empty and connected for every $x \in S_n$, then there exists a point $x_0 \in S_n$ such that $F(x_0)$ contains all straight lines in $E_n$ passing through $x_0$ and tangent to $S_n$.

Remark. The questions whether theorems 6 and 7 remain true for $m = 0$ is open.

References


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On function spaces

by

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In the present paper we are concerned with the study of the properties of topologies in function spaces; in particular, we shall consider the so-called $k$-topology (1). The following problems will be treated:

1° conditions regarding the spaces $X$ and $Y$ under which the space $Y^X$ with a $k$-topology is of the character $\leq m$ (2) in particular, conditions under which the space $Y^X$ is $m$-almost-metrizable (3);

2° conditions regarding $X$ and $Y$ under which there exists a topology for $Y^X$ which induces the continuous convergence of nets of functions (see [3], p. 241);

3° let $f$ be a function defined on the product $X \times T$ of topological spaces $X$ and $T$ with values from a topological space $Y$ and let $f$ be the function defined on $T$ whose value at a point $t$ is the function $f_t$ defined by the equality $f_t(x) = f(x, t)$. Clearly, the continuity of $f$ depends only on topologies in $X$, $Y$, $T$, and the continuity of $f$ depends on topologies in $T$ and $Y^X$. Is there a topology for $Y^X$ such that the continuity of $f$ with respect to this topology is equivalent to the continuity of $f$?

Known results relating to these problems may be listed as follows:

(1) A basis of $k$-topology for a function space $Y^X$ (= space of all continuous functions on $X$ to $Y$) consists of all sets of the form $W(C_1, \ldots, C_k; U_1, \ldots, U_k)$, where $C_i$ are compact subsets of $X$, $U_i$ are open subsets of $Y$, and

$$W(C_1, \ldots, C_k; U_1, \ldots, U_k) = \{f: Y^X, f(C_i) \subseteq U_i; i = 1, \ldots, k\}.$$ 

Clearly, sets of the form $W(C_i; U)$ form a subbasis for $k$-topology.

(2) The character of a point $x$ (in symbols: $x(m)$) is the least cardinal $m$ for which there is a basis of $x$ of the power $m$. The character of a space $X$ (in symbols: $x(X)$) is, by definition, the number $\sup x(x)$.

(3) A space $X$ is said to be $m$-almost-metrizable if there is a family $P = \{q_1, q_2, \ldots, q_m\}$ of pseudometrics on $X$ such that $A = \{x \in X: q_i(x, A) = 0 \text{ for each } i \text{ in } E\}$ for each $A \subseteq X$. It may always be assumed that $\max(q_1, q_2, \ldots, q_m) < \sup A$ in $P$. An $m$-almost-metrizable space is of the character $\leq m$ (see [4]).
1a. If \( Y \) is a metric space and \( X \) is hemicompact (i.e., \( X \) has a countable family \( S \) of compact subsets such that for every compact subset \( C \) of \( X \) there is a \( C_n \in S \) with \( C \subset C_n \)), then \( Y^X \) with a \( k \)-topology is metrizable (Arens [1]).

1b. If \( Y^X \) with a \( k \)-topology is first countable then \( X \) is hemicompact and \( Y \) is first countable (Arens [1]).

2a. If \( X \) is a completely regular locally bicompact space, then the \( k \)-topology in \( Y^X \) induces the continuous convergence of nets of sets (Arens [1]).

2b. If \( X \) is completely regular and \( Y \) is the unit interval \( [0,1] \) then there exists a smallest \((1)\) jointly continuous \((2)\) topology for \( Y^X \) if and only if \( X \) is locally compact (Arens [1]).

3a. If \( X \) is a completely regular locally compact space, \( Y \) and \( T \) are arbitrary completely regular spaces, then the continuity of \( f \) with respect to the \( k \)-topology in \( Y^X \) for every \( f \) which maps \( X \times T \) into \( Y \) (Fox [2]).

3b. If \( X \) and \( T \) are first countable, and \( Y \) is an arbitrary completely regular space, then the continuity of \( f \) is equivalent to the continuity of \( f \) with respect to the \( k \)-topology in \( Y^X \) for every \( f \) which maps \( X \times T \) into \( Y \) (Fox [2]).

3c. If \( X \) is separable metric, \( Y \) is the set of all reals, then there is a topology for \( Y^X \) such that the continuity of \( f \) is equivalent to the continuity of \( f \) with respect to that topology for every \( f \) which maps \( X \times T \) into \( Y \) if and only if \( X \) is locally compact (Fox [2]).

I. Definitions and notations. All topological spaces under consideration are supposed to be completely regular.

A space \( X \) is said to be \( m \)-compact if each open covering of \( X \) of the power \( \leq m \) contains a finite subcovering. A space is said to be compact (= bicom pact) if it is \( m \)-compact for each \( m \). A space is \( m \)-se m i c o m p a c t if it is the union of \( m \) compact subsets. A space is locally \( m \)-compact if each point of the space has a neighbourhood whose closure is \( m \)-compact.

The class of all cofinal subsets of a directed set \( D \) will be denoted by \( c(D) \).

\( D_n \) denotes the directed set consisting of all finite subsets of a set \( S \) of the power \( \leq n \). The partial ordering in \( D_n \) is the set-theoretical inclusion (i.e., \( n \leq m \) for \( n, m \in D_n \) if and only if \( n \subset m \)).

(1) By a topology we understand the family of all open sets. A topology \( \mathcal{S} \) is smaller than \( \mathcal{S}' \) if \( \mathcal{S} \subset \mathcal{S}' \).

(2) A topology \( \mathcal{S} \) for \( Y^X \) is said to be jointly continuous provided that the mapping \( F \) of \( Y^X \times X \) which carries \((f,x)\) into \( f(x) \) is continuous.

A net defined on a directed set \( D \) will be named \( D \)-net. A \( D_n \)-net will be called simply \( m \)-net. A \( D \)-net which assigns to an element \( n \in D \) an element \( x_n \) will be denoted by \((x_n, n \in D)\). A net \((x_n, n \in D)\) is said to be compact if there is an \( n_0 \in D \) such that the set \((x_n)_{n \geq n_0} \) has a compact closure.

Let \((W_D)\) be the following property of topological spaces:

\((W_D)\) Each convergent \( D \)-net of elements of the space is compact.

The property \((W_{D_{m}})\) will be denoted simply by \((W_m)\). A space \( X \) has the property \((W_D)\) if it has the property \((W_{D_{m}})\) for every directed set \( D \).

A class \( \mathcal{R} \) of compact subsets of a space \( X \) will be called a basis for compact sets if for each compact set \( C \subset X \) there is a \( C_n \in \mathcal{R} \) with \( C \subset C_n \). (Clearly, hemicompact spaces can be defined as spaces having an enumerably basis for compact sets.)

An open basis \( \mathcal{B} \) of a space \( X \) is said to be an \( m \)-basis if it is the union of \( m \) locally finite systems (see [3]). (A system of subsets of a space is said to be locally finite if each point of the space has a neighbourhood which intersects a finite number of members of the system.)

If \( f \) is a function defined on a set \( X \) whose values are functions, then \((f(t)(x))\) denotes the value of the function \( f(t) \) at the point \( x \).

If \( X, T, Y \) are topological spaces and \( \mathcal{S} \) is a topology for \( Y^X \), then we write \([X, Y, T, \mathcal{S}]\) if, for each function \( f \) which maps \( X \times T \) into \( Y \), the continuity of \( f \) is equivalent to the continuity of the function \( f \in (Y^X)^\mathcal{T} \),

where \( \mathcal{T} \) is defined by the equality \((f(t)(x)) = f(x, t) \).

The \( k \)-topology for \( Y^X \) will be denoted by \( k(Y^X) \).

\( f \) denotes the unit interval \((0,1)\).

II. Lemmas and auxiliary theorems.

Lemma 1. If \( \chi(x) \leq m \), then there exists a decreasing \( m \)-net \( \{U_n, n \in D_m\} \) (i.e. \( U_n \supset U_{n+1} \) for \( n \leq n' \)) of neighbourhoods of \( x \) such that the family \((U_n)_{n \in D_m} \) is a basis of \( x \) (such a basis will be called a special basis of \( x \)).

Proof. Let \( \{U_n\}_{n \leq x} \) be a basis of \( x \) of the power \( \leq m \). Let us set \( U_n = U_x \cap \ldots \cap U_{n-1} \) for \( n = \{n_1, \ldots, n_k\} \in D_m \).

Lemma 2. A space \( X \) is \( m \)-compact if and only if every \( m \)-net of elements of \( X \) has a cluster point.

Proof. If \( X \) is \( m \)-compact and \((x_n, n \in D_m)\) is an \( m \)-net of elements of \( X \), then there is a point \( x \) which belongs to the closures of all sets \( x_n = \{x_n\}_{n \in D_m} \). Since each neighbourhood of \( x \) intersects \( X_n \) for every \( n \in D_m \), \( x \) is cluster point of the net \((x_n, n \in D_m) \). Suppose \( X \) is such that each \( m \)-net has a cluster point and let \((A_n)_{n \leq x} \) (\( \equiv \leq m \)) be a centred family of closed subsets of \( X \). Let us set \( A_n = A_{n_1} \cap \ldots \cap A_{n_k} \) for \( n = \{n_1, \ldots, n_k\} \) and let \( x_n \) be an arbitrary point of \( A_n \). The net \((x_n, n \in D)\)
has a cluster point and it can easily be shown that this point is a common point of all \( A_n \).

**Lemma 3.** If \( \chi(x) \leq m \) and \( x \in X \), then there exists an \( m \)-net of elements of \( A \) which converges to \( x \).

**Proof.** Let \( \{ U_n, n \in D_n \} \) be a special basis of \( x \) and let \( x_n \) be an arbitrary point of \( A \cap U_n \). Clearly, the \( m \)-net \( \{ x_n, n \in D_n \} \) is convergent to \( x \).

Thus, if \( \chi(x) \leq m \), then the topology of \( X \) can be described in terms of convergence of \( m \)-nets.

**Lemma 4.** If \( X \) is \( m \)-compact and \( m \)-seminicompact, then \( X \) is compact.

**Proof.** Since \( X \) is \( m \)-seminicompact, each open covering of \( X \) contains a subcovering of the power \( \leq m \), thus, by \( m \)-compactness, it contains a finite subcovering.

**Theorem II.** If \( \chi(x) \leq m \) and \( X \) has the property \( (W_m) \), then \( X \) is locally \( m \)-compact.

**Proof.** Suppose that no neighbourhood of a point \( x_0 \in X \) has the \( m \)-compact closure and let \( (U_n, n \in D_n) \) be a special basis of \( x_0 \). For every \( n \in D_n \) there exists an \( m \)-net \( \{ x_{\alpha}, n \in D_n \} \) of elements of \( U_n \) which has no cluster point. Consider the net \( \{ x_{\alpha,n}, \langle n, \alpha \rangle \in D_n \times D_n \} \), where \( x_{\alpha,n} = x_{\alpha}(n) \). This is an \( m \)-net; indeed, the set \( D_n \times D_n \) is similar to \( D_n \). On the other hand, this net is convergent to \( x_0 \), but it is not compact.

**Corollary 1.** If \( X \) has the property \( (W_m) \) for each \( m \), then \( X \) is locally compact.

Clearly, a locally compact space has the property \( (W) \). Thus we obtain:

**Corollary 2.** A space \( X \) has the property \( (W) \) if and only if \( X \) has the property \( (W_m) \) for each \( m \).

**Corollary 3.** A space \( X \) is locally compact if and only if \( X \) has the property \( (W) \).

By lemma 4 and theorem I.1, we obtain:

**Theorem II.** If \( \chi(x) \leq m \) and \( X \) has the property \( (W_m) \) and is \( m \)-seminicompact, then \( X \) is locally compact.

**Lemma 5.** If \( X \) is compact and \( \chi(x) \leq m \) and \( \{ x_n, n \in D_n \} \) is an \( m \)-net of points of \( X \), then for each \( n \in D_n \) there is a \( k_n \geq n \) such that the \( m \)-net \( \{ x_{k_n}, n \in D_n \} \) converges to some point of \( X \).

**Proof.** By lemma 2, the net \( \{ x_n, n \in D_n \} \) has a cluster point \( x_0 \).

Let \( (U_n, n \in D_n) \) be a special basis of \( x_0 \). Then for each \( n \in D_n \) there is a \( k_n \geq n \) with \( x_{k_n} \in U_n \). Clearly, the net \( \{ x_{k_n}, n \in D_n \} \) converges to \( x_0 \).

**Lemma 6.** If \( \chi(X) \leq m \) and \( f \) is a function on \( X \) to \( X \), then \( f \) is continuous if and only if for each \( m \)-net \( \{ x_n, n \in D_n \} \) which converges to some point \( x_0 \in X \), the net \( \{ f(x_n), n \in D_n \} \) converges to \( f(x_0) \).

**Proof.** Suppose that \( f \) is not continuous. Then there is an \( A \in X \) such that \( f(A) \neq f(A) \); i.e., there is a point \( x_0 \in A \) with \( f(x_0) \neq f(A) \).

By lemma 3, there is an \( m \)-net \( \{ x_n, n \in D_n \} \) of elements of \( A \) which converges to \( x_0 \); on the other hand, there is a neighbourhood \( N \) of \( \{ f(A) \} \) such that \( N \cap f(A) = \emptyset \). Since \( f(x_n) \in N \) for each \( n \in D_n \), \( \{ f(x_n), n \in D_n \} \) does not converge to \( f(x_0) \).

Conversely, if \( \{ x_n, n \in D_n \} \) converges to \( x_0 \), then there is a neighbourhood \( V \) of \( \{ f(x_0) \} \) such that \( \forall n \in D_n \) : \( f(x_n) \in V \). Clearly, \( f(U) \subseteq V \) for each neighbourhood \( U \) of \( x_0 \), whence \( f \) is continuous.

**III. Character and almost-metrizability of \( Y \).** (In this section \( Y \) is supposed to carry a \( k \)-topology.)

**Theorem III.** If \( Y \) has an \( m \)-basis and \( X \) has a basis \( B \) for compact sets of the power \( \leq m \), then \( Y \) is of the character \( \leq m \).

**Proof.** Let \( B \) be an \( m \)-basis of \( Y \) and let \( B^* = \bigcup_{i \in \mathbb{N}} B^* \), where \( B^* \) are locally finite systems. Let \( f \in Y^X \). For every \( C \in B \) we denote by \( B_C \) the family of all \( U \in B \) for which \( f(C) \subseteq U \). Since \( f(C) \) is compact and \( B_C \) is locally finite, \( B_{w^*} \) is finite.

Let \( B^* = \bigcup_{i \in \mathbb{N}} B_{C^i} \) and let \( B_C \) be the family of all finite unions of members of \( B_C \). Clearly \( B_C \subseteq B \).

Let \( C = \bigcup_{i \in \mathbb{N}} (C \cap U_i) \cup B_i \). Clearly each member of \( C \) is compact and \( C \subseteq B \). We shall show that the family of all \( f(C) \subseteq U \), where \( C \in B \) and \( U \in B \) is a subbasis of \( f \). Let \( f(C) \subseteq U \), where \( C \) is an arbitrary compact subset of \( X \) and \( U \) is an arbitrary open subset of \( X \). There exists \( U \in B \) such that \( C \subseteq U \). For each \( x \in C \) there is a \( U_x \subseteq U \) such that \( f(x) \subseteq U_x \). Clearly \( f(C) \subseteq U \), whence \( f(C) \subseteq U \). Since \( x \in f(C) \subseteq U \), \( C_x \subseteq U \) and thus there is a finite system \( U_{x_0}, ..., U_{x_n} \) such that \( C_x \subseteq U_{x_0} \cap ... \cap U_{x_n} \). Let \( U = U_{x_0} \cup ... \cup U_{x_n} \). We have \( U \subseteq U \) and \( f(C) \subseteq U \). Repetitively reasoning, we infer that there is an \( U \subseteq B \) such that \( f(C) \subseteq U \subseteq U \subseteq U \). Let \( C = f(C) \subseteq f(C) \cap B \). Clearly \( C \subseteq B \).

Since \( C \subseteq B \subseteq B \), \( f(C) \subseteq B \). On the other hand, \( f(C) \subseteq f(C) \subseteq B \subseteq B \), whence \( f(C) \subseteq U \), where \( U \subseteq C \) and \( U \subseteq U \).

**Theorem III.** If \( X \) has a basis for compact sets of the power \( \leq m \) and \( X \) is \( m \)-almost-metrizable, then \( Y \) is \( m \)-almost-metrizable.

**Proof.** Let \( P = (\mathcal{B})_{\mathcal{B}_C} \) be a family of pseudometrics of \( Y \) and let \( B \) be a basis for compact sets in \( X \) of the power \( \leq m \). Let us set
We shall show that $P^*$ is a family of pseudometrics for $Y^X$. Let $V = \{ v \in E : g_0(f_a, f_b) < \varepsilon \}$. Since $C$ is compact, there is a finite system $C_1, \ldots, C_l$ of compact sets such that $C = C_1 \cup \cdots \cup C_l$ and $g_0(f_a(x), f_b(x')) < \frac{1}{l} \varepsilon$ for $x, x' \in C_l$. Let $x_0$ be an arbitrary point of $C_l$ and let $U_{x_0} = \{ y \in Y : g_0(f_{x_0}(y), y) < \varepsilon \}$. If $x \in C_l$, then $g_0(f_{x_0}(x), f_a(x)) < \frac{1}{l} \varepsilon$ and $f_a(x) \in U_{x_0}$, whence $f_{x_0} \subseteq C \subseteq U_{x_0}$, and thus $f_{x_0} \in W(C_1, \ldots, C_l; U_{x_0}, \ldots, U_{x_0})$. On the other hand, if $f \in W(C_1, \ldots, C_l; U_{x_0}, \ldots, U_{x_0})$ and $x \in C_l$, then $f(x) \in U_{x_0}$, and $g_0(f(x_0), f(x)) \leq g_0(f_{x_0}(x_0), f(x)) + g_0(f_{x_0}(x_0), f(x)) < \frac{1}{l} \varepsilon$, whence sup $g_0(f_{x_0}(x), f(x)) < \varepsilon$ and $V \subseteq U_{x_0}$. Finally, $f_{x_0} \in W(C_1, \ldots, C_l; U_{x_0}, \ldots, U_{x_0}) \subseteq V$. Conversely, let $W(C_1, \ldots, C_l; U_{x_0}, \ldots, U_{x_0})$ be an arbitrary neighborhood of $f_{x_0}$. Since $f_{x_0} \subseteq C \subseteq U_{x_0}$, there is a pseudometric $g_0$ in $P$ such that $g_0(f_{x_0}(C), Y, U_{x_0}) < \varepsilon$. Let $g_0 = \min(g_0, \varepsilon)$ and $V = \{ x \in X : g_0(f_{x_0}(x), x) < \varepsilon \}$. Since $f_{x_0} \subseteq C \subseteq U_{x_0}$, there is a $C$ in $\mathbf{R}$ such that $C \subseteq \cdots \subseteq C_l \subseteq C$. Let $V = \{ v \in E : g_0(f_a, f_b) < \varepsilon \}$. If $f \in V$ and $x \in C_l$, then $g_0(f_{x_0}(C), f(x)) < \varepsilon$, whence $f(x) \in U_{x_0}$ and $f(x) \in U_{x_0}$, and finally $V \subseteq W(C_1, \ldots, C_l; U_{x_0}, \ldots, U_{x_0})$. Theorem 3. 3. If $F$ is of the character $\leq m$, then $X$ has a basis for compact sets of the power $\leq m$.

Proof. Let $f$ be a function which is identically equal to 0 and let $U_{x_0} = \{ y \in E : 0 < y < \varepsilon \}$. Since $x_0 \notin X$, there is a family $S$ of compact sets of the power $\leq m$ consisting of compact subsets of $X$ with the following property: for each compact set $C \subseteq X$ there are $C_1, \ldots, C_m$ in $S$ and open sets $U_{x_1}, \ldots, U_{x_m} \subseteq C$ such that $f \in W(C_1, \ldots, C_m; U_{x_1}, \ldots, U_{x_m}) \subseteq W(C_1, \ldots, C_m)$. But $f \in W(C_1, \ldots, C_m; U_{x_1}, \ldots, U_{x_m}) \subseteq W(C_1, \ldots, C_m)$ implies $C \subseteq C \subseteq \cdots \subseteq C_m$. In fact, if there is a point $x_0 \in C \subseteq C \subseteq \cdots \subseteq C_m$, then there is a function $f$ with $f(x_0) = 1$ and $f(x) = 0$ for $x \in C \subseteq \cdots \subseteq C_m$. Clearly $f \in W(C_1, \ldots, C_m; U_{x_1}, \ldots, U_{x_m})$ and $f \notin W(C_1, \ldots, C_m)$. This contradicts $W(C_1, \ldots, C_m; U_{x_1}, \ldots, U_{x_m}) \subseteq W(C_1, U_{x_0})$. It follows that the family of all finite unions of members of $S$ is a basis for compact sets.

IV. Continuous convergence of nets of functions. We recall that a net $(f_a, a \in D)$ in $Y^X$ is said to be continuously convergent to $f$ in $Y^X$ provided that for each net $(x_n, n \in D)$ in $X$ which converges to a point $x \in X$, the net $(f_a(x_n), n \in D)$ converges to $f(x)$. It may easily be shown that a topology $\mathcal{T}$ in $Y^X$ is jointly continuous if and only if each net in $Y^X$ which converges to some $f \in Y^X$ with respect to the topology $\mathcal{T}$ converges to $f$ in $Y^X$. Theorem 4. 2. If $X$ has a basis for compact sets of the power $\leq m$ and the $k$-topology for $Y^X$ induces the continuous convergence of $m$-nets in $Y^X$, then $X$ has the property $(W_m)$. Proof. Let $R = (C_i)_{i \in I} = \{ x_i \} = \{ x_m \}$ be a basis for compact sets. Let us set $C_0 = C_1 \cup \cdots \cup C_m$ for $n \in D_n, n = \{ s_1, \ldots, s_k \}$. If $X$ does not have the property $(W_m)$, then there is an $m$-net $(x_n, n \in D_n)$ which converges
to some point \( x_0 \in X \) such that for each \( n \in D_m \) there is \( k_n \geq n \) with \( x_{k_n} \approx x_0 \). Then there is a continuous function \( f_n \) on \( X \) to \( I \) such that \( f_n(x_0) = 1 \) and \( f_n(x) = 0 \) for \( x \neq x_0 \). Let \( f_n \) be a function which is identically equal to 0 on \( X \). Since \( (x_{k_n}, n \in D_m) \) converges to \( x_0 \) and \( f_n(x_{k_n}) = 1 \) for each \( n \in D_m \), \( f_n(x_{k_n}) \) does not converge to \( f_n \) continuously.

On the other hand, if \( W(G, U) \) is a neighbourhood of \( f_n \), then there is an \( n_0 \in D_m \) such that \( C \subseteq C_{n_0} \). Since \( C_{n_0} \subseteq C \) for \( n \geq n_0 \), \( x \in W(G, U) \) for \( n \geq n_0 \); thus \( f_n(x_{k_n}) \) converges to \( f_n \) with respect to \( k \)-topology.

**Corollary 1.** If \( \gamma(X) \leq m \), \( f^2 \) is a topology of the character \( \leq m \) and the \( k \)-topology for \( f^2 \) induces the continuous convergence of \( m \)-nets, then \( X \) is locally compact.

**Corollary 2.** Under the assumptions of the previous corollary, the \( k \)-topology for \( f^2 \) is a topology of continuous convergence.

**V. Continuity of functions with values from a function space.**

**Theorem V. 1.** If \( \gamma(X) \leq m \), \( \gamma(T) \leq m \) and \( X \) has the property \( (W_m) \), then \( \{X, Y, T, k(X^2)\} \) for an arbitrary completely regular space \( Y \).

**Proof.** Suppose that \( f \) maps \( X \times T \) into \( Y \) and let \( f \) be defined by the equality \( f(t)(x) = f(x, t) \). We shall show that the continuity of \( f \) is equivalent to that of \( f \). Suppose that \( f \) is continuous. Clearly, the values of \( f \) are continuous functions of the variable \( x \); thus \( f \) maps \( T \) into \( Y^X \).

Let \( (t_n, n \in D_n) \) be an \( m \)-net of elements of \( T \) which converges to some point \( t \in T \). Let \( g_n = f(t_n) \), \( g_n = f(t_n) \). If \( (x_n, n \in D_n) \) is a \( m \)-net of elements of \( X \) which converges to some point \( x_n \in X \), then, by the continuity of \( f \), \( (g_n(x_n), n \in D_n) \) converges to \( f(x_n, t) = g_n(x_n) \), whence \( (g_n, n \in D_n) \) converges to \( g_n \) continuously; thus, by Theorem IV. 1, \( (g_n, n \in D_n) \) converges to \( g_n \) with respect to \( k \)-topology, i.e., \( \{g_n, n \in D_n\} \) converges to \( g_n \) if, by lemma 6, that \( f \) is continuous.

Conversely, suppose that \( f \) is continuous, and let \( \{x_n, t_n, n \in D_n\} \) be an arbitrary \( m \)-net of elements of \( X \times T \) which converges to some point \( (x, t) \in X \times T \). Since \( (t_n, n \in D_n) \) converges to \( t \), \( f \) is continuous, \( \{f(t_n), n \in D_n\} \) converges to \( f(t) \), i.e., the net \( \{g_n, n \in D_n\} \), where \( g_n = f(t_n) \), converges to \( g = f(t) \) with respect to \( k \)-topology.

By Theorem IV. 1, \( \{g_n, n \in D_n\} \) converges to \( g_n \) continuously. Since \( (x_n, n \in D_n) \) converges to \( x_n \), \( (g_n(x_n), n \in D_n) \) converges to \( g(x_n) = f(x_n, t_n) \).

But \( g_n(x) = f(x_n, t_n) \), and thus \( f(x_n, t_n, n \in D_n) \) converges to \( f(x_n, t_n) \). Since \( \gamma(X \times T) \leq m \), \( f \) is continuous by lemma 6.

**Theorem V. 2.** If \( X \) has a basis for compact sets of the power \( \leq m \), \( \gamma(X) \leq m \) and \( [X, I, T, k(X^2)] \) for each \( T \) with \( \gamma(T) \leq m \), then \( X \) has the property \( (W_m) \).

**Proof.** Let \( T \) be the space consisting of all elements of \( D_n \) and of an "ideal" element \( a \). Points of \( D_n \) are isolated in \( T \), neighbourhoods of \( a \) are of the form \( \{a\} \cup \{x \in D_n: n \geq m \} \) for some \( m \in D_n \). Clearly, \( \gamma(T) \leq m \).

Let \( S = (C)_x, x \in m \), be a basis for compact sets in \( X \), and let \( C_a = C_0 \cup \ldots \cup C_{n_a} \) for \( n \geq n_a \), \( n = \{n_1, \ldots, n_2\} \). If \( X \) does not have the property \( (W_m) \), then there is a net \( \{x_n, n \in D_n\} \) which converges to some point \( x_0 \in X \) and is such that for each \( n \in D_n \), there is \( k_n \geq n \) with \( x_{k_n} \approx x_0 \). Let \( g_n \) be a continuous function on \( X \) to \( I \) such that \( g_n(x_0) = 1 \) and \( g_n(x) = 0 \) for \( x \in C_n \). Let \( f \) be a function defined on \( X \times T \) by the equalities \( f(x, a) = g_n(x) \), \( f(x, a) = 0 \), and let \( f \) be the function on \( T \) to \( f^2 \) defined by the equality \( f(t)(x) = f(x, t) \). Since \( \{x_{n_a}, n \in D_n\} \) converges to \( \{x_0, a\} \), \( f(x_{n_a}, n) = 1 \), \( f(x_{n_a}, t) = 0 \), \( f \) is not continuous.

On the other hand, let \( \{t_n, n \in D_n\} \) be an \( m \)-net in \( T \) which converges to \( a \) and let \( W(G, U) \) be a neighbourhood of \( f(a) \). There is an \( m \)-net \( d_n \in D_n \) with \( C_{n_a} \supseteq C \) and there is \( n_a \in D_n \) such that \( t_n \not\approx m_n \) for \( n \geq n_a \). It can be assumed that \( t_n \not\approx a \). Since \( C_{n_a} \supseteq C \) for \( n \geq n_a \), \( g_n(x) = 0 \) for \( x \in C \), whence \( f(t) = g_n \in W(G, U) \) for \( n \geq n_a \) and it means that \( f(t_n, a) \) converges to \( f(a) \); thus \( f \) is continuous.

**Corollary.** If \( \{X, I, T, k(X^2)\} \) for an arbitrary topological space \( T \), then \( X \) is locally compact.

A stronger result is given by the following:

**Theorem V. 3.** If there is a topology \( \mathcal{S} \) for \( X^2 \) such that \( \mathcal{S}(X, I, T, 3) \) for an arbitrary topological space \( T \), then \( X \) is locally compact.

**Proof.** It follows from the proof of the preceding theorem that if \( \gamma(X) \leq m \) and \( \gamma(T) \leq m \) for an arbitrary topological space \( T \) with \( \gamma(T) \leq m \), then \( \mathcal{S} \) induces the continuous convergence of \( m \)-nets in \( X^2 \), whence, if \( \{X, I, T, 3\} \) for an arbitrary topological space \( T \), then \( \mathcal{S} \) is a topology of continuous convergence (\( 1 \)). But (see remarks in IV), the space \( X^2 \) has a topology of uniform convergence if and only if \( X \) is locally compact.

\(^{(1)}\) If a topology \( \mathcal{S} \) for \( X^2 \) induces the continuous convergence of all \( m \)-nets (for arbitrary cardinal \( m \)), then \( \mathcal{S} \) is a topology of continuous convergence. Indeed, in this case, the mapping \( f(t)(x) = f(x, t) \) is continuous, i.e., \( \mathcal{S} \) is jointly continuous. If \( \mathcal{S} \) is another jointly continuous topology smaller than \( \mathcal{S} \), then each \( m \)-net in \( X^2 \) convergent with respect to \( \mathcal{S} \) is convergent continuously, i.e., it is convergent with respect to \( \mathcal{S} \).

On the other hand, since \( \mathcal{S} \) is smaller than \( \mathcal{S} \), each \( m \)-net which is convergent with respect to \( \mathcal{S} \) is also convergent with respect to \( \mathcal{S} \). Thus \( \mathcal{S} \) and \( \mathcal{S} \) agree on \( m \)-nets, whence, by lemma 3, \( \mathcal{S} \), \( \mathcal{S} \). We see that \( \mathcal{S} \) is the smallest jointly continuous topology, i.e., it is a topology of continuous convergence.
On free groups of motions and decompositions of the Euclidean space

by

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The purpose of this paper is to prove two theorems given in section 1 (1). These theorems are solutions of some problems proposed to the authors by T. Dekker.

1. A group \( \Phi \) of 1-1 transformations of a set \( E \) onto itself is called without fixed points if for every \( p \in E \) and every \( \varphi \in \Phi \), we have \( \varphi(p) \neq p \).

The rank of a free group is the potency of a set of free generators of this group.

The sense-preserving isometries of the Euclidean space \( \mathcal{E} \), i.e., the superpositions of rotations and translations are called motions.

Theorem 1. There exists a free group of the rank \( 2^{\aleph_0} \) of motions of \( \mathcal{E} \) without fixed points.

The proof of this theorem follows in sections 3-9 (it is an effective construction, which does not use the axiom of choice). The relations of Theorem 1 with known results are given in section 2.

An application of this theorem (the construction of a set \( E \subset \mathcal{E} \)) which is congruent to \((E \setminus A) \cup B\) for any at most denumerable sets \( A, B \subset \mathcal{E} \) is given in [7]. Another application, of a well known character (compare [2], [3], [4], [8]), is the theorem 2 of this paper. It states that, for any system of congruence relations, \( \mathcal{E} \) can be divided into disjoint sets satisfying that system. For an exact description of this theorem we take the following notations:

\( M \) and \( N \) are non-empty sets.

\( (P_i)_{i \in M} \) and \( (Q_i)_{i \in N} \) are arbitrary systems of subsets of \( N \), all different from \( \emptyset \) and \( N \). (We do not suppose that \( \mu_1 \neq \mu_2 \) implies \( P_{\mu_1} \neq P_{\mu_2} \) or \( Q_{\mu_1} \neq Q_{\mu_2} \).)

\( \simeq \) denotes congruence of point sets realizable by a motion.

(1) They were announced in [8] by the first author, but the original proof was faulty, and the proof presented here was worked out by both authors.