On free groups of motions and decompositions of the Euclidean space

by

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The purpose of this paper is to prove two theorems given in section 1 [1]. These theorems are solutions of some problems proposed to the authors by T. Dekker.

1. A group Φ of 1-1 transformations of a set E onto itself is called *without fixed points* if for every $p \in E$ and every $\varphi \in \Phi \setminus (e)$ we have $\varphi(p) \neq p$.

The *rank* of a free group is the potency of a set of free generators of this group.

The sense-preserving isometries of the Euclidean space $\mathcal{E}^3$, *i. e.*, the superpositions of rotations and translations are called *motions*.

THEOREM 1. *There exists a free group of the rank $2^{\aleph_0}$ of motions of $\mathcal{E}^3$ without fixed points.*

The proof of this theorem follows in sections 3-9 (it is an effective construction, which does not use the axiom of choice). The relations of Theorem 1 with known results are given in section 2.

An application of this theorem (the construction of a set $E \subset \mathcal{E}^3$ which is congruent to $(E \setminus A) \cup B$ for any at most denumerable sets $A, B \subset \mathcal{E}^3$) is given in [7]. Another application, of a well known character (compare [2], [3], [4], [8]), is the theorem 2 of this paper. It states that, for any system of congruence relations, $\mathcal{E}^3$ can be divided into disjoint sets satisfying that system. For an exact description of this theorem we take the following notations:

M and N are non-empty sets.

$\{P_\mu\}_{\mu \in M}$ and $\{Q_\mu\}_{\mu \in M}$ are arbitrary systems of subsets of N, all different from $\emptyset$ and N. (We do not suppose that $\mu_1 \neq \mu_2$ implies $P_{\mu_1} \neq P_{\mu_2}$ or $Q_{\mu_1} \neq Q_{\mu_2}$.)

$\simeq$ denotes congruence of point sets realizable by a motion.

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(1) They were announced in [6] by the first author, but the original proof was faulty, and the proof presented here was worked out by both authors.

THEOREM 2. *If $\overline{\overline{M}} \leqslant 2^{\aleph_0}$ and $\overline{\overline{N}} \leqslant 2^{\aleph_0}$ then the space $\mathcal{E}^3$ can be decomposed into $\overline{\overline{N}}$ disjoint sets A_ν $(\nu \in N)$ satisfying the system of congruences*

$$\bigcup_{\nu \in P_\mu} A_\nu \simeq \bigcup_{\nu \in Q_\mu} A_\nu \qquad (\mu \in M). \tag{1}$$

Moreover all the pieces A_ν can be non-empty.

The proof of this theorem follows in section 10. The relations of Theorem 2 with known results are given in section 2. Note here the following applications:

(A) $\mathcal{E}^3$ is the sum of a sequence $A_1, A_2, \ldots$ of disjoint sets such that

$$\bigcup_{n \in N_1} A_n \simeq \bigcup_{n \in N_2} A_n$$

for any sets $N_1, N_2 \subset \{1, 2, \ldots\}$ different from $\emptyset$ and $\{1, 2, \ldots\}$.

(B) For any order type α of potency $\leqslant 2^{\aleph_0}$ without the upper end, $\mathcal{E}^3$ is the sum of a family of distinct sets, ordered by the relation $\subset$ isomorphically to α and congruent by motions each to the other.

Indeed let N be ordered in the type α by the relation $\prec$ and $\nu_0 \in N$ and $M = N$. Take for (1) the system

$$\bigcup_{\nu \in N, \nu \prec \mu} A_\nu \simeq A_{\nu_0} \qquad (\mu \in N).$$

(C) There exists such a set $E \subset \mathcal{E}^3$ that for any cardinal $\mathfrak{m}$, such that $2 \leqslant \mathfrak{m} \leqslant 2^{\aleph_0}$, $\mathcal{E}^3$ is a sum of $\mathfrak{m}$ disjoint sets each congruent by a motion with E.

Indeed let $\overline{\overline{N}} = 2^{\aleph_0}$, $N_{\mathfrak{k}} \subset N$, $\overline{\overline{N}}_{\mathfrak{k}} = \mathfrak{k}$ for any $\mathfrak{k} < 2^{\aleph_0}$ and $\nu_0 \in N$. Take for (1) the system

$$A_{\nu_0} \simeq A_\nu \qquad (\nu \in N),$$

$$A_{\nu_0} \simeq \bigcup_{\nu \in N \setminus N_{\mathfrak{k}}} A_\nu \qquad (\mathfrak{k} < 2^{\aleph_0}).$$

Then we put $E = A_{\nu_0}$ and (C) is obvious.

At last note that the generalization of Theorems 1 and 2 to any space $\mathcal{E}^n$ with $n \geqslant 3$ follows immediately.

2. Concerning Theorem 1 note that the existence of a free group of the rank $2^{\aleph_0}$ of rotations of $\mathcal{E}^3$ around a fixed point is well known (Sierpiński [12], p. 238 Lemme 1). Sierpiński's proof was simplified and related results were obtained by J. de Groot [5]. This theorem easily follows from our Theorem 1 (by the method given in section 5). These proofs are effective, *i. e.*, they do not use the axiom of choice. Non-effective theorems on the existence of free subgroups in topological groups, generalizing Sierpiński's theorem are given in [1].

As for Theorem 2, it is analogous to theorems known for spheres and non-Euclidean spaces of dimension $\geqslant 2$ ([2], [3]). It permits us to complete the two tables given in [3], p. 107 in all points concerning the Euclidean spaces. Theorem 2 follows from our Theorem 1 and a theorem of T. Dekker ([2], 2.2.2), but we give a direct proof in section 10 because in our case it is simpler. The applications (A)-(C) are analogous to the statements about the sphere $\mathcal{S}_2$ (instead of $\mathcal{E}^3$) proved in [4] and [8]. Of course Theorem 2 for the space $\mathcal{E}^3$ without one point follows from the analogous result concerning the sphere $\mathcal{S}_2$, but then motions are not sufficient to realize all the congruences — in general reflections are needed.

Theorem 1 does not hold for $\mathcal{E}^1$ and $\mathcal{E}^2$ because the group of motions of the plane is solvable ([9], p. 10) and thus cannot contain any free group of rank>1. Neither Theorem 2 holds for the line and the plane, as has been proved by T. Dekker ([2], p. 584).

3. We put

$$A_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix},$$

$$R_{\varphi\psi} = A_\varphi B_\psi A_\varphi^{-1},$$

$$f(x) = \sum_{n=1}^{\infty} 2^{2^{[nx]} - 2^{n^2}} \quad \text{for} \quad x > 0,$$

$$\varphi(x) = 2 \operatorname{arctg} f(x). \tag{2}$$

Let T_φ be a translation of $\mathcal{E}^3$ obtained by adding to every point of $\mathcal{E}^3$ the point-vector $A_\varphi(1, 0, 0)$ = the point $(1, 0, 0)$ transformed by A_φ.

The following theorem clearly implies Theorem 1:

THEOREM 1′. *The motions $T_{\varphi(x)} R_{\varphi(x)\varphi(1)}$ with $0 < x < 1$ are free generators of a free group without fixed points.*

Occasionally it is easy to derive the following corollary to this theorem.

COROLLARY. *The rotations $R_{\varphi(x)\varphi(1)}$ with $0 < x < 1$ are free generators of a free group.*

(This was proved by J. de Groot [5], Theorem II.)

4. For proving Theorem 1′ we need some lemmas.

LEMMA 1. *The values of the function $f(x)$ for $x > 0$ are algebraically independent numbers, i. e., if we put different values of this function (which is strictly increasing) in the places of the arguments of a non-constant rational function with integral coeficients, then we obtain a transcendental number.*

This is a theorem of J. von Neumann [10].

LEMMA 2. *Any product of the form*

$$A_\varphi^{k_1} B_\varphi^{l_1} A_\varphi^{k_2} B_\varphi^{l_2} \ldots A_\varphi^{k_n} B_\varphi^{l_n},$$

where $n \geqslant 1$ and k_i and l_i are integers different from 0, except k_1 and l_n one of which can be equal to 0, is a non-constant function of φ.

This Lemma is proved in [5], p. 257-8.

We put for brevity

(3) $$\varphi = \varphi(x_i) \quad (i = 1, \ldots, n)$$

and suppose that

(4) $$x_i \neq x_{i+1} \quad \text{and} \quad 0 < x_i < 1.$$

LEMMA 3. *Any product of the form*

$$R_{\varphi_1\psi}^{k_1} R_{\varphi_2\psi}^{k_2} \ldots R_{\varphi_n\psi}^{k_n},$$

where $n \geqslant 1$ and k_i are integers different from 0 is a non-constant function of ψ.

Proof. We write the product more explicitly

$$P_\psi = A_{\varphi_1} B_\psi^{k_1} A_{\varphi_1}^{-1} A_{\varphi_2} B_\psi^{k_2} A_{\varphi_2}^{-1} \ldots A_{\varphi_n} B_\psi^{k_n} A_{\varphi_n}^{-1}.$$

Of course

$$P_0 = I\ (^2).$$

Then for proving the lemma it is enough to verify that

$$P_{\varphi(x)} \neq I \quad \text{if} \quad x \neq x_i \quad \text{for} \quad i = 1, \ldots, n.$$

By (2) we have

$$\sin\varphi(u) = \frac{2f(u)}{1+\big(f(u)\big)^2}, \quad \cos\varphi(u) = \frac{1-\big(f(u)\big)^2}{1+\big(f(u)\big)^2}.$$

Therefore the elements of the matrix $P_{\varphi(x)}$ are rational functions of the arguments $f(x_1), \ldots, f(x_n), f(x)$. By Lemma 1 it is enough to verify that one of them is a non-constant function, because then it gives a transcendental value. This is equivalent to the assertion that the product

$$P_{\psi,\zeta_1,\zeta_2,\ldots} = A_{\zeta_{j_1}} B_\psi^{k_1} A_{\zeta_{j_1}}^{-1} A_{\zeta_{j_2}} B_\psi^{k_2} A_{\zeta_{j_2}}^{-1} \ldots A_{\zeta_{j_n}} B_\psi^{k_n} A_{\zeta_{j_n}}^{-1}$$

where $j_r = j_s$ if and only if $x_r = x_s$, is a non-constant function of the variables $\psi, \zeta_1, \zeta_2, \ldots$

(²) I denotes the unity-matrix.

We have

$$P_{\varphi,\varphi,2\varphi,3\varphi,\ldots} = A_\varphi^{j_1} B_\varphi^{k_1} A_\varphi^{j_2-j_1} B_\varphi^{k_2} A_\varphi^{j_3-j_2} \ldots A_\varphi^{j_n-j_{n-1}} B_\varphi^{k_n} A_\varphi^{j_n}$$

where by (4) $j_r - j_{r-1} \neq 0$. Then by Lemma 2 $P_{\varphi,\varphi,2\varphi,3\varphi,\ldots}$ is a non-constant function of φ; which concludes the proof.

5. We introduce the notations

$$o = (0, 0, 0), \quad S_{\varphi\psi} = T_\varphi R_{\varphi\psi}, \quad p_\varphi = A_\varphi(1, 0, 0).$$

For any $p \in \mathcal{E}^3$ we denote by $[p]$ the translation defined by

$$[p](q) = q + p\ (^3).$$

Then $T_\varphi = [p_\varphi]$ and

(5) $$S_{\varphi\psi} = [p_\varphi] R_{\varphi\psi}.$$

(6) p_φ is an eigenvector of $R_{\varphi\psi}$

(because $(1, 0, 0)$ lies on the axis of the rotation B_ψ).

For any motion M of $\mathcal{E}^3$ we consider the *canonical decomposition*

$$M = [p] R$$

where $p = M(o)$ and R is a rotation around o.

(7) M is without fixed points if and only if $p \neq o$ and p is not perpendicular to the axis of R (if $R \neq e$).

We have the following rules for the multiplication of canonical decompositions:

(8) If p is an eigenvector of R, then $([p]R)^k = [kp]R^k$ for $k = 0, \pm 1, \pm 2, \ldots$

(9) $$[p_1]R_1[p_2]R_2 = [p_1 + R_1(p_2)]R_1R_2.$$

LEMMA 4. *For any integers $k_1, \ldots, k_n$ we have the canonical decomposition*

(10) $$S_{\varphi_1\psi}^{k_1} \ldots S_{\varphi_n\psi}^{k_n} = [q_\psi] Q_\psi,$$

where

$$q_\psi = \sum_{i=1}^{n} k_i R_{\varphi_1\psi}^{k_1} \ldots R_{\varphi_{i-1}\psi}^{k_{i-1}}(p_{\varphi_i}), \quad Q_\psi = R_{\varphi_1\psi}^{k_1} \ldots R_{\varphi_n\psi}^{k_n}.$$

Proof. Of course (5) is a canonical decomposition of $S_{\varphi\psi}$. Then applying (6), (8) and (9) we obtain the lemma.

6. Let v_ψ denote the *angular velocity vector* of the rotation Q_ψ at the moment ψ (ψ is the time variable).

(³) $+$ denotes the vector addition.

LEMMA 5. *$q_\psi = v_\psi$ and this vector either is a non-constant function of ψ or is a constant $\neq 0$.*

Proof. v_ψ is the only vector which satisfies the equation

$$\frac{\partial}{\partial\psi} Q_\psi(p) = v_\psi \times Q_\psi(p) \text{ [4]} \quad \text{for each} \quad p \in \mathcal{E}^3 .$$

Now

$$\frac{\partial}{\partial\psi} R^{k_i}_{\varphi_i\psi}(p) = \frac{\partial}{\partial\psi} A_{\varphi_i} B^{k_i}_\psi A^{-1}_{\varphi_i}(p) = A_{\varphi_i} \frac{\partial}{\partial\psi} B^{k_i}_\psi\big(A^{-1}_{\varphi_i}(p)\big)$$
$$= A_{\varphi_i}\big((k_i, 0, 0) \times B^{k_i}_\psi A^{-1}_{\varphi_i}(p)\big) = k_i p_{\varphi_i} \times R^{k_i}_{\varphi_i\psi}(p)$$

and then

$$\frac{\partial}{\partial\psi} Q_\psi(p) = \frac{\partial}{\partial\psi}\big(R^{k_1}_{\varphi_1\psi} \dots R^{k_n}_{\varphi_n\psi}(p)\big)$$
$$= \sum_{i=1}^{n} R^{k_1}_{\varphi_1\psi} \dots R^{k_{i-1}}_{\varphi_{i-1}\psi}\left(\frac{\partial}{\partial\psi} R^{k_i}_{\varphi_i\psi}\right) R^{k_{i+1}}_{\varphi_{i+1}\psi} \dots R^{k_n}_{\varphi_n\psi}(p)$$
$$= \sum_{i=1}^{n} R^{k_1}_{\varphi_1\psi} \dots R^{k_{i-1}}_{\varphi_{i-1}\psi}\big(k_i p_{\varphi_i} \times R^{k_i}_{\varphi_i\psi} \dots R^{k_n}_{\varphi_n\psi}(p)\big)$$
$$= \sum_{i=1}^{n} k_i R^{k_1}_{\varphi_1\psi} \dots R^{k_{i-1}}_{\varphi_{i-1}}(p_{\varphi_i}) \times Q_\psi(p)$$
$$= q_\psi \times Q_\psi(p) .$$

Then the equality of Lemma 5 follows. By Lemma 3 it cannot be $v_\psi \equiv 0$, which concludes the proof.

7. Of course the elements of the matrix Q_ψ are analytic functions of the variable ψ and $Q_0 = I$. Then by Lemma 3 there exists such an open non-empty interval $(0, \alpha)$ that

$$Q_\psi \neq I \quad \text{for} \quad \psi \in (0, \alpha) .$$

Let us denote by l_ψ any eigenvector $\neq o$ of the rotation Q_ψ (for $\psi \in (0, \alpha)$).

LEMMA 6. *There exists such a $\psi \in (0, \alpha)$ that $v_\psi \neq o$ and $\sphericalangle(v_\psi, l_\psi) \neq \frac{1}{2}\pi$.*

Proof. Since the coordinates of v_ψ are analytic functions of ψ, by Lemma 5 there exists such an $\alpha' \in (0, \alpha)$, that

$$v_\psi \neq o \quad \text{for} \quad \psi \in (0, \alpha') .$$

Suppose that

(11) $$\sphericalangle(v_\psi, l_\psi) = \tfrac{1}{2}\pi \quad \text{for each} \quad \psi \in (0, \alpha') .$$

[4] $\times$ denotes the vector product.

Then, since $l_\psi = Q_\psi(l_\psi)$,

(12) $$\sphericalangle\big((v_\psi, Q_\psi(l_\psi)\big) = \tfrac{1}{2}\pi .$$

Now both derivatives $\partial v_u/\partial u$, $\partial Q_u(l_\psi)/\partial u$ exist at the point $u = 0$ (by analyticity of the functions). Then for every positive ε there exists such a $\delta \in (0, \alpha')$ that if

(13) $$0 < \chi < \psi < \delta$$

then

$$\sphericalangle(v_\psi, v_\chi) < \varepsilon \quad \text{and} \quad \sphericalangle\big(Q_\psi(l_\psi), Q_\chi(l_\psi)\big) < \varepsilon .$$

Consequently, by (12), for every positive ε there exists such a $\delta \in (0, \alpha)$ that if (13) holds then

$$v_\psi \times Q_\psi(l_\psi) \neq o, \quad v_\chi \times Q_\chi(l_\psi) \neq 0 ,$$
$$\sphericalangle\big((v_\psi \times Q_\psi(l_\psi), v_\chi \times Q_\chi(l_\psi)\big) < \varepsilon .$$

That is

$$\left[\frac{\partial}{\partial u} Q_u(l_\psi)\right]_{u=\psi} \neq 0 , \quad \left[\frac{\partial}{\partial u} Q_u(l_\psi)\right]_{u=\chi} \neq 0 ,$$
$$\sphericalangle\left(\left[\frac{\partial}{\partial u} Q_u(l_\psi)\right]_{u=\psi}, \left[\frac{\partial}{\partial u} Q_u(l_\psi)\right]_{u=\chi}\right) < \varepsilon .$$

This shows that for some $\varepsilon < \frac{1}{2}\pi$ and $0 < \chi < \psi < \delta(\varepsilon)$ the projection of $[\partial Q_u(l_\psi)/\partial u]_{u=\chi}$ on $[\partial Q_u(l_\psi)/\partial u]_{u=\psi}$ is positive. This implies that

$$Q_0(l_\psi) \neq Q_\psi(l_\psi) \quad \text{i. e.} \quad l_\psi \neq l_\psi .$$

Then (11) is inconsistent, which proves the Lemma.

8. Of course Q_ψ can be represented by an orthogonal matrix

$$Q_\psi = (\alpha_{ij})_{i,j=1,2,3} .$$

Let $(\beta_{ij}) = (\alpha_{ij}) - I$ and let γ_{ij} denote the algebraic complement of β_{ij} in the matrix (β_{ij}).

It is clear that

(14) the vectors $l^{(i)}_\psi = (\gamma_{i1}, \gamma_{i2}, \gamma_{i3})$, $i = 1, 2, 3$, are eigenvectors of Q_ψ and that if $Q_\psi \neq I$, at least one of them is different from o.

LEMMA 7. *The sum*

(15) $$\Sigma_\psi = \sum_{i=1}^{3} (v_\psi \cdot l^{(i)}_\psi)^2 \text{ [5]}$$

either is a non-constant function of ψ or is a constant $\neq 0$.

[5] $\cdot$ denotes the scalar product.

Proof. Taking a number ψ satisfying Lemma 6 we obviously have $\Sigma_\psi \neq 0$.

9. Proof of Theorem 1′. It is enough to show that every product of the form (10), with $k_i \neq 0$, $n \geqslant 1$, and $\psi = \varphi(1)$ is a motion without fixed points. By Lemma 7 the expression Σ_ψ, which is a rational function of the arguments $f(x_1), \ldots, f(x_n)$ and $\operatorname{tg}\frac{1}{2}\psi$ (see (2), (3) and Lemma 5) is a non-constant function of ψ or a constant $\neq 0$. Then in the first case by Lemma 1 and in the other case also,

$$\Sigma_{\varphi(1)} \neq 0 \tag{16}$$

(because $x_i \neq 1$ for $i = 1, \ldots, n$). In the same way, by Lemmas 3 and 4,

$$Q_{\varphi(1)} \neq I \quad \text{and} \quad q_{\varphi(1)} \neq o\,.$$

By Lemma 5, (14) and (16) the vector $q_{\varphi(1)}$ is not orthogonal to the axis of the rotation $Q_{\varphi(1)}$, which proves (see (7)) that the motion $[q_{\varphi(1)}]Q_{\varphi(1)}$ is without fixed points; q. e. d.

10. Now we shall prove theorem 2.

We adopt the notations introduced in section 1. Moreover:

Let $\{\varphi_\mu\}_{\mu\in M}$ $(\varphi_{\mu_1} \neq \varphi_{\mu_2}$ if $\mu_1 \neq \mu_2)$ be a set of free generators of a free group Φ.

Then every $a \in \Phi$ has a unique factorization

$$a = \varphi_{\mu_1}^{k_1} \ldots \varphi_{\mu_n}^{k_n}$$

where $k_i = \pm 1$, and $\varphi_{\mu_i}^{k_i} \neq \varphi_{\mu_{i+1}}^{-k_{i+1}}$.

Lemma. *There exists a decomposition of Φ into $\overline{\overline{N}}$ disjoint sets $\{S_\nu\}_{\nu\in N}$, one of which — say A_{ν_0} — is non-empty and satisfying the system of equalities*

$$\varphi_\mu\Big(\bigcup_{\nu\in P_\mu} S_\nu\Big) = \bigcup_{\nu\in Q_\mu} S_\nu\,, \qquad \mu \in M\,. \tag{17}$$

Proof. We take the notations

$$P_\mu^1 = P_\mu\,, \quad Q_\mu^1 = Q_\mu\,, \quad P_\mu^{-1} = Q_\mu\,, \quad Q_\mu^{-1} = P_\mu\,.$$

Then we must have

$$\varphi_\mu^k\Big(\bigcup_{\nu\in P_\mu^k} S_\nu\Big) = \bigcup_{\nu\in Q_\mu^k} S_\nu \quad \text{for} \quad k = \pm 1\,. \tag{18}$$

We begin by putting e into S_{ν_0}. Now if a has been put into a set S_{ν_1} and $\beta = \varphi_\mu^k a$ where φ_μ^k does not cancel with the first factor of a, then we put β into a set S_{ν_2} such that $(\nu_1 \in P_\mu^k \,\&\, \nu_2 \in Q_\mu^k)$ or $(\nu_1 \in N \setminus P_\mu^k \,\&\, \nu_2 \in N \setminus Q_\mu^k)$.

Consequently (18) holds. Then the whole group Φ is decomposed ([6]), $S_{\nu_0} \neq \emptyset$ and (17) is satisfied; q. e. d.

([6]) The axiom of choice is used here.

Proof of Theorem 2 ([7]). Take the free group Φ without fixed points, with free generators $\{\varphi_\mu\}_{\mu\in M}$ given by Theorem 1, and the factor space $\mathcal{E}^3/\Phi$. For any $E \in \mathcal{E}^3/\Phi$ we take a point $p_E \in E$ ([6]).

Now using the Lemma we put

$$A_\nu = \bigcup_E S_\nu(p_E) = \{p\colon\ p = a(p_E),\ a \in S_\nu,\ E \in \mathcal{E}^3/\Phi\}\,.$$

Then, since Φ is without fixed points the pieces A_ν are disjoint and

$$\varphi_\mu\Big(\bigcup_{\nu\in P_\mu} A_\nu\Big) = \bigcup_E \Big(\varphi_\mu \bigcup_{\nu\in P_\mu} S_\nu(p_E)\Big) = \bigcup_E \bigcup_{\nu\in Q_\mu} S_\nu(p_E) = \bigcup_{\nu\in Q_\mu} A_\nu\,.$$

This proves the first part of the theorem.

It is easy to see that we can suppose that $\overline{\overline{\mathcal{E}^3/\Phi}} = 2^{\aleph_0}$ (removing from Φ some of the generators). Then for different $E \in \mathcal{E}^3/\Phi$ we can take different ν_0 in such a way that, if $\overline{\overline{N}} \leqslant 2^{\aleph_0}$, all the pieces A_ν are non-empty; which completes the proof of Theorem 2.

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([7]) Compare [11] and [4].