On function spaces

by

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In the present paper we are concerned with the study of the properties of topologies in function spaces; in particular, we shall consider the so-called k-topology (¹). The following problems will be treated:

1° conditions regarding the spaces X and Y under which the space Y^X with a k-topology is of the character $\leqslant \mathfrak{m}$ (²) in particular, conditions under which the space Y^X is $\mathfrak{m}$-almost-metrizable (³);

2° conditions regarding X and Y under which there exists a topology for Y^X which induces the continuous convergence of nets of functions (see [3], p. 241);

3° let f be a function defined on the product $X \times T$ of topological spaces X and T with values from a topological space Y and let $\mathfrak{f}$ be the function defined on T whose value at a point t_0 is the function f_{t_0} defined by the equality $f_{t_0}(x) = f(x, t_0)$. Clearly, the continuity of f depends only on topologies in X, Y, T, and the continuity of $\mathfrak{f}$ depends on topologies in T and Y^X. Is there a topology for Y^X such that the continuity of $\mathfrak{f}$ with respect to this topology is equivalent to the continuity of f?

Known results relating to these problems may be listed as follows:

(¹) A basis of k-topology for a function space Y^X (= space of all continuous functions on X to Y) consists of all sets of the form $W(C_1, \ldots, C_k; U_1, \ldots, U_k)$, where C_i are compact subsets of X, U_i are open subsets of Y, and

$$W(C_1, \ldots, C_k; U_1, \ldots, U_k) = \{f \in Y^X: f(C_i) \subset U_i;\ i = 1, \ldots, k\}.$$

Clearly, sets of the form $W(C; U)$ form a subbasis for k-topology.

(²) The character of a point x (in symbols: $\chi(x)$) is the least cardinal $\mathfrak{m}$ for which there is a basis of x of the power $\mathfrak{m}$. The character of a space X (in symbols: $\chi(X)$) is, by definition, the number $\sup_{x \in X} \chi(x)$.

(³) A space X is said to be $\mathfrak{m}$-almost-metrizable if there is a family $P = \{\varrho_\xi\}_{\xi \in \Xi}$ ($\overline{\overline{\Xi}} = \mathfrak{m}$) of pseudometrics on X such that $\bar{A} = \{x \in X: \varrho_\xi(x, A) = 0$ for each ξ in $\Xi\}$ for each $A \subset X$. It may always be assumed that $\max\{\varrho_{\xi'}, \varrho_{\xi''}\} \in P$ for each $\varrho_{\xi'}, \varrho_{\xi''}$ in P. An $\mathfrak{m}$-almost-metrizable space is of the character $\leqslant \mathfrak{m}$ (see [4]).

1a. If Y is a metric space and X is hemicompact (*i. e.*, X has a countable family $\mathfrak{K}$ of compact subsets such that for every compact subset C of X there is a C_0 in $\mathfrak{K}$ with $C \subset C_0$), then Y^X with a k-topology is metrizable (Arens [1]).

1b. If Y^X with a k-topology is first countable then X is hemicompact and Y is first countable (Arens [1]).

2a. If X is a completely regular locally bicompact space, then the k-topology in Y^X induces the continuous convergence of nets of sets (Arens [1]).

2b. If X is completely regular and Y is the unit interval $\langle 0,1\rangle$ then there exists a smallest (⁴) jointly continuous (⁵) topology for Y^X if and only if X is locally compact (Arens [1]).

3a. If X is a completely regular locally compact space, Y and T are arbitrary completely regular spaces, then the continuity of f is equivalent to the continuity of $\mathfrak{f}$ with respect to the k-topology in Y^X for every f which maps $X\times T$ into Y (Fox [2]).

3b. If X and T are first countable, and Y is an arbitrary completely regular space, then the continuity of f is equivalent to the continuity of $\mathfrak{f}$ with respect to the k-topology in Y^X for every f which maps $X\times T$ into Y (Fox [2]).

3c. If X is separable metric, Y is the set of all reals, then there is a topology for Y^X such that the continuity of f is equivalent to the continuity of $\mathfrak{f}$ with respect to that topology for every f which maps $X\times T$ into Y if and only if X is locally compact (Fox [2]).

I. Definitions and notations. All topological spaces under consideration are supposed to be completely regular.

A space X is said to be $\mathfrak{m}$-*compact* if each open covering of X of the power $\leqslant \mathfrak{m}$ contains a finite subcovering. A space is said to be *compact* (= *bicompact*) if it is $\mathfrak{m}$-compact for each $\mathfrak{m}$. A space is $\mathfrak{m}$-*semicompact* if it is the union of $\mathfrak{m}$ compact subsets. A space is *locally* $\mathfrak{m}$-*compact* if each point of the space has a neighbourhood whose closure is $\mathfrak{m}$-compact.

The class of all cofinal subsets of a directed set D will be denoted by $\mathrm{cf}(D)$.

$D_\mathfrak{m}$ denotes the directed set consisting of all finite subset of a set Ξ of the power $\leqslant \mathfrak{m}$. The partial ordering in $D_\mathfrak{m}$ is the set-theoretical inclusion (*i. e.*, $n \leqslant m$ for $n, m \in D_\mathfrak{m}$ if and only if $n \subset m$).

(⁴) By a topology we understand the family of all open sets. A topology $\mathfrak{I}'$ is smaller than $\mathfrak{I}$ if $\mathfrak{I}' \subset \mathfrak{I}$.

(⁵) A topology $\mathfrak{I}$ for Y^X is said to be jointly continuous provided that the mapping F of $Y^X\times X$ which carries $\langle f, x\rangle$ into $f(x)$ is continuous.

A net defined on a directed set D will be named D-*net*. A $D_\mathfrak{m}$-net will be called simply $\mathfrak{m}$-net. A D-net which assigns to an element $n \in D$ an element x_n will be denoted by $\{x_n, m \in D\}$. A net $\{x_n, n \in D\}$ is said to be *compact* if there is an $n_0 \in D$ such that the set $\{x_n\}_{n\geqslant n_0}$ has a compact closure.

Let $(\mathbf{W}_D)$ be the following property of topological spaces:

$(\mathbf{W}_D)$ *Each convergent D-net of elements of the space is compact.*

The property $(\mathbf{W}_{D_\mathfrak{m}})$ will be denoted simply by $(\mathbf{W}_\mathfrak{m})$. A space X has the property $(\mathbf{W})$ if it has the property $(\mathbf{W}_D)$ for every directed set D.

A class $\mathfrak{K}$ of compact subsets of a space X will be called a *basis for compact sets* if for each compact set $C \subset X$ there is a $C_0 \in \mathfrak{K}$ with $C \subset C_0$. (Clearly, hemicompact spaces can be defined as spaces having an enumerable basis for compact sets.)

An open basis $\mathfrak{B}$ of a space X is said to be an $\mathfrak{m}$-basis if it is the union of $\mathfrak{m}$ locally finite systems (see [5]). (A system of subsets of a space is said to be *locally finite* if each point of the space has a neighbourhood which intersects a finite number of members of the system.)

If $\mathfrak{f}$ is a function defined on a set T whose values are functions, then $[\mathfrak{f}(t)](x)$ denotes the value of the function $\mathfrak{f}(t)$ at the point x.

If X, T, Y are topological spaces and $\mathfrak{I}$ is a topology for Y^X, then we write $[X, Y, T, \mathfrak{I}]$ if, for each function f which maps $X\times T$ into Y, the continuity of f is equivalent to the continuity of the function $\mathfrak{f} \in (Y^X)^T$, where $\mathfrak{f}$ is defined by the equality $[\mathfrak{f}(t)](x) = f(x, t)$.

The k-topology for Y^X will be denoted by $k(Y^X)$.

$\boldsymbol{I}$ denotes the unit interval $\langle 0, 1\rangle$.

II. Lemmas and auxiliary theorems.

LEMMA 1. *If $\chi(x) \leqslant \mathfrak{m}$ then there exists a decreasing $\mathfrak{m}$-net $\{U_n, n \in D_\mathfrak{m}\}$ (i. e., $U_n \supset U_{n'}$ for $n \leqslant n'$) of neighbourhoods of x such that the family $\{U_n\}_{n\in D_\mathfrak{m}}$ is a basis of x (such a basis will be called a special basis of x).*

Proof. Let $\{U_\xi\}_{\xi\in\Xi}$ be a basis of x of the power $\leqslant\mathfrak{m}$. Let us set $U_n = U_{\xi_1} \cap \ldots \cap U_{\xi_k}$ for $n = \{\xi_1, \ldots, \xi_k\} \in D_\mathfrak{m}$.

LEMMA 2. *A space X is $\mathfrak{m}$-compact if and only if every $\mathfrak{m}$-net of elements of X has a cluster point.*

Proof. If X is $\mathfrak{m}$-compact and $\{x_n, n \in D_\mathfrak{m}\}$ is an $\mathfrak{m}$-net of elements of X, then there is a point x which belongs to the closures of all sets $X_n = \{x_m\}_{m\geqslant n}$. Since each neighbourhood of x intersects X_n for every $n \in D_\mathfrak{m}$, x is cluster point of the net $\{x_n, n \in D_\mathfrak{m}\}$. Suppose X is such that each $\mathfrak{m}$-net has a cluster point and let $\{A_\xi\}_{\xi\in\Xi}$ $(\Xi = \mathfrak{m})$ be a centred family of closed subsets of X. Let us set $A_n = A_{\xi_1} \cap \ldots \cap A_{\xi_k}$ for $n = \{\xi_1, \ldots, \xi_k\}$ and let x_n be an arbitrary point of A_n. The net $\{x_n, n \in D\}$

has a cluster point and it can easily be shown that this point is a common point of all A_ξ.

LEMMA 3. *If* $\chi(x) \leqslant \mathfrak{m}$ *and* $x \in \bar{A}$, *then there exists an* $\mathfrak{m}$-*net of elements of* A *which converges to* x.

Proof. Let $\{U_n,\ n \in D_\mathfrak{m}\}$ be a special basis of x and let x_n be an arbitrary point of $A \cap U_n$. Clearly, the $\mathfrak{m}$-net $\{x_n,\ n \in D_\mathfrak{m}\}$ is convergent to x.

Thus, if $\chi(X) \leqslant \mathfrak{m}$, then the topology of X can be described in terms of convergence of $\mathfrak{m}$-nets.

LEMMA 4. *If* X *is* $\mathfrak{m}$-*compact and* $\mathfrak{m}$-*semicompact, then* X *is compact.*

Proof. Since X is $\mathfrak{m}$-semicompact, each open covering of X contains a subcovering of the power $\leqslant \mathfrak{m}$, thus, by $\mathfrak{m}$-compactness, it contains a finite subcovering.

THEOREM II. 1. *If* $\chi(X) \leqslant \mathfrak{m}$ *and* X *has the property* $(\mathbf{W}_\mathfrak{m})$, *then* X *is locally* $\mathfrak{m}$-*compact.*

Proof. Suppose that no neighbourhood of a point $x_0 \in X$ has the $\mathfrak{m}$-compact closure and let $\{U_n,\ n \in D_\mathfrak{m}\}$ be a special basis of x_0. For every $n \in D_\mathfrak{m}$ there exists an $\mathfrak{m}$-net $\{x_m^{(n)},\ m \in D_\mathfrak{m}\}$ of elements of $\bar{U}_n$ which has no cluster point. Consider the net $\{x_{\langle n,m\rangle},\ \langle n, m\rangle \in D_\mathfrak{m} \times D_\mathfrak{m}\}$, where $x_{\langle n,m\rangle} = x_m^{(n)}$. This net is an $\mathfrak{m}$-net; indeed, the set $D_\mathfrak{m} \times D_\mathfrak{m}$ is similar to $D_\mathfrak{m}$. On the other hand, this net is convergent to x_0, but it is not compact.

COROLLARY 1. *If* X *has the property* $(\mathbf{W}_\mathfrak{m})$ *for each* $\mathfrak{m}$, *then* X *is locally compact.*

Clearly, a locally compact space has the property $(\mathbf{W})$. Thus we obtain:

COROLLARY 2. *A space* X *has the property* $(\mathbf{W})$ *if and only if* X *has the property* $(\mathbf{W}_\mathfrak{m})$ *for each* $\mathfrak{m}$.

COROLLARY 3. *A space* X *is locally compact if and only if* X *has the property* $(\mathbf{W})$.

By lemma 4 and theorem I. 1., we obtain:

THEOREM II. 2. *If* $\chi(X) \leqslant \mathfrak{m}$, X *has the property* $(\mathbf{W}_\mathfrak{m})$ *and is* $\mathfrak{m}$-*semicompact, then* X *is locally compact.*

LEMMA 5. *If* X *is compact and* $\chi(X) \leqslant \mathfrak{m}$ *and* $\{x_n,\ n \in D_\mathfrak{m}\}$ *is an* $\mathfrak{m}$-*net of points of* X, *then for each* $n \in D_\mathfrak{m}$ *there is a* $k_n \geqslant n$ *such that the* $\mathfrak{m}$-*net* $(x_{k_n},\ n \in D_m)$ *converges to some point of* X.

Proof. By lemma 2, the net $\{x_n,\ n \in D_\mathfrak{m}\}$ has a cluster point x_0. Let $\{U_n,\ n \in D_m\}$ be a special basis of x_0. Then for each $n \in D_\mathfrak{m}$ there is a $k_n \geqslant n$ with $x_{k_n} \in U_n$. Clearly, the net $\{x_{k_n},\ n \in D_\mathfrak{m}\}$ converges to x_0.

LEMMA 6. *If* $\chi(X) \leqslant \mathfrak{m}$ *and* f *is a function on* X *to* Y, *then* f *is continuous if and only if for each* $\mathfrak{m}$-*net* $\{x_n,\ n \in D_\mathfrak{m}\}$ *which converges to some point* $x_0 \in X$, *the net* $\{f(x_n),\ n \in D_\mathfrak{m}\}$ *converges to* $f(x_0)$.

Proof. Suppose that f is not continuous. Then there is an $A \subset X$ such that $f(\bar{A}) \not\subset \overline{f(A)}$; *i. e.*, there is a point $x_0 \in \bar{A}$ with $f(x_0) \notin \overline{f(A)}$. By lemma 3, there is an $\mathfrak{m}$-net $\{x_n,\ n \in D_\mathfrak{m}\}$ of elements of A which converges to x_0; on the other hand, there is a neighbourhood V of $f(x_0)$ such that $V \cap f(A) = 0$. Since $f(x_n) \notin U$ for each n in $D_\mathfrak{m}$, $\{f(x_n),\ n \in D_\mathfrak{m}\}$ does not converge to $f(x_0)$. Conversely, if $\{x_n,\ n \in D_\mathfrak{m}\}$ converges to x_0 and $\{f(x_n),\ n \in D_\mathfrak{m}\}$ does not converge to $f(x_0)$, then there is a neighbourhood V of $f(x_0)$ such that $\{n \in D_\mathfrak{m}\colon f(x_n) \notin V\} \in \mathrm{cf}(D_\mathfrak{m})$. Clearly, $f(U) \not\subset V$ for each neighbourhood U of x, whence f is not continuous.

III. Character and almost-metrizability of Y^X. (In this section Y^X is supposed to carry a k-topology.)

THEOREM III. 1. *If* Y *has an* $\mathfrak{m}$-*basis and* X *has a basis* $\mathfrak{K}$ *for compact sets of the power* $\leqslant \mathfrak{m}$, *then* Y^X *is of the character* $\leqslant \mathfrak{m}$.

Proof. Let $\mathfrak{B}$ be an $\mathfrak{m}$-basis of Y and let $\mathfrak{B} = \bigcup_{\xi \in \Xi} \mathfrak{B}_\xi (\overline{\overline{\Xi}} = \mathfrak{m})$, where $\mathfrak{B}_\xi$ are locally finite systems. Let $f_0 \in Y^X$. For every $C \in \mathfrak{K}$ we denote by $\mathfrak{B}_{\xi C}$ the family of all $U \in \mathfrak{B}_\xi$ for which $f_0(C) \cap U \neq 0$. Since $f_0(C)$ is compact and $\mathfrak{B}_\xi$ is locally finite, $\mathfrak{B}_{\xi C}$ is finite. Let $\mathfrak{B}^* = \bigcup_{\xi \in \Xi, C \in \mathfrak{K}} \mathfrak{B}_{\xi C}$ and let $\mathfrak{B}'$ be the family of all finite unions of members of $\mathfrak{B}^*$. Clearly $\overline{\overline{\mathfrak{B}'}} \leqslant \mathfrak{m}$. Let $\mathfrak{K}' = \{f^{-1}(U) \cap C\}_{U \in \mathfrak{B}', C \in \mathfrak{K}'}$. Clearly each member of $\mathfrak{K}'$ is compact and $\mathfrak{K}' \leqslant \mathfrak{m}$. We shall show that the family of all $f_0 \in W(C, U)$, where $C \in \mathfrak{K}'$ and $U \in \mathfrak{B}'$ is a subbasis of f_0. Let $f_0 \in W(C_0, U_0)$, where C_0 is an arbitrary compact subset of X and U_0 is an arbitrary open subset of Y. There exists $C' \in \mathfrak{K}$ such that $C_0 \subset C'$. For each $x \in C_0$ there is a $\xi_x \in \Xi$ and $U_x \in \mathfrak{B}_{\xi_x}$ such that $f_0(x) \in U_x \subset U_0$. Clearly $f_0(C') \cap U_x \neq 0$, whence $U_x \in \mathfrak{B}'$. Since $x \in f_0^{-1}(U_x)$, $C_0 \subset \bigcup_{x \in C_0} f_0^{-1}(U_x)$, thus there is a finite system $U_{x_1}, \ldots, U_{x_j}$, such that $C_0 \subset f_0^{-1}(U_{x_1}) \cup \ldots \cup f_0^{-1}(U_{x_j})$. Let $U = U_{x_1} \cup \ldots \cup U_{x_j}$. We have $U \in \mathfrak{B}'$ and $f_0(C_0) \subset U$. Repeating the above reasoning, we infer that there is an $U' \in \mathfrak{B}'$ such that $f_0(C_0) \subset U' \subset \bar{U}' \subset U$. Let $C = f_0^{-1}(U') \cap C'$. Clearly $C \in \mathfrak{K}'$. Since $C_0 \subset f_0^{-1}(U') \cap C'$, $C_0 \subset C$. On the other hand, $f_0(C) = f_0(\overline{f_0^{-1}(U') \cap C}) \subset \overline{f_0(f_0^{-1}(U') \cap C)} \subset \bar{U}' \subset U$, whence $f_0 \in W(C, U)$. But $C_0 \subset C$ and $U \subset U_0$, and thus $W(C, U) \subset W(C_0, U_0)$.

THEOREM III. 2. *If* X *has a basis for compact sets of the power* $\leqslant \mathfrak{m}$ *and* Y *is* $\mathfrak{m}$-*almost-metrizable, then* Y^X *is* $\mathfrak{m}$-*almost-metrizable.*

Proof. Let $P = \{\varrho_\xi\}_{\xi \in \Xi}$ $(\overline{\overline{\Xi}} = \mathfrak{m})$ be a family of pseudometrics of Y and let $\mathfrak{K}$ be a basis for compact sets in X of the power $\leqslant \mathfrak{m}$. Let us set

$P^* = \{\varrho^*_{\xi C}\}_{\xi\in\Xi, C\in\mathfrak{K}}$ where

$$\varrho^*_{\xi C}(f, g) = \sup_{x\in C} \varrho_\xi\big(f(x), g(x)\big).$$

We shall show that P^* is a family of pseudometrics for Y^X. Let $V = \{f;\ \varrho_{\xi C}(f_0, f) < \varepsilon\}$. Since C is compact, there is a finite system $C_1, \ldots, C_k$ of compact sets such that $C = C_1 \cup \ldots \cup C_k$ and $\varrho_\xi\big(f_0(x), f_0(x')\big) < \frac{1}{3}\varepsilon$ for x, x' in C_i. Let x_i be an arbitrary point of C_i and let $U_i = \{y \in Y:\ \varrho_{\xi_i}\big(f_0(x_i), y\big) < \frac{1}{3}\varepsilon\}$. If $x \in C_i$, then $\varrho_\xi\big(f_0(x_i), f_0(x)\big) < \frac{1}{3}\varepsilon$ and $f_0(x) \in U_i$, whence $f_0(C_i) \subset U_i$, and thus $f_0 \in W(C_1, \ldots, C_k;\ U_1, \ldots, U_k)$. On the other hand, if $f \in W(C_1, \ldots, C_k;\ U_1, \ldots, U_k)$ and $x \in C_i$, then $f(x) \in U_i$ and $\varrho_\xi\big(f_0(x), f(x)\big) \leqslant \varrho_\xi\big(f_0(x), f_0(x_0)\big) + \varrho_\xi\big(f_0(x_0), f(x)\big) < \frac{2}{3}\varepsilon$, whence $\sup_{x\in C} \varrho_\xi\big(f_0(x), f(x)\big) < \varepsilon$ and $f \in V$. Finally, $f_0 \in W(C_1, \ldots, C_k;\ U_1, \ldots, U_k) \subset V$. Conversely, let $W(C_1, \ldots, C_k;\ U_1, \ldots, U_k)$ be an arbitrary neighbourhood of f_0. Since $f_0(C_i) \subset U_i$ and $f_0(C_i)$ is compact, there is a pseudometric ϱ_{ξ_i} in P such that $\varrho_{\xi_i}\big(f_0(C_i), Y \setminus U_i\big) = \varepsilon_i > 0$. Let $\varrho_\xi = \max\{\varrho_{\xi_1}, \ldots, \varrho_{\xi_k}\}$ and $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_k\}$. Moreover, there is a C in $\mathfrak{K}$ such that $C_1 \cup \ldots \cup C_k \subset C$. Let $V = \{f:\ \varrho_{\xi C}(f_0, f) < \varepsilon\}$. If $f \in V$ and $x \in C_i$, then $\varrho_{\xi_i}\big(f_0(C_i), f(x)\big) \leqslant \varrho_\xi\big(f_0(C_i), f(x)\big) \leqslant \varrho_\xi\big(f_0(x), f(x)\big) < \varepsilon$, whence $f(x) \in U_i$ and $f(C_i) \subset U_i$; thus $f \in W(C_1, \ldots, C_k;\ U_1, \ldots, U_k)$ and finally $V \subset W(C_1, \ldots, C_k;\ U_1, \ldots, U_k)$.

Theorem III. 3. *If I^X is of the character $\leqslant \mathfrak{m}$, then X has a basis for compact sets of the power $\leqslant \mathfrak{m}$.*

Proof. Let f_0 be a function which is identically equal to 0 and let $U_0 = \{y \in I:\ 0 \leqslant y < 1\}$. Since $\chi(f_0) \leqslant \mathfrak{m}$, there is a family $\mathfrak{K}_0$ of the power $\leqslant \mathfrak{m}$ consisting of compact subsets of X with the following property: for each compact set $C \subset X$ there are $C_1, \ldots, C_k$ in $\mathfrak{K}_0$ and open sets $U_1, \ldots, U_k \subset I$ such that $f_0 \in W(C_1, \ldots, C_k;\ U_1, \ldots, U_k) \subset W(C;\ U_0)$. But $f_0 \in W(C_1, \ldots, C_k;\ U_1, \ldots, U_k) \subset W(C;\ U_0)$ implies $C \subset C_1 \cup \ldots \cup C_k$. In fact, if there is a point $x_0 \in C \setminus (C_1 \cup \ldots \cup C_k)$, then there is a function f with $f(x_0) = 1$ and $f(x) = 0$ for $x \in C_1 \cup \ldots \cup C_k$. Clearly $f \in W(C_1, \ldots, C_k;\ U_1, \ldots, U_k)$ and $f \notin W(C;\ U_0)$, and this contradics $W(C_1, \ldots, C_k;\ U_1, \ldots, U_k) \subset W(C;\ U_0)$. It follows that the family of all finite unions of members of $\mathfrak{K}_0$ is a basis for compact sets.

IV. Continuous convergence of nets of functions. We recall that a net $\{f_n,\ n \in D\}$ $(f_n \in Y^X)$ is said to be continuously convergent to $f \in Y^X$ provided that for each net $\{x_n,\ n \in D\}$ $(x_n \in X)$ which converges to a point $x \in X$, the net $\{f_n(x_n),\ n \in D\}$ converges to $f(x)$. It may easily be shown that a topology $\mathfrak{J}$ for Y^X is jointly continuous if and only if each net in Y^X which converges to some $f \in Y^X$ with respect to the topology $\mathfrak{J}$ converges to f continuously. Arens [1] has shown that there is a smallest jointly continuous topology for I^X if and only if X is locally compact.

We shall say that a topology $\mathfrak{J}$ for Y^X is a topology of continuous convergence if each net in Y^X converges to $f \in Y^X$ with respect to the topology $\mathfrak{J}$ if and only if it converges to f continuously. Then from Arens' result it follows that there exists a topology of continuous convergence for I^X if and only if X is locally compact. On the other hand, Arens also has shown that if X is locally compact, then the k-topology for Y^X is a topology of continuous convergence for arbitrary Y; hence, in this case, the k-topology induces the continuous convergence of all nets in Y^X. The following theorems explain when k-topology induces the continuous convergence of nets of some special sort.

Theorem IV. 1. *If X has the property $(\mathbf{W}_\mathfrak{m})$ and $\chi(X) \leqslant \mathfrak{m}$, then the k-topology induces the continuous convergence of $\mathfrak{m}$-nets in Y^X for arbitrary Y.*

Proof. Suppose that an $\mathfrak{m}$-net $\{f_n,\ n \in D_\mathfrak{m}\}$ converges to f_0 continuously and does not converge to f_0 with respect to k-topology. Then there is a neighbourhood $W(C,\ U)$ of f_0 with the following property: for each $n \in D_\mathfrak{m}$ there is a $k_n \in D_\mathfrak{m}$ such that $k_n \geqslant n$ and $f_{k_n} \notin W(C, U)$, i. e., there is a point $x_n \in C$ such that $f_{k_n}(x_n) \notin U$. But $\chi(C) \leqslant \mathfrak{m}$ and C is compact, whence by lemma 5 there is a net $\{l_n,\ n \in D_\mathfrak{m}\}$ such that $l_n \geqslant n$ for each n in $D_\mathfrak{m}$ and the net $\{x_{l_n},\ n \in D_\mathfrak{m}\}$ converges to some point $x_0 \in C$. Clearly, the net $\{f_{k_{l_n}},\ n \in D_\mathfrak{m}\}$ converges to f_0 continuously, whence $\{f_{k_{l_n}}(x_{l_n}),\ n \in D_\mathfrak{m}\}$ converges to $f_0(x_0)$, but $f_{k_{l_n}}(x_{l_n}) \notin U$ for each $n \in D_\mathfrak{m}$, which it leads to a contradiction.

Conversely, suppose that $\{f_n,\ n \in D_\mathfrak{m}\}$ does not converge to f_0 continuously. Then there is an $\mathfrak{m}$-net $\{x_n,\ n \in D_\mathfrak{m}\}$ which converges to some point $x_0 \in X$ such that $\{f_n(x_n),\ n \in D_\mathfrak{m}\}$ does not converge to $f_0(x_0)$. It follows that there is a neighbourhood U of $f_0(x_0)$ such that for each $m \in D_\mathfrak{m}$ there is an $n \geqslant m$ with $f_n(x_n) \notin U$. But X has the property $(\mathbf{W}_\mathfrak{m})$, whence there is an $m_0 \in D_\mathfrak{m}$ such that the set $C = \overline{\{x_n\}}_{n\geqslant m_0}$ is compact. We see that for each $m \in D_\mathfrak{m}$ there is an $n \geqslant m$ such that $x_n \in C$ and $f_n(x_n) \notin U$, i. e., $f_n(C) \not\subset U$; thus $\{f_n,\ n \in D_\mathfrak{m}\}$ does not converge to f_0 with respect to k-topology.

Theorem IV. 2. *If X has a basis for compact sets of the power $\leqslant \mathfrak{m}$ and the k-topology for I^X induces the continuous convergence of $\mathfrak{m}$-nets, then X has the property $(\mathbf{W}_\mathfrak{m})$.*

Proof. Let $\mathfrak{K} = \{C_\xi\}_{\xi\in\Xi}(\overline{\overline{\Xi}} = \mathfrak{m})$ be a basis for compact sets. Let us set $C_n = C_{\xi_1} \cup \ldots \cup C_{\xi_k}$ for $n \in D_\mathfrak{m}$, $n = \{\xi_1, \ldots, \xi_k\}$. If X does not have the property $(\mathbf{W}_\mathfrak{m})$, then there is an $\mathfrak{m}$-net $\{x_n,\ n \in D_\mathfrak{m}\}$ which converges to some point $x_0 \in X$ such that for each $n \in D_m$ there is a $k_n \geqslant n$ with $x_{k_n} \notin C_n$. Then there is a continuous function f_n on X to I such that $f_n(x_{k_n}) = 1$ and $f_n(x) = 0$ for x in C_n. Let f_0 be a function which is identically equal to 0 on X. Since $\{x_{k_n}, n \in D_m\}$ converges to x_0 and $f_n(x_{k_n}) = 1$ for each n in D_m, $\{f_n, n \in D_m\}$ does not converge to f_0 continuously. On the other hand, if $W(C, U)$ is a neighbourhood of f_0, then there is an n_0 in D_m such that $C \subset C_{n_0}$. Since $C_{n_0} \subset C_n$ for $n \geqslant n_0$, $f_n \in W(C, U)$ for $n \geqslant n_0$; thus $\{f_n, n \in D_m\}$ converges to f_0 with respect to k-topology.

COROLLARY 1. *If $\chi(X) \leqslant \mathfrak{m}$, I^X with a k-topology is of the character $\leqslant \mathfrak{m}$ and the k-topology for I^X induces the continuous convergence of $\mathfrak{m}$-nets, then X is locally compact.*

COROLLARY 2. *Under the assumptions of the previous collorary, the k-topology for I^X is a topology of continuous convergence.*

V. Continuity of functions with values from a function space.

THEOREM V. 1. *If $\chi(X) \leqslant \mathfrak{m}$, $\chi(T) \leqslant \mathfrak{m}$ and X has the property $(\mathbf{W}_\mathfrak{m})$, then $[X, Y, T, k(Y^X)]$ for an arbitrary completely regular space Y.*

Proof. Suppose that f maps $X \times T$ into Y and let $\mathfrak{f}$ be defined by the equality $[\mathfrak{f}(t)](x) = f(x, t)$. We shall show that the continuity of f is equivalent to that of $\mathfrak{f}$. Suppose that f is continuous. Clearly, the values of $\mathfrak{f}$ are continuous functions of the variable x; thus $\mathfrak{f}$ maps T into Y^X. Let $\{t_n, n \in D_m\}$ be an $\mathfrak{m}$-net of elements of T which converges to some point $t_0 \in T$. Let $g_n = \mathfrak{f}(t_n)$, $g_0 = \mathfrak{f}(t_0)$. If $\{x_n, n \in D_m\}$ is an $\mathfrak{m}$-net of elements of X which converges to some point $x_0 \in X$, then, by the continuity of f, $\{g_n(x_n), n \in D_m\}$ converges to $f(x_0, t_0) = g_0(x_0)$, whence $\{g_n, n \in D_m\}$ converges to g_0 continuously; thus, by Theorem IV. 1, $\{g_n, n \in D_m\}$ converges to g_0 with respect to k-topology, *i. e.*, $\{\mathfrak{f}(t_n), n \in D_m\}$ converges to $\mathfrak{f}(t_0)$. If follows, by lemma 6, that $\mathfrak{f}$ is continuous.

Conversely, suppose that $\mathfrak{f}$ is continuous, and let $\{\langle x_n, t_n\rangle, n \in D_m\}$ be an arbitrary m-net of elements of $X \times T$ which converges to some point $\langle x_0, t_0\rangle \in X \times T$. Since $\{t_n, n \in D_m\}$ converges to t_0, and $\mathfrak{f}$ is continuous, $\{\mathfrak{f}(t_n), n \in D_m\}$ converges to $\mathfrak{f}(t_0)$, *i. e.*, the net $\{g_n, n \in D_m\}$, where $g_n = \mathfrak{f}(t_n)$, converges to $g_0 = \mathfrak{f}(t_0)$ with respect to k-topology. By Theorem IV. 1, $\{g_n, n \in D_m\}$ converges to g_0 continuously. Since $\{x_n, n \in D_m\}$, converges to x_0, $\{g_n(x_n), n \in D_m\}$ converges to $g_0(x_0) = f(x_0, t_0)$. But $g_n(x_n) = f(x_n, t_n)$, and thus $\{f(x_n, t_n), n \in D_m\}$ converges to $f(x_0, t_0)$. Since $\chi(X \times T) \leqslant m$, f is continuous by lemma 6.

THEOREM V. 2. *If X has a basis for compact sets of the power $\leqslant \mathfrak{m}$, $\chi(X) \leqslant \mathfrak{m}$ and $[X, I, T, k(Y^X)]$ for each T with $\chi(T) \leqslant \mathfrak{m}$, then X has the property $(\mathbf{W}_\mathfrak{m})$.*

Proof. Let T be the space consisting of all elements of D_m and of an "ideal" element a. Points of D_m are isolated in T, neighbourhoods of a are of the form $\{a\} \cup \{n \in D_m: n \geqslant m\}$ for some m in D_m. Clearly, $\chi(T) \leqslant \mathfrak{m}$.

Let $\mathfrak{K} = \{C_\xi\}_{\xi \in \Xi}$, $\overline{\Xi} = \mathfrak{m}$, be a basis for compact sets in X, and let $C_n = C_{\xi_1} \cup \ldots \cup C_{\xi_k}$ for $n \in D_m$, $n = \{\xi_1, \ldots, \xi_k\}$. If X does not have the property $(\mathbf{W}_\mathfrak{m})$, then there is an $\mathfrak{m}$-net $\{x_n, n \in D_m\}$ which converges to some point $x_0 \in X$ and is such that for each $n \in D_m$ there is a $k_n \geqslant n$ with $x_{k_n} \notin C_n$. Let g_n be a continuous function on X to I such that $g_n(x_{k_n}) = 1$ and $g_n(x) = 0$ for $x \in C_n$. Let f be a function defined on $X \times T$ by the equalities $f(x, n) = g_n(x)$, $f(x, a) = 0$ and let $\mathfrak{f}$ be the function on T to I^X defined by the equality $[\mathfrak{f}(t)](x) = f(x, t)$. Since $\{\langle x_{k_n}, n\rangle, n \in D_m\}$ converges to $\langle x_0, a\rangle$, $f(x_{k_n}, n) = 1$, $f(x_0, t_0) = 0$, f is not continuous.

On the other hand, let $\{t_n, n \in D_m\}$ be an $\mathfrak{m}$-net in T which converges to a nad let $W(C, U)$ be a neighbourhood of $f(a)$. There is an $m_0 \in D_m$ with $C_{m_0} \supset C$ and there is an $n_0 \in D_m$ such that $t_n \geqslant m_0$ for $n \geqslant n_0$. It can be assumed that $t_n \neq a$. Since $C_{t_n} \supset C$ for $n \geqslant n_0$, $g_{t_n}(x) = 0$ for $x \in C$, whence $f(t_n) = g_{t_n} \in W(C; U)$ for $n \geqslant n_0$ and it means that $\{\mathfrak{f}(t_n), n \in D_m\}$ converges to $\mathfrak{f}(a)$; thus $\mathfrak{f}$ is continuous.

COROLLARY. *If $[X, \mathbf{I}, T, k(\mathbf{I}^X)]$ for an arbitrary topological space T, then X is locally compact.*

A stronger result is given by the following

THEOREM V. 3. *If there is a topology $\mathfrak{I}$ for $\mathbf{I}^X$ such that $[X, \mathbf{I}, T, \mathfrak{I}]$ for an arbitrary topological space T, then X is locally compact.*

Proof. It follows from the proof of the preceding theorem that if $\chi(X) \leqslant \mathfrak{m}$ and $[X, \mathbf{I}, T, \mathfrak{I}]$ for an arbitrary topological space T with $\chi(T) \leqslant \mathfrak{m}$, then $\mathfrak{I}$ induces the continuous convergence of $\mathfrak{m}$-nets in $\mathbf{I}^X$, whence, if $[X, \mathbf{I}, T, \mathfrak{I}]$ for an arbitrary topological space T, then $\mathfrak{I}$ is a topology of continuous convergence (6). But (see remarks in IV), the space I^X has a topology of uniform convergence if and only if X is locally compact.

(6) If a topology $\mathfrak{I}$ for Y^X induces the continuous convergence of all $\mathfrak{m}$-sets (for arbitrary cardinal $\mathfrak{m}$), then $\mathfrak{I}$ is a topology of continuous convergence. Indeed, in this case, the mapping $F(f, x) = f(x)$ is continuous, *i. e.*, $\mathfrak{I}$ is jointly continuous. If $\mathfrak{I}'$ is another jointly continuous topology smaller than $\mathfrak{I}$, then each $\mathfrak{m}$-net in Y^X convergent with respect to $\mathfrak{I}'$ is convergent continuously, *i. e.*, it is convergent with respect to I. On the other hand, since $\mathfrak{I}'$ is smaller than $\mathfrak{I}$, each $\mathfrak{m}$-net which is convergent with respect to $\mathfrak{I}$ is also convergent with respect to $\mathfrak{I}'$. Thus $\mathfrak{I}$ and $\mathfrak{I}'$ agree on $\mathfrak{m}$-nets, whence, by lemma 3, $\mathfrak{I} = \mathfrak{I}'$. We see that $\mathfrak{I}$ is the smallest jointly continuous topology, *i. e.*, it is a topology of continuous convergence.

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On free groups of motions and decompositions of the Euclidean space

by

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The purpose of this paper is to prove two theorems given in section 1 (1). These theorems are solutions of some problems proposed to the authors by T. Dekker.

1. A group Φ of 1-1 transformations of a set E onto itself is called *without fixed points* if for every $p \in E$ and every $\varphi \in \Phi \setminus (e)$ we have $\varphi(p) \neq p$.

The *rank* of a free group is the potency of a set of free generators of this group.

The sense-preserving isometries of the Euclidean space $\mathcal{E}^3$, *i. e.*, the superpositions of rotations and translations are called *motions*.

THEOREM 1. *There exists a free group of the rank $2^{\aleph_0}$ of motions of $\mathcal{E}^3$ without fixed points.*

The proof of this theorem follows in sections 3-9 (it is an effective construction, which does not use the axiom of choice). The relations of Theorem 1 with known results are given in section 2.

An application of this theorem (the construction of a set $E \subset \mathcal{E}^3$ which is congruent to $(E \setminus A) \cup B$ for any at most denumerable sets $A, B \subset \mathcal{E}^3$) is given in [7]. Another application, of a well known character (compare [2], [3], [4], [8]), is the theorem 2 of this paper. It states that, for any system of congruence relations, $\mathcal{E}^3$ can be divided into disjoint sets satisfying that system. For an exact description of this theorem we take the following notations:

M and N are non-empty sets.

$\{P_\mu\}_{\mu \in M}$ and $\{Q_\mu\}_{\mu \in M}$ are arbitrary systems of subsets of N, all different from $\emptyset$ and N. (We do not suppose that $\mu_1 \neq \mu_2$ implies $P_{\mu_1} \neq P_{\mu_2}$ or $Q_{\mu_1} \neq Q_{\mu_2}$.)

$\simeq$ denotes congruence of point sets realizable by a motion.

(1) They were announced in [6] by the first author, but the original proof was faulty, and the proof presented here was worked out by both authors.