About sets with strange isometrical properties (II)

by

Jan Mycielski (Wrocław)

It is the purpose of this paper to give some generalizations of the results of part (I) [5]. Theorem 1 of this paper was announced in [6]. This paper was intended first to be a continuation of [5] because in the proof of theorem 2 the continuum hypothesis was used. Since that time we have elaborated with S. Balcerzyk [1] a method which enables me to eliminate the continuum hypothesis from that proof. Therefore theorem 2 of this paper is a generalization of the main theorem 2 of [5]. Let us mention also that problem II of [5] was solved by E. G. Straus, who proved [7] that a plane set E contains at most one point p such that

$$E \simeq E \setminus (p) \quad (\simeq \text{ denotes congruence of sets}).$$

In our proofs the axiom of choice is used without restrictions.

The theorems of [5] concerning the sphere $\mathcal{S}_2$ and the group of rotations are transferred here for arbitrary analytic manifolds and groups of analytic homeomorphisms.

All manifolds considered are supposed to be analytic, real and connected. For two manifolds M and M', a mapping $f\colon M \to M'$ is called *analytic* if the local coordinates of the point $f(p) \in M'$ are analytic functions of the local coordinates of the point $p \in M$. $\langle M, G\rangle$ is called an *analytic system* if M is a manifold and G a connected Lie group of analytic homeomorphisms of M onto itself and the mapping $r\colon (G\times M)\to M$ defined by $r(h,p) = h(p)$ is analytic.

THEOREM 1. *Let M be a manifold and U a free group of rank 2 of analytic homeomorphisms of M onto itself. Then there exists such a denumerable set $E \subset M$ that for every finite set $F \subset E$ there exists a $h \in U$ such that*

$$h(E) = E \setminus F.$$

THEOREM 2. *Let $\langle M, G\rangle$ be an analytic system and suppose that G contains a free subgroup of rank 2. Then there exists such a set $H \subset M$ of potency $2^{\aleph_0}$ that for every at most denumerable set $D \subset H$ there exists a $h \in G$ such that*

$$h(H) = H \setminus D.$$

In order to apply these theorems to more concrete spaces let us remember that every connected compact non-abelian and every connected locally compact non-solvable topological group contains a free subgroup of the rank $2^{\aleph_0}$ (see [2]). Consequently the groups of isometries of the sphere $(x^2+y^2+z^2 = 1)$ and of the elliptic plane contain free subgroups of rank 2. It has also been proved by T. Dekker [3], [4] that the group of isometries of the hyperbolic plane contains a free subgroup of rank 2 (and $2^{\aleph_0}$). Then our theorems imply, that the sphere, the elliptic plane, and the hyperbolic plane contain such a denumerable set E and such a set H of potency $2^{\aleph_0}$ that

$E \simeq E \setminus F$ for every finite set $F \subset E$ [1],

$H \simeq H \setminus D$ for every at most denumerable set $D \subset H$.

To prove theorems 1 and 2 we need some notions and lemmas.

A group of transformations G acting on a space M is said to be *free at a point* $p \in M$ if for every $g, h \in G$

$$g(p) = h(p) \quad \text{implies} \quad g = h.$$

For any subgroup $U \subset G$, $x \in G$ and $p \in M$, x is called *free at U and p* if for every sequences of elements $a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_n \in U$ ($a_i, b_j \neq e$ for $i \neq 1, m$, $j \neq 1, n$) and of integers $k_1, \ldots, k_m, l_1, \ldots, l_n$ ($k_i, l_i \neq 0$) the equality

$$a_0x^{k_1}a_1x^{k_2}a_2\ldots x^{k_m}a_m(p) = b_0x^{l_1}b_1x^{l_2}b_2\ldots x^{l_n}b_n(p) \tag{1}$$

implies

$$m = n,\quad k_i = l_i \quad\text{and}\quad a_i = b_i \quad\text{for}\quad i = 0, 1, \ldots, m. \tag{2}$$

For any set $K \subset G$ we put

$$K(p) = \{f(p)\colon f \in K\}.$$

LEMMA 1. *If G is free at p then for every $K \subset G$ we have*

$$(G\setminus K)(p) = G(p)\setminus K(p).$$

The proof is obvious.

LEMMA 2. *Let G be a group of analytic homeomorphisms of a manifold M onto itself. If $\overline{\overline{G}} < 2^{\aleph_0}$, then there exists such a point $p \in M$ that G is free at p.*

Proof. If $g, h \in G$ and $g \neq h$ there exists such a $p \in M$ that

$$g(p) \neq h(p).$$

[1] This theorem for the hyperbolic plane was proved in a more direct way by T. Viola [8].

Then the set $A_{gh} = \{p: g(p) = h(p)\}$ is an analytic surface in M (see [1]). By a theorem of [1] the union $S = \bigcup_{g,h\in G, g\neq h} A_{gh}$ is a border set in M. Then there exists a point $p \in M \setminus S$, as required.

LEMMA 3. *Suppose that the hypothesis of theorem 2 is satisfied. Then there exists a free subgroup U of G, of rank $2^{\aleph_0}$ and a point $p \in M$ such that U is free at p.*

Proof. There exist a free group $G' \subset G$ of rank 2 and, by lemma 2, a point $p \in M$ on which G' is free. Then

(3) *There exists a free group $U_0 \subset G$ of the rank $\aleph_0$ which is free at a point $p \in M$.*

Indeed if a, b are free generators of G' then $ab, a^2b^2, a^3b^3, \ldots$ are free generators of such a group U_0.

Now we shall prove the following assertion:

(4) *If U_ξ is a free group of infinite rank, $\overline{\overline{U}}_\xi < 2^{\aleph_0}$ and U_ξ is free at p, then there exists an $x \in G$ which is free at U_ξ and p.*

Consider the equation (1) with $a_i, b_i \in U_\xi$ and suppose that (2) is not satisfied. Then (1) is not satisfied for some $x \in G$ (it is enough to take for x one of the free generators of U_ξ which does not occur in the reduced expressions of a_i and b_i). Then the set $\{x: x \in G,\ x \text{ satisfies } (1)\}$ is an analytic surface in G. The number of these surfaces is $< 2^{\aleph_0}$, then their union is a border set in G. Consequently there exists an $x \in G$ which satisfies an equation of the form (1) only if the corresponding relations (2) hold. This proves (4).

The statements (3) and (4) clearly implies the lemma 3.

LEMMA 4. *A free group G of the rank 2 contains such a denumerable set P, that for every finite set $F \subset P$ there exists such an $u \in G$, that*

$$uP = P \setminus F .$$

For the proof see [5] or [7] (it does not use the axiom of choice).

LEMMA 5. *A free group U of the rank $2^{\aleph_0}$ contains such a set R of the power $2^{\aleph_0}$, that for every at most denumerable set $D \subset R$ there exists such an $u \in G$, that*

$$uR = R \setminus D .$$

For the proof see [5] or [7] (it uses the axiom of choice).

Proof of Theorem 1. Applying the lemmas 2 and 4, we put $E = P(p)$. By Lemma 1 E satisfies Theorem 1.

Proof of Theorem 2. Applying the lemmas 3 and 5, we put $H = R(p)$. By Lemma 1 H satisfies Theorem 2.

References

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Reçu par la Rédaction le 18. 7. 1957