About sets invariant with respect to denumerable changes

by

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1. In this paper I develop the results given in [4]. The main results were announced in [6]. Other generalizations of the theorems of [4] were given in [5], but the point of view adopted here is different. We are studying the algebraic aspect of the problem and the geometrical applications appear only in the last part of the paper. An analogous task concerning another geometrical problem — the (paradoxal) decompositions of the sphere — was performed by T. Dekker [2].

The main theorem of this paper is the following: *There exists in the 3-dimensional Euclidean space a set which is congruent to each set obtained by taking out of it and adding to it any at most denumerable sets.* Sets with this property exist also in the n-dimensional Euclidean space $\mathcal{E}^n$ (for $n \geqslant 3$), the n-dimensional sphere $\mathcal{S}_n(x_0^2+x_1^2+\ldots+x_n^2=1)$ (for $n\geqslant 2$, $n\neq 4$), the n-dimensional elliptic space $\mathcal{L}^n$ (for $n \geqslant 2$, $n\neq 4$), and the n-dimensional hyperbolic space $\mathcal{H}^n$ (for $n\geqslant 2$).

The problem of the existence of such sets in $\mathcal{S}_4$ and $\mathcal{L}^4$ remains open. The problem of the existence of such sets in $\mathcal{E}^2$ was solved negatively by E. G. Straus, who proved [8] that a plane set E contains at most one point p such that

(1.1) $$E \setminus (p) \simeq E$$

($\simeq$ denote congruence of sets). But $\mathcal{S}_4$ and $\mathcal{L}^4$ contain sets E satisfying (1.1) for each $p \in E$ (see [5] and Theorem 5 of this paper.)

The geometrical theorems of this paper are obtained by means of the general (algebraic) results developed here and on more special results concerning the groups of motions of $\mathcal{E}^n$ (Jan Mycielski and S. Świerczkowski [7]) and $\mathcal{S}_n$, $\mathcal{L}^n$, $\mathcal{H}^n$ (T. Dekker [2], [3]).

2. Notation. Let G be a group of 1-1 transformations of a set S onto itself (*e. g.* $S = G$, and the elements of G are treated as transformations of G in the usual sense).

1 denotes the unity of groups, which are all multiplicatively written.

For any $H \subset G$ and $P \subset S$ we use the notation:

(2.1) $$HP = \{h(p)\colon\ h \in H,\ p \in P\}.$$

(p) denotes the set consisting of a single point p; thus, for $p \in S$, the set $H(p)$ is also well defined.

$[H]$ denotes the subgroup of G generated by H.

(2.2) We say that H is *free at* P if every element of HP is uniquely expressible in the form $h(p)$ where $h \in H$, $p \in P$ ([1]).

It is easy to verify that

(2.3) If H is free at P and $U, V \subset H$, then

$$(U\cup V)P = UP \cup VP, \quad (U\cap V)P = UP\cap VP, \quad (U\setminus V)P = UP\setminus VP.$$

(2.4) An element $x \in G$ is said to be *free at* H if for every $a_0, a_n \in [H]$ and $a_1, a_2, \ldots, a_{n-1} \in [H]\setminus(1)$, and every integers $k_1, \ldots, k_n$ different from 0, the equality

$$a_0 x^{k_1} a_1 \ldots a_{n-1} x^{k_n} a_n = 1$$

implies $n = 0$ (this means that x is of infinite order and the group $[(x)\cup H]$ is a free product of the groups $[(x)]$ and $[H]$).

(2.5) An element $x \in G$ is said to be *free at* H *and* P if x is free at H and $[(x)\cup H]\setminus[H]$ is free at P.

3. The main algebraic theorem.

THEOREM 1. *Let S be a set and G a free group of 1-1 transformations of S onto itself, with a well ordered set of free generators $\{\varphi_\xi\}_{\xi<\alpha}$. Two disjoint sets $C_1, C_2 \subset S$ and three sequences $\{F_\xi\}_{\xi<\alpha}$, $\{A_\xi\}_{\xi<\alpha}$, $\{B_\xi\}_{\xi<\alpha}$ of subsets of S are given.*

We suppose that for every $\xi < \alpha$, φ_ξ is free at $[\{\varphi_\tau\}_{\tau<\xi}]$ and $C_1 \cup C_2 \cup \bigcup_{\tau\leqslant\xi} F_\tau$ and φ_ξ is free at $[\{\varphi_\tau\}_{\tau<\xi}]$ and $C_1 \cup C_2 \cup \bigcup_{\tau\leqslant\xi}(A_\tau \cup B_\tau)$ (see (2.5)).

Then there exist such sets $E_1, E_2 \subset S$ that for every $\xi < \alpha$

$$\varphi_\xi E_1 = E_1 \dot{-} F_\xi\,(^2), \qquad \varphi_\xi E_2 = (E_2 \setminus A_\xi)\cup B_\xi,$$

and moreover

$$C_1 \subset E_i \subset S\setminus C_2 \quad \textit{for} \quad i = 1, 2;$$

$$\overline{\overline{E}}_1 = \aleph_0\overline{\overline{\alpha}} + \overline{\overline{C}}_1 + \sum_{\xi<\alpha}\overline{\overline{F}}_\xi .$$

(the hypothesis on $\{F_\xi\}$ is not necessary for the construction of E_2 and the hypothesis on $\{A_\xi\}$ and $\{B_\xi\}$ is not necessary for the construction of E_1).

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(1) This is a basic notion also in my papers [4] and [5].

(2) $\dot{-}$ denotes the symmetric difference of sets, *i. e.*, $X \dot{-} Y = (X \cup Y)\setminus(X \cap Y)$.

For the proof of this theorem we need some definitions and a lemma.

Let $[M]$ be a free group with a set of free generators M. Let $\varphi \in M$. For any $\tau \in [M]\setminus(1)$ consider the canonical representation

$$\tau = \varphi_1^{k_1}\varphi_2^{k_2}\ldots\varphi_m^{k_m}$$

where $\varphi_i \in M$, $k_i = \pm 1$ and $\varphi_i^{k_i} \neq \varphi_{i+1}^{-k_{i+1}}$.

(3.1)
Φ^+ is the set of all those τ in which $\varphi_m = \varphi$ and $k_m = 1$;
Φ^- is the set of all those τ in which $\varphi_m = \varphi$ and $k_m = -1$;
$\Phi^\pm$ is the set of all those τ in which $\varphi_m = \varphi$.

Lemma 1. *We have the following equalities:*

(3.2) $$\varphi\Phi^+ = \Phi^+\setminus(\varphi),$$

(3.3) $$\varphi\Phi^- = \Phi^-\cup(1),$$

(3.4) $$\varphi\Phi^\pm = \Phi^\pm\setminus(\varphi)\cup(1),$$

(3.5) $$\psi\Phi^\iota = \Phi^\iota \quad \textit{for} \quad \iota = +,-,\pm \textit{ and any } \psi \in M\setminus(\varphi).$$

Proof. We see that $\varphi^{-1}\tau \in \Phi^+$ if and only if $\tau \in \Phi^+$ and $\tau \neq \varphi$. This proves (3.2). We see that $\varphi^{-1}\tau \in \Phi^-$ if and only if $\tau \in \Phi^-$ or $\tau = 1$. This proves (3.3). (3.4) follows from (3.2) and (3.3) since $\Phi^\pm = \Phi^+\cup\Phi^-$. The equalities (3.5) are obvious; q. e. d.

Proof of Theorem 1. 1. The construction of E_1. For $\alpha = 0$ we put $E_1 = C_1$ and the theorem is satisfied. Then suppose that $\alpha > 0$.

We will construct a sequence of subsets of S

$$R_{-1}, R_0, R_1, R_2, \ldots, R_\xi, \ldots, \qquad \xi < \alpha$$

such that if $0 \leqslant \xi \leqslant \zeta < \alpha$, then

(3.6) $$R_\zeta \subset [\{\varphi_\tau\}_{\tau\leqslant\zeta}](C_1 \cup \bigcup_{\tau\leqslant\zeta} F_\tau),$$

(3.7) $$\overline{\overline{R}}_\zeta = \aleph_0(\bar{\zeta}+1) + \overline{\overline{C}}_1 + \sum_{\tau\leqslant\zeta}\overline{\overline{F}}_\tau,$$

(3.8) $$C_1 \subset R_{-1} \subset R_\xi \subset R_\zeta \subset S\setminus C_2,$$

(3.9) $$F_\xi \cap R_\zeta = F_\xi \cap R_\xi,$$

(3.10) $$\varphi_\xi R_\zeta = R_\zeta \dot{-} F_\xi.$$

Then we obtain a set E_1 satisfying the theorem if we put

$$E_1 = \bigcup_{\xi<\alpha} R_\xi.$$

In fact: $C_1 \subset E_1 \subset S\setminus C_2$ by (3.8), $\overline{\overline{E}}_1 = \aleph_0\bar{\alpha} + \overline{\overline{C}}_1 + \sum_{\xi<\alpha}\overline{\overline{F}}_\xi$ by (3.7) and for every $\xi < \alpha$

$$\begin{aligned}
\varphi_\xi E_1 &= \varphi_\xi \bigcup_{\tau\geqslant\xi} R_\tau && \text{(by (3.8))}\\
&= \bigcup_{\tau\geqslant\xi}\{(R_\tau\cup F_\xi)\setminus(R_\tau\cap F_\xi)\} && \text{(by (3.10))}\\
&= \{\bigcup_{\tau\geqslant\xi}(R_\tau\cup F_\xi)\}\setminus(E_1\cap F_\xi) && \text{(by (3.9))}\\
&= E_1 \dot{-} F_\xi, && \text{q. e. d.}
\end{aligned}$$

Now we give an inductive definition of the sequence $\{R_\xi\}_{\xi<\alpha}$.

We put $R_{-1} = C_1$. Suppose that for some $\zeta < \alpha$ the sequence $\{R_\xi\}_{\xi<\zeta}$ is already defined and satisfies (3.6)-(3.10). We will construct R_ζ satisfying these conditions.

We use the notation

(3.11) Φ_ι $(\iota = +,-,\pm)$ is defined by (3.1) with $M = \{\varphi_\xi\}_{\xi<\zeta}$, $\varphi = \varphi_\zeta$;

(3.12) $$R = \bigcup_{\xi<\zeta} R_\xi;$$

(3.13) $$A = R\cap F_\zeta, \quad B = F_\zeta\setminus R, \quad C = R\setminus F_\zeta.$$

We put

(3.14) $$R_\zeta = R \cup \Phi^+A \cup \Phi^-B \cup \Phi^\pm C.$$

Obviously R_ζ fulfills (3.6) and (3.7). The hypothesis of the theorem implies that

(3.15) $$\varphi_\zeta \text{ is free at } [\{\varphi_\xi\}_{\xi<\zeta}] \text{ and } [\{\varphi_\xi\}_{\xi<\zeta}](C_1\cup C_2\cup\bigcup_{\xi\leqslant\zeta}F_\xi).$$

Then (3.8) and (3.9) are also satisfied, *e. g.* $C_2\cap R_\zeta = \emptyset$ because $\Phi^\pm$ is free on $R\cup F_\zeta\cup C_2$, $C_2\cap R = \emptyset$, and $1 \notin \Phi^\pm$. For proving (3.10) we use Lemma 1. By (3.15) for $\xi \leqslant \zeta$ the sets $R\cup F_\xi$, Φ^+A, Φ^-B, $\Phi^\pm C$ are disjoint. Indeed as to the sets Φ^+A, Φ^-B it follows from (3.15) and from that $\Phi^+\cap\Phi^- = \emptyset$. Other cases are to be treated similarly. Then (3.10) for $\xi < \zeta$ follows from (3.5), and the inductive hypothesis. In order to prove (3.10) for $\xi = \zeta$ we remark that $R = A\cup C$ and then by (2.3) we have

(3.16) $$R_\zeta = \Phi^+A \cup A \cup \Phi^-B \cup \Phi^\pm C \cup C.$$

By (3.15) the sets $A, B, C, \Phi^+A, \Phi^-B, \Phi^\pm C$ are disjoint. Therefore, by (3.2)-(3.4)

$$\begin{aligned}
\varphi_\zeta R_\zeta &= \big(\Phi^+\setminus(\varphi_\zeta)\big)A \cup \varphi_\zeta A \cup \big(\Phi^-\cup(1)\big)B \cup \big(\Phi^\pm\setminus(\varphi_\zeta)\cup(1)\big)C \cup \varphi_\zeta C\\
&= \Phi^+A \cup (\Phi^-B\cup B) \cup (\Phi^\pm C\cup C) \qquad \text{(by (2.3))},\\
&= R_\zeta \dot{-} F_\zeta \qquad \text{(by (3.13))}.
\end{aligned}$$

Then the sequence $\{R_\xi\}_{\xi<a}$ is constructed and satisfies (3.6)-(3.10). Consequently on account of the previous remark the set E_1 is constructed.

2. The construction of E_2. For $a=0$ we put $E_2=C_1$ and the theorem is satisfied. Then suppose that $a>0$.

We will construct a sequence of subsets of S

$$R_{-1}, R_0, R_1, R_2, \ldots, R_\xi, \ldots, \qquad \xi<a,$$

such that if $0\leqslant\xi\leqslant\zeta<a$, then the conditions (3.6), (3.8) are satisfied and

$$\varphi_\xi R_\zeta=(R_\zeta\setminus A_\xi)\cup B_\xi. \tag{3.17}$$

Then we obtain a set E_2 satisfying the theorem by putting

$$E_2=\bigcup_{\xi<a} R_\xi.$$

This can be verified in the same way as for E_1.

Now we give an inductive definition of the sequence $\{R_\xi\}_{\xi<a}$.

We put $R_{-1}=C_1$. Suppose that for some $\zeta<a$ the sequence $\{R_\xi\}_{\xi<\zeta}$ is already defined and satisfies (3.6), (3.8) and (3.17).

We use the notation of (3.11), (3.12) and

$$A=R\cap A_\zeta, \quad B=B_\zeta, \quad C=R\setminus A_\zeta.$$

We define R_ζ by the formula (3.14). We can verify that R_ζ satisfies (3.6) and (3.8) (in the same way as in the construction of E_1). For obtaining (3.17) it is enough to use Lemma 1 and the identity (3.16), which holds here also.

Consequently on account of these remarks the set E_2 is constructed, which completes the proof of Theorem 1.

Let us note the following corollary to Theorem 1:

COROLLARY 1. *Let $[M]$ be a free group of* 1-1 *transformations of a set S onto itself, with a set M of free generators. Let $R\subset S$ be a set on which $[M]$ is free and $\boldsymbol{F}$ any family of subsets of R such that $\overline{\overline{\boldsymbol{F}}}\leqslant\overline{\overline{M}}$. Then there exists such a set E that $R\subset E\subset S$ and for each $F\in\boldsymbol{F}$ there exists a $\varphi\in M$ such that $\varphi E=E\setminus F$ and moreover $\overline{\overline{E}}=\aleph_0+\overline{\overline{R}}+\overline{\overline{\boldsymbol{F}}}$.*

Proof. We put in the Theorem 1

$$C_1=R, \quad C_2=\emptyset, \quad \{\varphi_\xi\}=M, \quad \{F_\xi\}=\boldsymbol{F}.$$

Then $E=E_1$ satisfies the corollary.

(Another proof of this corollary, similar to the proof of Theorem 1 in [4], will be given in section 6. This Corollary and Lemma 2 of [4] implies Theorem 1 of [4]. Other applications, especially for analytic systems (see [5]) are also possible.)

4. Some applications.

(4.1) A group G of 1-1 transformations of a set S onto itself is called *locally commutative* if any two transformations $h_1, h_2\in G$ which have a common fixed point are commutative, *i. e.* the existence of such a $p\in S$ that $h_1(p)=h_2(p)=p$ implies $h_1h_2=h_2h_1$ (3).

We introduce the following condition on a family of sets $\boldsymbol{F}$ and a cardinal number $\mathfrak{m}$:

(4.2) $\mathfrak{m}$ is infinite, $\overline{\overline{\boldsymbol{F}}}\leqslant\mathfrak{m}$ and for every $F\in\boldsymbol{F}$ there is $\overline{\overline{F}}<\mathfrak{m}$.

We need the following

LEMMA 2. *If $\boldsymbol{F}$ and $\mathfrak{m}$ satisfy* (4.2) *then $\boldsymbol{F}$ can be well ordered into a sequence $\{F_\xi\}$ such that for every F_{ξ_0} there is*

$$\xi_0<\mathfrak{m} \quad \textit{and} \quad \sum_{\xi\leqslant\xi_0}\overline{\overline{F}}_\xi<\mathfrak{m}\ (^4).$$

Proof. Of course it is enough to consider the case when for each cardinal number $\mathfrak{n}<\mathfrak{m}$ there exist $\mathfrak{m}$ sets $F\in\boldsymbol{F}$ of potency $\mathfrak{n}$. Therefore we can suppose that

$$\boldsymbol{F}=\{F_{\chi\xi}\}_{\bar\chi,\bar\xi<\mathfrak{m}} \quad \text{and} \quad \overline{\overline{F}}_{\chi\xi}=\bar\chi \quad \text{for each} \quad \bar\chi,\bar\xi<\mathfrak{m}.$$

We take the well ordering of $\boldsymbol{F}$ given by the conditions

$F_0=F_{00}$,

$F_\zeta=F_{\chi\xi}$ where $F_{\chi\xi}\notin\{F_\varkappa\}_{\varkappa<\zeta}$, $\chi+\xi$ is minimal and χ is minimal.

It is clear that then $\{F_\xi\}$ satisfies Lemma 2, q. e. d.

THEOREM 2. *Let S be a set, G a free group of* 1-1 *transformations of S onto itself with a set of free generators M, and $\boldsymbol{F}$ a family of subsets of S. To any $F\in\boldsymbol{F}$ correspond two sets A_F, B_F and $A_F\cup B_F=F$.*

We suppose that $\boldsymbol{F}$ and $\overline{\overline{M}}$ satisfy (4.2) *and that G is locally commutative on S.*

Then there exist such sets $E_1, E_2\subset S$ that for every $F\in\boldsymbol{F}$ there exists such a $\varphi\in M$ that

$$\varphi E_1=E_1\dot{-}F, \quad \varphi E_2=(E_2\setminus A_F)\cup B_F$$

and moreover $\overline{\overline{E}}_1=\aleph_0\sum_{F\in\boldsymbol{F}}\overline{\overline{F}}$.

Proof. We take the well ordering $\boldsymbol{F}=\{F_\xi\}_{\xi<a}$ given by Lemma 2. By Theorem 1 it is enough to construct a sequence $\{\varphi_\xi\}_{\xi<a}\subset M$, such that for any $\xi<a$, φ_ξ is free at $[\{\varphi_\tau\}_{\tau<\xi}]$ and $\bigcup_{\tau\leqslant\xi}F_\tau$. We give an inductive definition of such a sequence.

(3) Terminology of T. Dekker [2].

(4) I own this lemma to S. Świerczkowski.

Suppose that for some $\zeta < \alpha$ the sequence $\{\varphi_\xi\}_{\xi<\zeta} = Z$ is already defined ($Z \subset M$).

Let f_x be any function on G which is of the form

$$f_x = a_0 x^{k_1} a_1 \ldots a_{n-1} x^{k_n} a_n$$

where $a_0, a_n \in [Z]$, $a_1, \ldots, a_{n-1} \in [Z] \setminus (1)$, $k_1, \ldots, k_n$ are integers different from 0 and $n \geqslant 1$.

We shall prove that for any such function f_x and any $p_1, p_2 \in S$ the set

$$S_{p_1 p_2 f_x} = \{\varphi \colon \varphi \in M \setminus Z,\ f_\varphi(p_1) = p_2\}$$

consists of two elements at most. Suppose a contrario that

$$f_{\varphi_1}(p_1) = p_2, \quad f_{\varphi_2}(p_1) = p_2, \quad f_\psi(p_1) = p_2$$

where $\varphi_1, \varphi_2, \psi \in M \setminus Z$ and $\varphi_1 \neq \varphi_2 \neq \varphi \neq \varphi_1$. Then

$$f_\psi^{-1} f_{\varphi_1}(p_1) = p_1, \quad f_\psi^{-1} f_{\varphi_2}(p_1) = p_1, \quad f_\psi^{-1} f_{\varphi_1} \neq 1 \neq f_\psi^{-1} f_{\varphi_2}.$$

But then the transformations $f_\psi^{-1} f_{\varphi_1}$, $f_\psi^{-1} f_{\varphi_2}$ have a common fixed point and they are commutative. This contradicts the hypothesis that M is a set of free generators.

We put

$$K = [Z]\Big(\bigcup_{\tau \leqslant \zeta} F_\tau\Big).$$

By (4.2) $\bar{\bar{\zeta}} < \bar{\bar{M}}$ and $\bar{\bar{K}} < \bar{\bar{M}}$. Then there exists an element

$$\varphi_\zeta \in M \setminus \Big(Z \cup \bigcup_{p_1, p_2 \in K, f_x} S_{p_1 p_2 f_x}\Big).$$

Of course φ_ζ is free on $[Z]$ and K. Then the Theorem 2 is proved.

Corollary 2. *Let U be a group, $[M]$ a free subgroup of U with a set of free generators M and let $\boldsymbol{F}$ be a family of subsets of U. We suppose that $\boldsymbol{F}$ and $\bar{\bar{M}}$ satisfy* (4.2).

Then there exists such a set $E \subset U$ that $\bar{\bar{E}} = \aleph_0 \sum_{F \in \boldsymbol{F}} \bar{\bar{F}}$ and for every $F \in \boldsymbol{F}$ there exists a $\varphi \in M$ such that

$$\varphi E = E \dot{-} F.$$

Proof. By Theorem 2, since M is locally commutative (without fixed points) on U.

(This Corollary implicitly contains Lemma 4 of [4], Lemma 4 of [5] and Theorem 2 of [8]. Recall that every connected compact non-abelian and every connected locally compact non-solvable topological group contains a free subgroup of the rank $2^{\aleph_0}$, (see [1]), which gives some applications of Corollary 2 to such groups.)

Corollary 3. *A free group U of any rank >1 contains such a set E that for any finite sets $A, B \subset U$ there exists such a $\varphi \in U$ that*

$$\varphi E = (E \setminus A) \cup B.$$

Proof. By Corollary 2. (This Corollary implies Lemma 4 of [5].)

5. The geometrical theorem.

Theorem 3. *Each of the spaces*

$\mathcal{E}^n$ *for* $n \geqslant 3$,
$\mathcal{S}_n$ *for* $n \geqslant 2,\ n \neq 4$,
$\mathcal{L}^n$ *for* $n \geqslant 2,\ n \neq 4$,
$\mathcal{H}^n$ *for* $n \geqslant 2$,

contains such a set E that for any at most denumerable subsets A, B of this space we have

$$E \simeq (E \setminus A) \cup B.$$

Proof. It was proved [7] that there exist free groups of motions of $\mathcal{E}^n$ ($n \geqslant 3$) of the rank $2^{\aleph_0}$, without fixed points. Of course these groups are locally commutative (see (4.1)). The existence of such groups of isometries for the other spaces enumerated in the theorem is proved in [2], [3].

Since the families of at most denumerable subsets of these spaces and the cardinal $2^{\aleph_0}$ satisfy the condition (4.2), our theorem follows from Theorem 2.

(The Theorem 3 for $\mathcal{S}_2$ is a refinement of the main Theorem 2 of [4].)

6. Remarks and problems. 1. The existence of sets invariant with respect to denumerable changes in the spaces $\mathcal{S}_4$ and $\mathcal{L}^4$ remains an open problem, because it is an open problem if there exist free locally commutative groups of the rank $2^{\aleph_0}$ (also of the rank 2) of isometries of these spaces (see [3]).

2. In this paper the axiom of choice was used in the following cases:

(a) The proof of Corollary 1 (the well-ordering of M and $\boldsymbol{F}$).
(b) The proof of Lemma 2.
(c) The proof of Theorem 2 (the construction of $\{\varphi_\xi\}_{\xi<\alpha}$).
(d) The proof of Theorem 3.

Concerning (a) we give here an independent effective proof of Corollary 1:

By hypothesis there exists a 1-1 mapping φ_F ($F \in \boldsymbol{F}$) of $\boldsymbol{F}$ into M. For every $\boldsymbol{P} \subset \boldsymbol{F}$ we put

$$\Phi_{\boldsymbol{P}} = [\{\varphi_F\}_{F \in \boldsymbol{F}}] \setminus \bigcup \Phi^-$$

where Φ^- runs over all sets defined by (3.1) with $\varphi = \varphi_F$ and $F \in \boldsymbol{P}$.

Then by (3.3) and (3.5) we have

$$(6.1)\qquad 1 \in \Phi_{\boldsymbol{P}};$$

$$(6.2)\qquad \varphi_F\Phi_{\boldsymbol{P}} = \Phi_{\boldsymbol{P}}\setminus(1) \quad \text{if} \quad F \in \boldsymbol{P};$$

$$(6.3)\qquad \varphi_F\Phi_{\boldsymbol{P}} = \Phi_{\boldsymbol{P}} \quad \text{if} \quad F \in \boldsymbol{F}\setminus\boldsymbol{P}.$$

(compare [4] Lemma 1).

Now we put for every $p \in R$

$$\boldsymbol{P}(p) = \{F:\ p \in F \in \boldsymbol{F}\},$$

and

$$E = \bigcup_{p\in R} \Phi_{\boldsymbol{P}(p)}(p).$$

It is clear that $\overline{\overline{E}} = \aleph_0 + \overline{\overline{R}} + \overline{\overline{F}}$, and that $R \subset E$. Since $[M]$ is free on R and by (6.1), (6.2) and (6.3) we have

$$\begin{aligned}\varphi_F E &= \bigcup_{p\in F} \varphi_F\Phi_{\boldsymbol{P}(p)}(p) \cup \bigcup_{p\in R\setminus F} \varphi_F\Phi_{\boldsymbol{P}(p)}(p)\\ &= \bigcup_{p\in F} \big(\Phi_{\boldsymbol{P}(p)}\setminus(1)\big)(p) \cup \bigcup_{p\in R\setminus F} \Phi_{\boldsymbol{P}(p)}(p)\\ &= E\setminus F, \qquad \text{q. e. d.}\end{aligned}$$

Concerning (b), (c) and (d) it seems that the use of the axiom of choice is essential. To sum up, in the proof of Theorem 3 the axiom of choice is used twice: 1° the well-ordering of the family of at most denumerable subsets of the continuum, 2° the well-ordering of a set of free generators of a free locally commutative group of potency $2^{\aleph_0}$.

3. Note that the following theorem may be proved without using the axiom of choice:

THEOREM 4. *Each of the spaces $\mathcal{S}_2$, $\mathcal{L}^2$ and $\mathcal{H}^2$ contains such a denumerable set E and such a set H of potency $2^{\aleph_0}$ that*

$$E\setminus F \simeq E \quad \textit{and} \quad H\setminus D \simeq H$$

for each finite set $F \subset E$ and each at most denumerable set $D \subset H$.

Such an effective proof of this theorem can be obtained by a fuller exploitation of the material given in [8] and [2], [3].

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