of p such that at least one of the relations $(Z(f) \cap U) \supseteq (Z(g) \cap U)$, $(Z(f) \cap U) \subseteq (Z(g) \cap U)$ is valid.*

Proof. Let p be a βF-point, and let $f, g \in C(X)$ satisfy $\hat{f}(p) = \hat{g}(p) = 0$. Set $k = |f| - |g|$. Then $\hat{k}(p) = 0$, so there is an X-neighborhood U of p such that either $k \geq 0$ or $k \leq 0$ on U. If $k \geq 0$ on U, then for any $x \in U$, $f(x) = 0$ implies $g(x) = 0$, *i. e.*, $(Z(f) \cap U) \subseteq (Z(g) \cap U)$; and similarly if $k \leq 0$ on U.

Conversely, if p is not a βF-point, let $f \in C(X)$ satisfy $\hat{f}(p) = 0$ and change sign in every X-neighborhood of p. Define $g = \max\{f, 0\}$, $h = \min\{f, 0\}$. For every X-neighborhood U of p, there exist $x, y \in U$ such that $g(x) \neq 0$, $h(y) \neq 0$, whence $g(y) = 0$, $h(x) = 0$. Thus, neither $(Z(g) \cap U) \supseteq (Z(h) \cap U)$ nor $(Z(g) \cap U) \subseteq (Z(h) \cap U)$ is valid.

Let $p \in \beta X$. If p is a non-isolated point of X, we define M'^p to be the set of all $f \in C(X)$ such that $Z(f)$ meets every deleted neighborhood of p. Otherwise, we set $M'^p = M^p$. Evidently, $M^p \supseteq M'^p \supseteq N^p$.

THEOREM 5.4. *If p is a βF-point, then M'^p is a prime ideal.*

Proof. Since M^p is a prime ideal, it suffices to consider the case where p is a non-isolated point of X. Let $f, g \in M'^p$. By 5.3, there is an X-neighborhood U of p such that (say) $(Z(f) \cap U) \supseteq (Z(g) \cap U)$. Let V' be an arbitrary deleted X-neighborhood of p. Since $U \cap V'$ is a deleted X-neighborhood of p, there is a point x in $Z(g) \cap U \cap V'$. Now we have also $x \in Z(f) \cap U \cap V'$. But $Z(f-g) \supseteq Z(f) \cap Z(g)$; so $x \in Z(f-g) \cap U \cap V'$. Hence $Z(f-g)$ meets V'. It follows that $f - g \in M'^p$.

It is clear that M'^p is closed under multiplication by arbitrary elements of $C(X)$, and that the complement of M'^p is a multiplicative system. Hence M'^p is a prime ideal.

We note that if p is a $\beta P'$-point, then $M'^p = N^p$; and if p is a βF-point such that $M'^p = N^p$, then p is a $\beta P'$-point.

When p is not a βF-point, M'^p need not even be an ideal. For example, let $X = R$, $p = 0$, and let $f, g \in C(X)$ be defined by $f(x) = x \sin^2 1/x$, $x \neq 0$, $f(0) = 0$, and $g(x) = x \cos^2 1/x$, $x \neq 0$, $g(0) = 0$. Then $f, g \in M'^p$, but $f + g \notin M'^p$. On the other hand, M'^p can be an ideal at a non-βF-point. For example, let X be a linearly ordered space, $p \in X$ a point with character c_{10} (see, *e. g.*, [3], § 6). Then it is easily seen that p is not a βF-point. But $M'^p = M^p$, since every continuous function which is zero at p is zero on a whole interval to the left of p; so M'^p is an ideal.

We point out next that when p is a βF-point but not a $\beta P'$-point (whence $N^p \neq M'^p$), then either $M'^p = M^p$ or $M'^p \neq M^p$ can occur. In both our examples, we shall make use of the space $E = N \cup \{e\}$ defined as follows ([4], 8.5): $N = \{e_1, e_2, \ldots\}$ is the denumerable discrete space, and $e \in \beta N - N$. Thus every e_n is an isolated point, while deleted neighborhoods of e are the members of some free ultra-filter on N (*i. e.*, maximal filter on N with total intersection void). It follows that e is a $\beta P'$-point of E ([4], 8.6).

For the first case, let X be the space described in [4], 8.10: $X=E\cup W(\omega_1)$, where all points of $N\cup W(\omega_1)$ are isolated, and a neighborhood of e in X is the union of a neighborhood of e in E with the complement in $W(\omega_1)$ of a countable subset of $W(\omega_1)$. Then e is a βF-point but not a $\beta P'$-point, and $M'=M^e$.

The second case is illustrated by the following example (due to L. Gillman).

EXAMPLE 5.5. For $n=1,2,\ldots$, let $E_n=\{e_{n1},e_{n2},\ldots,e_n\}$ be a copy of E (with e_n corresponding in E_n to e in E), and $X=\big(\bigcup_n E_n\big)\cup E=\big(\bigcup_n E_n\big)\cup\{e\}$, with the following topology: each e_{nm} is isolated; the neighborhoods of e_n are its neighborhoods in E_n; while each neighborhood of e is the union of a neighborhood U of e in E with a neighborhood U_n of e_n in E_n for each $e_n\in U$.

Then e is a βF-point. For consider any $f\in C(X)$ satisfying $f(e)=0$. If there is a deleted neighborhood V of e such that $f>0$ or $f<0$ on $V\cap E$, then the defining property obviously holds for f. If no such neighborhood exists, then, since e is a $\beta P'$-point of E, there is a neighborhood W of e such that $f=0$ on $W\cap E$. Set

$W_1=\{e_n\in W\colon f>0$ on some deleted neighborhood of $e_n\}$,
$W_2=\{e_n\in W\colon f<0$ on some deleted neighborhood of $e_n\}$,
$W_3=\{e_n\in W\colon f=0$ on some deleted neighborhood of $e_n\}$.

Precisely one of W_1, W_2, W_3 is a deleted neighborhood of e in E. The union of this set with suitable neighborhoods of its elements in the associated E_n's is a neighborhood of e in X on which $f\geqslant 0$ or $f\leqslant 0$.

The function $g\in C(X)$ defined by $g(e_n)=g(e)=0$, $g(e_{nm})=1/m$ $(m,n=1,2,\ldots)$ shows that e is not a $\beta P'$-point.

Finally, $M'^e\neq M^e$, as shown by the function $h\in C(X)$ defined as follows: $h(e_n)=h(e_{nm})=1/n$ $(m,n=1,2,\ldots)$, $h(e)=0$.

We now investigate the properties of the quotient-rings $C(X)/N^p$ and $C(X)/M'^p$.

THEOREM 5.6. *Let p be a βF-point.* (a) *The ring $C(X)/N^p$ is an ordered integral domain containing R, in which the image of M^p forms the unique maximal ideal; and $C(X)/N^p$ has infinitely large elements if and only if $p\notin vX$.*

(b) *The corresponding statement for $C(X)/M'^p$ also holds.*

Proof. If $f\in C(X)$, $g\in N^p$, and $0\leqslant f\leqslant g$, then, trivially, $f\in N^p$. For any $h\in C(X)$, $h^2-|h|^2=0\in N^p$. By Theorem 5.2, N^p is prime; so at least one of the congruences $h\equiv|h|\ (\mathrm{mod}\ N^p)$, $h\equiv-|h|\ (\mathrm{mod}\ N^p)$ is valid. Thus, by [2], Theorem 4.4, $C(X)/N^p$ is an ordered integral domain. Explicitly, the image of $f\in C(X)$ in $C(X)/N^p$ is positive if $f\geqslant 0$ on some X-neighborhood of p (*i. e.*, $f\equiv|f|\ (\mathrm{mod}\ N^p)$), but $f=0$ on no X-neighborhood of p (*i. e.*, $f\not\equiv 0\ (\mathrm{mod}\ N^p)$). Finally, it is clear that the functions which are constant in some X-neighborhood of p map into a subset of $C(X)/N^p$ which is isomorphic to R.

Since M^p is the only maximal ideal of $C(X)$ containing N^p, it is evident that $C(X)/N^p$ has a unique maximal ideal, namely, the image of M^p.

If $p\in vX$, then for each $f\in C(X)$, there is an $r\in R$ such that $\hat f(p)=r$. Thus, the image of f differs from the element corresponding to r by at most an infinitely small element. Conversely, if $p\notin vX$, there is a $g\in C(X)$, $g\geqslant 0$, such that $\hat g(p)=\infty$. Hence g is unbounded on every X-neighborhood of p; so the image of g is an infinitely large element.

The proof for $C(X)/M'^p$ is similar.

A commutative ring A with identity is a *valuation ring* if for any $a,b\in A$, either a divides b or b divides a.

THEOREM 5.7. *If p is a βF-point, then $C(X)/M'^p$ is a valuation ring. If p is a $\beta P'$-point, or if X is an F-space, then $C(X)/N^p$ is a valuation ring.*

Proof. It is easily seen that every element of $C(X)/N^p$ which is not infinitely small has an inverse. The same statement then follows for its homomorphic image $C(X)/M'^p$. Thus, to show that either $C(X)/N^p$ or $C(X)/M'^p$ is a valuation ring, it suffices to consider an arbitrary pair of distinct infinitely small elements.

Let p be a βF-point, and let $\gamma,\delta\in C(X)/M'^p$ be infinitely small positive elements such that $\gamma<\delta$. It will be shown that δ divides γ.

When p is isolated or $p\notin X$, then $M'^p=M^p$, so the desired conclusion is obvious. We therefore suppose that p is a non-isolated point of X.

Let $f,g\in C(X)$ map into γ, δ, respectively. We show first that it may be assumed that $Z(f)=Z(g)=\{p\}$, and that $0\leqslant f\leqslant g\leqslant 1$ on X.

Since $f\geqslant 0$ on some X-neighborhood of p, we may suppose that $0\leqslant f\leqslant 1$ on X, replacing f by $\min\{|f|,1\}$ if necessary. Now $f>0$ on some deleted X-neighborhood of p, so $Z(f)-\{p\}$ is a closed subset of X. By complete regularity, there is an $f_1\in C^*(X)$ such that $0\leqslant f_1\leqslant 2$, $f_1(x)=0$ for $x\in Z(f)-\{p\}$, and $f_1(p)=2$. Set $h=f+f_1$. Then $0\leqslant h\leqslant 3$, $Z(h)=Z(f)-\{p\}$, and $h(p)=2$. Let $k=1-\min\{h,1\}$. Then $k(x)=1$ when $x\in Z(h)$, and $k(x)=0$ when $h(x)\geqslant 1$; so $Z(k)$ is a closed X-neighborhood of p. Thus, $Z(f+k)=\{p\}$, and $0\leqslant f+k\leqslant 1$; moreover, $f+k$ is congruent to f modulo N^p, and hence modulo M'^p.

Similarly, we may suppose that $0\leqslant g\leqslant 1$, and that $Z(g)=\{p\}$. Finally, since $\gamma<\delta$ in $C(X)/M'^p$, we have $f\leqslant g$ on some X-neighborhood of p. Thus, $\min\{f,g\}\equiv f\ (\mathrm{mod}\ M'^p)$, so replacing f by $\min\{f,g\}$ if necessary, we may assume that $f\leqslant g$ everywhere.

The remainder of this part of the proof is a simple modification of the argument used in [4], Theorem 2.3, III. On $X-\{p\}$, we define:

(1) $d=f/g$.

For every real r, define a function $\mu_r \in C^*(X)$ by

(2) $\mu_r(x)=f(x)-rg(x)$.

Obviously, if $r>s$, then $\mu_r \leqslant \mu_s$ (since $g \geqslant 0$). Furthermore, $\mu_r(p)=0$ for every real r.

We have $\mu_0=f \geqslant 0$. Now there is a deleted neighborhood of p not meeting $Z(f-g)$. Thus, since $f-g \leqslant 0$, for each neighborhood U of p, there is a $y \in U$ such that $\mu_1(y)=f(y)-g(y)<0$. We may put

(3) $d(p)=\sup\{r\colon \mu_r \geqslant 0$ on some neighborhood of $p\}$.

It must be shown that d is continuous at p. By (3), for every $r>d(p)$, and for every neighborhood U of p, there is an $x \in U$ such that $\mu_r(x)<0$. Since $\mu_r(p)=0$, $\mu_r \leqslant 0$ on some neighborhood of p. From this point we may follow the proof of [4], 2.3, exactly, changing only the notation. We then have: For every $\varepsilon>0$, there is a neighborhood U of p such that $|d(x)-d(p)| \leqslant \varepsilon$, $x \in U-\{p\}$. Thus, $d \in C(X)$. Clearly $f=dg$, so $f \equiv dg$ (mod M'^p). This concludes the proof that $C(X)/M'^p$ is a valuation ring.

If p is a $\beta P'$-point, then $M'^p=N^p$, so $C(X)/N^p$ is a valuation ring by the result just established.

Now suppose X is an F-space. Given a pair of distinct infinitely small positive elements of $C(X)/N^p$, let a, b be functions in $C(X)$ which map into these elements. It is easily seen that we may assume that $0 \leqslant a \leqslant b \leqslant 1$. Clearly $Z(a) \supseteq Z(b)$. Set $c=a/b$ on $X-Z(b)$. Then $c \in C^*\big(X-Z(b)\big)$. By [4], Theorem 2.6, c has a continuous extension $c' \in C^*(X)$. Clearly $a=c'b$, so $a \equiv c'b$ (mod N^p). Thus, $C(X)/N^p$ is a valuation ring.

Corollary 5.8. *If p is a βF-point, then the prime ideals containing M'^p form a chain. If p is a $\beta P'$-point, or if X is an F-space, then the set of all prime ideals contained in M^p form a chain.*

Proof. It is easily seen that the set of all ideals in a valuation ring form a chain. The sets in question are the sets of inverse images, under the natural mapping, of the prime ideals in $C(X)/M'^p$, $C(X)/N^p$, respectively (cf. also [2], Theorem 3.10).

We shall show next that the conclusions of the second parts of 5.7 and 5.8 never hold when p fails to be a βF-point.

Theorem 5.9. *If p is not a βF-point, then the prime ideals contained in M^p do not form a chain.*

Proof. By Theorem 5.2, if p is not a βF-point, then N^p is not prime. From [4], Theorem 1.4, we have that the intersection of the prime ideals contained in M^p is not a prime ideal. But then the prime ideals contained in M^p do not form a chain (cf. [2], Theorem 3.9).

Corollary 5.10. *If p is not a βF-point, then $C(X)/N^p$ is not a valuation ring.*

Proof. By 5.9 there are incomparable prime ideals contained in M^p (and containing N^p); these map into incomparable prime ideals in $C(X)/N^p$. As already noted, the ideals of a valuation ring form a chain.

The following alternative proof of 5.9 seems interesting. By 5.3, if p is not a βF-point, there are functions $g, h \in C(X)$ satisfying $\hat{g}(p)=\hat{h}(p)=0$, and such that $Z(g)$, $Z(h)$ are incomparable in every X-neighborhood of p. We show that $\{g, g^2, \ldots, g^n, \ldots\} \cap (N^p, h)$ is empty. Suppose not; then there exist $d \in N^p$, $k \in C(X)$ and a positive integer m such that $g^m=d+kh$. Let V be an X-neighborhood of p on which $d=0$. Then $Z(g) \cap V=Z(g^m) \cap V=\big([Z(k) \cup Z(h)] \cap V\big) \supseteq \big(Z(h) \cap V\big)$, a contradiction. Similarly, $\{h, h^2, \ldots, h^n, \ldots\} \cap (N^p, g)$ is empty.

By [15], Lemma 2, p. 105, there are prime ideals P, Q containing (N^p, g), (N^p, h), respectively, and disjoint from the multiplicative systems $\{h, h^2, \ldots, h^n, \ldots\}$, $\{g, g^2, \ldots, g^n, \ldots\}$, respectively. Since P and Q contain N^p, they are contained in M^p; and they are clearly incomparable.

This method can be used to obtain a prime ideal contained properly in M^p whenever p is not a P-point with respect to X (cf. [3], Theorem 3.5). That is, we choose a function $f \in M^p-N^p$, and let P be an ideal which is maximal in the class of ideals containing N^p and disjoint from the multiplicative system $\{f, f^1, \ldots, f^n, \ldots\}$. Now P is never the entire complement of $\{f, f^2, \ldots, f^n, \ldots\}$ in M^p. For, let $g=\max\{f, 0\}$, $h=\min\{f, 0\}$; then the relation $g+h=f \notin P$ implies that not both g and h are in P, while $gh=0 \in P$ implies that at least one of the elements g, h is in P. Thus, exactly one of g, h is in P. From this, we conclude that $|f|=g-h \notin P$. Thus $|f|^{1/k} \notin P$, so that $|f|^{1/k} \in M^p-(P \cup \{f, f^2, \ldots, f^n, \ldots\})$, $(k=2,3,\ldots)$.

Theorem 5.11. *If p is a βF-point, and I is a proper $\mathfrak{Z}$-ideal of $C(X)$ containing M'^p, then I is a prime ideal. If p is a $\beta P'$-point or if X is an F-space, and I is a $\mathfrak{Z}$-ideal of $C(X)$ containing N^p, then I is a prime ideal.*

Proof. By Lemma 2.2, I is an intersection of proper prime ideals. From 5.8, the prime ideals containing I form a chain. Thus, any intersection of prime ideals containing I is a prime ideal.

Finally, we consider a type ofi deal which may be viewed as a generalization of N^p.

Lemma 5.12. *Let h be a fixed element of a maximal ideal M^p of $C(X)$, and let I be the set of all f in M^p such that $\big(Z(f) \cap U\big) \supseteq \big(Z(h) \cap U\big)$, where U is an X-neighborhood of p (depending on f). Then I is an ideal of $C(X)$.*

Proof. Given $f, g \in I$, choose X-neighborhoods U, V of p satisfying $\big(Z(f)\cap U\big) \supseteq \big(Z(h)\cap U\big)$, $\big(Z(g)\cap V\big) \supseteq \big(Z(h)\cap V\big)$. Then $\big(Z(f-g)\cap U\cap V\big) \supseteq \big(Z(f)\cap Z(g)\cap U\cap V\big) \supseteq \big(Z(h)\cap U\cap V\big)$; so $f-g \in I$. Since it is clear that I is closed under multiplication by arbitrary elements of $C(X)$, I is an ideal.

THEOREM 5.13. *If p is a $\beta P'$-point or if X is an F-space, then the ideal I defined in 5.12 is prime.*

Proof. Since I is clearly a $\mathfrak{z}$-ideal containing N^p, this follows from 5.11.

6. P-spaces, and prime ideals of M^p. We take up first some questions related to [K], § 5. An example was given there of a ring possessing an ideal of an ideal which fails to be an ideal of the whole ring. We give now an example of a ring of continuous functions having the same property. Let $X = R$, and let i be the identity function: $i(x) = x$, for all $x \in X$. Let I be the ideal $\{gi\colon\ g \in M^0\}$ of $C(X)$; and let J be the ideal $\{gi^2 + ni^2\colon\ g \in M^0,\ n \text{ an integer}\}$ of I. Then $i^2 \in J$, but $i^2/2 \notin J$; so J is not an ideal of $C(X)$.

The theorem which follows shows that for rings of continuous functions, this is the usual situation.

THEOREM 6.1. *Let I be a proper ideal of $C(X)$, and let J be a proper ideal of I. Then J is invariably an ideal of $C(X)$ if and only if X is a P-space. In particular, if $p \in X$ is not a P-point, then there is an ideal of an ideal of $C(X)$, contained in M^p, which fails to be an ideal of $C(X)$.*

Proof. If X is a P-space, then $C(X)$ is a regular ring ([3], Theorem 5.3). Let $j \in J$ be given, and let $a \in C(X)$ be arbitrary. There is a $b \in C(X)$ such that $j^2b = j$. Then $jba \in I$; so $ja = j(jba) \in J$. Thus, J is an ideal of $C(X)$.

Conversely, suppose X is not a P-space. Let p be a point of X which is not a P-point, and let $f \in M^p - N^p$. Let I be the ideal $\{gf\colon\ g \in M^p\}$ of $C(X)$; and let J be the ideal $\{gf^2 + nf^2\colon\ g \in M^p,\ n \text{ an integer}\}$ of I. Then $f^2 \in J$; but $f^2/2 \notin J$, since a continuous function which vanishes at p cannot assume the value 1/2 in every neighborhood of p. Thus, J is not an ideal of $C(X)$.

It was also noted in [K], § 5 that a prime ideal of an ideal need not be prime in the whole ring. Now if P and Q are prime ideals of a ring A, $P\cap Q$ is a prime ideal of P; and if P and Q are incomparable, $P\cap Q$ is not prime in A. Thus, an example in function rings may be obtained from any $C(X)$, where X has more than one point; if $q \neq p$, the ideal $M^p \cap M^q$ is a prime ideal of M^p which is not prime in $C(X)$.

By [3], Lemma 3.2, if a prime ideal P of M^p is to be prime in $C(X)$, it is necessary that P contain N^p. In the next theorem, we see that this condition is also sufficient.

THEOREM 6.2. *Let p be any point in βX. Then the prime ideals of $C(X)$ containing N^p coincide with the prime ideals of M^p containing N^p.*

Proof. If P is a prime ideal of $C(X)$ containing N^p, it is contained in M^p; and it is easily seen to be a prime ideal of M^p.

Now let P be a prime ideal of M^p containing N^p. We assume that P is proper, the case $P = M^p$ being trivial. By [K], Theorem 5.1, the set $Q = \{f \in C(X)\colon\ fM^p \subseteq P\}$ is a prime ideal of $C(X)$ such that $P = Q \cap M^p$. Since $Q \supseteq N^p$, we have $Q \subseteq M^p$ ([3], Theorem 3.3). Hence $P = Q$; so P is a prime ideal of $C(X)$.

We close with a result about subfamilies of $\mathcal{Z}\big(C(X)\big)$, the family of zero-sets of $C(X)$. By a *$\mathcal{Z}$-filter on X*, we mean a non-empty subfamily $\mathcal{F}$ of $\mathcal{Z}\big(C(X)\big)$ having the finite intersection property, and such that $A \in \mathcal{F}$, $B \in \mathcal{Z}\big(C(X)\big)$, $A \subseteq B$ imply $B \in \mathcal{F}$. It is well known that there is a natural correspondence between the proper ideals I of $C(X)$ and the $\mathcal{Z}$-filters $\mathcal{F}$ on X, namely, $I \to \mathcal{F} = \mathcal{Z}(I)$ ([8], Theorem 36). We give first an example to show that in general this correspondence is not one-to-one. Let $X = R$, let i be the identity function: $i(x) = x$, for all $x \in X$, and let $I = (i)$, $J = (i^2)$. Clearly $\mathcal{Z}(I) = \mathcal{Z}(J) = \mathcal{Z}(M^0)$. But $i \notin J$, so $I \neq J$.

Again, our theorem on this question shows that the example illustrates the normal situation.

THEOREM 6.3. *The proper ideals of $C(X)$ and the $\mathcal{Z}$-filters on X are in one-to-one correspondence if and only if X is a P-space. In particular, if $p \in X$ is not a P-point, there are distinct ideals contained in M^p having the same $\mathcal{Z}$-filter on X.*

Proof. Let X be a P-space, and let I, J be proper ideals of $C(X)$ such that $\mathcal{Z}(I) = \mathcal{Z}(J)$. Then $\Delta(I) = \Delta(J)$ (see section 2). Now in a P-space, every ideal is the intersection of all the maximal ideals containing it ([3], Theorem 5.3). Hence $I = \bigcap_{p \in \Delta(I)} M^p = \bigcap_{p \in \Delta(J)} M^p = J$. It follows that the correspondence between ideals of $C(X)$ and $\mathcal{Z}$-filters on X is one-to-one.

Conversely, suppose X is not a P-space; let p be a point of X which is not a P-point, let f be any function in $M^p - N^p$, and let $I = (f)$, $J = (f^2)$. Since $Z(f) = Z(f^2)$, and in view of the relation $Z(gh) = Z(g) \cup Z(h)$, we have $\mathcal{Z}(I) = \mathcal{Z}(J)$. But $f \notin J$, since a continuous function g cannot satisfy $g(x)f(x) = 1$ for values of x arbitrarily near p; so $I \neq J$. It follows that the correspondence between ideals of $C(X)$ and $\mathcal{Z}$-filters on X is not one-to-one.

When X is a P-space, it is clear that the correspondence discussed in Theorem 6.3 is a lattice isomorphism.

References

[1] J. Abdelhay, *On a theorem of representation*, Bull. Amer. Math. Soc. 55 (1949), p. 408-417.

[2] L. Gillman, *Rings with Hausdorff structure space*, this volume, p. 1-16.

[3] — and M. Henriksen, *Concerning rings of continuous functions*, Trans. Amer. Math. Soc. 77 (1954), p. 340-362.

[4] — — *Rings of continuous functions in which every finitely generated ideal is principal*, Trans. Amer. Math. Soc. 82 (1956), p. 366-391.

[5] — — and M. Jerison, *On a theorem of Gelfand and Kolmogoroff concerning maximal ideals in rings of continuous functions*, Proc. Amer. Math. Soc. 5 (1954), p. 447-455.

[6] L. J. Heider, *A characterization of function-lattices*, Duke Math. J. 23(1956), p. 297-301.

[7] M. Henriksen, *On the prime ideals of the ring of entire functions*, Pacific J. Math. 3 (1953), p. 711-720.

[8] E. Hewitt, *Rings of real-valued continuous functions, I*, Trans. Amer. Math. Soc. 64 (1948), p. 45-99.

[9] N. Jacobson, *A topology for the set of primitive ideals in an arbitrary ring*, Proc. Nat. Acad. Sci. U. S. A. 31 (1945), p. 333-338.

[10] R. V. Kadison, *A representation theory for commutative topological algebra*, Amer. Math. Soc. Memoir No. 7, 1951.

[11] J. L. Kelley, *General topology*, Van Nostrand, New York 1955.

[12] C. W. Kohls, *On the embedding of a generalized regular ring in a ring with identity*, Michigan Math. J. 3 (1956), p. 165-168.

[13] — *The space of prime ideals of a ring*, this volume, p. 17-27.

[14] L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York 1953.

[15] N. H. McCoy, *Rings and ideals*, Mathematical Association of America, 1948.

[16] J. D. McKnight Jr., *On the characterization of rings of functions*, Purdue doctoral thesis, 1953 (unpublished).

[17] M. E. Shanks, *Rings of functions on locally compact spaces* (Preliminary report), Bull. Amer. Math. Soc. 57 (1951), p. 295, Abstract No. 365.

PURDUE UNIVERSITY

Reçu par la Rédaction le 17.6.1956

Sur l'unicohérence, les homéomorphies locales et les continus irréductibles

par

A. Lelek (Wrocław)

§ 1. Introduction. Soient X et Y deux espaces métriques compacts et f une fonction dont la variable x parcourt X et dont Y est l'ensemble des valeurs. Appelons f *homéomorphie locale au sens large* lorsqu'il existe, pour tout point $x \in X$, un entourage (ensemble ouvert contenant ce point) U_x tel que la fonction partielle $f|U_x$ est une homéomorphie. Lorsqu'il en existe, pour tout $x \in X$, dont les images $f(U_x)$ sont en outre ouverts (donc des entourages ouverts de $f(x)$ dans Y), la fonction f est dite (voir [2], p. 35) *homéomorphie locale* tout court. Appelons enfin la fonction f *recouvrement* de Y par X (voir le livre [11] de Pontriagin, p. 352, définition 45 (¹)), lorsqu'il existe, pour tout point $y \in Y$, un entourage V_y tel que l'ensemble $f^{-1}(V_y)$, c'est-à-dire celui des x pour lesquels $f(x) \in V_y$, est somme d'une famille d'ensembles ouverts disjoints,

$$f^{-1}(V_y) = \sum_i U_i,$$

sur lesquels les fonctions partielles $f|U$ sont des homéomorphies et $f(U_i) = V_y$.

Les fonctions de ces trois classes sont donc continues par définition.

Toute homéomorphie locale en est trivialement une au sens large, mais pas réciproquement, même lorsque X et Y sont compacts. Par exemple, la fonction $f(x) = e^{ix}$ transforme le segment $0 \leqslant x \leqslant 2\pi$ en circonférence par l'homéomorphie locale au sens large, sans qu'elle soit une homéomorphie locale; en effet, aucun ensemble ouvert dans une circonférence n'est l'image homéomorphe d'un ensemble qui est ouvert dans le segment et en contient le bout $x = 0$.

Le même exemple montre qu'une homéomorphie locale au sens large peut augmenter l'*ordre* d'un point, à savoir transformer le bout (donc point d'ordre 1) d'un segment en un point de circonférence (donc point d'ordre 2); mais il sera démontré qu'elle ne peut le diminuer (voir § 3, théorème 2).

(¹) Dans [9] et [11], c'est l'espace X qui est dit *recouvrement de l'espace* Y.