Some consequences of the Vietoris Mapping Theorem

by

J. W. Jaworowski (Warszawa)

1. We know several generalizations of theorems concerning continuous mappings in the case of multi-valued mappings, *i. e.*, of mappings which take each point of a space X into a non-empty closed subset of a space Y. The continuity of a multi-valued mapping $F\colon X\to Y$ is defined as the upper semi-continuity of the mapping F regarded as a single-valued mapping of X into the space 2^Y of non-empty closed subsets of Y. For instance, S. Eilenberg and D. Montgomery have extended in this way the Lefschetz formula concerning fixed points of mappings (see [3]); in [4] a similar generalization of Borsuk's theorem on antipodes is given. The Vietoris Mapping Theorem is an important tool in such generalizations. It ensures that a multi-valued continuous mapping F from X to Y, such that the sets $F(x)$ are acyclic, induces a homomorphism of homology groups of X into those of Y.

W. L. Strother studied in [7] the multi-valued mappings which are continuous as mappings from X to 2^Y. The author introduced the notion of multi-valued homotopy of such mappings. However, the identity mapping of the sphere S_n is "multi-homotopic" to zero in this sense, so that the "multi-homotopy groups" based on this notion of homotopy are trivial. We consider in this paper a similar notion of homotopy for multi-valued mappings $F\colon X\to Y$ which are continuous in the previous sense (*i. e.*, upper semi-continuous). We show (Theorem 3) that if two multi-valued mappings $F, G\colon X\to Y$ are homotopic and if during such a multi-valued homotopy $\Phi\colon X\times I\to Y$ the sets $\Phi(x,\tau)$ are acyclic, then F and G induce the same homomorphism of homology groups. It follows, in particular, that there exists no acyclic homotopy joining the identity mapping of the sphere with the constant mapping. Moreover, this theorem and the above mentioned formula of Lefschetz for multi-valued mappings yield a generalization of the classical theorem of Poincaré-Brouwer concerning vector fields on the sphere.

2. Let X be a compact metric space and let $\varepsilon>0$. By a k-dimensional ε-simplex of X we understand a set $\{p_0, p_1, \ldots, p_k\}$ of k points of X of diameter $<\varepsilon$. The oriented ε-simplex with the vertices $p_0, p_1, \ldots, p_k$ will be denoted by $(p_0, p_1, \ldots, p_k)$. In the known way we introduce the notion of ε-chains of X over a group $\mathfrak{G}$ of coefficients. The boundary of a chain $\varkappa$ of X will be denoted by $\partial\varkappa$. Let us point out that by the boundary of a 0-dimensional simplex consisting of a single point of X we understand the number 1, so that 0-dimensional cycles are only those in which the sum of coefficients is zero; the elements of the group $\mathfrak{G}$ may be considering as (-1)-dimensional cycles. Two k-dimensional ε-cycles γ_1^k and γ_2^k are said to be η-homologous in X (in notation $\gamma_1^k \underset{\eta}{\sim} \gamma_2^k$ in X), if there exists in X a $(k+1)$-dimensional η-chain $\varkappa^{k+1}$ such that $\partial\varkappa^{k+1} = \gamma_1^k - \gamma_2^k$.

A k-dimensional true chain is a sequence $\varkappa = \{\varkappa_i\}$ of k-dimensional ε_i-cycles $\varkappa_i$ of X such that $\varepsilon_i \to 0$. A true chain $\gamma = \{\gamma_i\}$ is called a true cycle if $\partial\gamma = \{\partial\gamma_i\} = 0$. A true cycle $\{\gamma_i\}$ of X is said to be convergent provided that for every $\eta > 0$ there exists an index i_0 such that, for every $i > i_0$, $\gamma_i \underset{\eta}{\sim} \gamma_{i+1}$ in X.

We shall denote by $H_k(X, \mathfrak{G})$ the Vietoris homology groups of X over $\mathfrak{G}$ based on convergent cycles [1]. Only the cases in which $\mathfrak{G}$ is the field of rational numbers or a group J_m of integers modulo m, with $m \geqslant 2$, will be considered. The homology groups of X over the field of rational numbers will be denoted simply by $H_k(X)$.

The space X is called *n-acyclic* (resp. *acyclic*), if $H_k(X) = 0$, for every $-1 \leqslant k \leqslant n$ (resp. for every k). Thus X is (-1)-acyclic if and only if it is not empty; it is 0-acyclic if and only if it is a continuum.

Let f be a continuous mapping of a compact space X into another compact space Y. The homomorphism $H_k(X, \mathfrak{G}) \to H_k(Y, \mathfrak{G})$ induced by f will be denoted by $\bar{f}_k$.

VIETORIS MAPPING THEOREM. *Let f be a continuous mapping of a compact space X into Y such that $H_k(f^{-1}(y), \mathfrak{G}) = 0$, for every $-1 \leqslant k \leqslant n$ and $y \in Y$. Then f induces an isomorphism $\bar{f}_k\colon H_k(X, \mathfrak{G}) \approx H_k(Y, \mathfrak{G})$, for every $k \leqslant n$* (see [8] or [2]).

COROLLARY. *Let $\varphi\colon X \to A$ be a retraction of a compact space X onto a closed set $A \subset X$ such that $H_k(\varphi^{-1}(x), \mathfrak{G}) = 0$, for every $-1 \leqslant k \leqslant n$ and $y \in A$, and let $i\colon A \to X$ be the injection mapping. Then $\bar{i}_k\bar{\varphi}_k$ is the identity automorphism e of the group $H_k(X, \mathfrak{G})$.*

Proof. Since $\varphi i(x) = x$ for every $x \in A$, $\bar{\varphi}_k\bar{i}_k$ is the identity on $H_k(A, \mathfrak{G})$. By the Vietoris theorem, $\bar{\varphi}_k$ is an isomorphism onto, whence

$$\bar{i}_k\bar{\varphi}_k = \bar{\varphi}_k^{-1}\bar{\varphi}_k\bar{i}_k\bar{\varphi}_k = \bar{\varphi}_k^{-1}\bar{\varphi}_k = e.$$

[1] Hence $H_0(X, \mathfrak{G})$ denotes the "reduced" homology group.

3. Let X and Y be compact spaces and let $F\colon X \to Y$ be a *multi-valued mapping* from X to Y, *i. e.*, for every $x \in X$, $F(x)$ is a subset of Y. The set

$$W = \underset{(x,y)}{E}[y \in F(x)] \subset X \times Y$$

is called the *graph* of F. The multi-valued mapping F is said to be *continuous* if the graph of F is closed in $X \times Y$. The continuity of F implies that the sets $F(x)$ are compact. A continuous multi-valued mapping $F\colon X \to Y$ may be regarded as an upper semi-continuous mapping of X into the space 2^Y of non-empty closed subsets of Y. If $F\colon X \to Y$ is single-valued, *i. e.*, if every set $F(x)$ consists of a single point $f(x)$, then the continuity of F means the ordinary continuity of the single-valued mapping f.

A single-valued continuous mapping $f\colon X \to Y$ is said to be a *cross-section* of a multi-valued mapping $F\colon X \to Y$ if the graph of F contains the graph of f, *i. e.*, if $f(x) \in F(x)$ for every $x \in X$. If the identity mapping of X is a cross section of a multi-valued continuous mapping $F\colon X \to Y$, *i. e.*, if $x \in F(x)$ for every $x \in X$, then F is said to be a *multi-valued identity*.

A multi-valued continuous mapping $F\colon X \to Y$ is said to be *n-acyclic* (resp. *acyclic*) provided that the sets $F(x)$ are n-acyclic (resp. acyclic), for every $x \in X$.

Let $F\colon X \to Y$ be a multi-valued continuous n-acyclic mapping of a compact space X and let W be the graph of F. Let $r\colon W \to X$, $s\colon W \to Y$ be the projections defined by

$$r(x, y) = x, \quad s(x, y) = y. \tag{1}$$

Hence r and s are continuous and the sets $r^{-1}(x)$ are homeomorphic to $F(x)$. It follows that $r^{-1}(x)$ is n-acyclic, for every $x \in X$, and, by the Vietoris theorem, $\bar{r}_k$ is an isomorphism onto, for every $k \leqslant n$. Let us consider the homomorphism

$$\bar{F}_k = \bar{s}_k\bar{r}_k^{-1}\colon H_k(X) \to H_k(Y) \tag{2}$$

which is defined for every $k \leqslant n$. The homomorphism $\bar{F}_k$ is said to be *induced* by the multi-valued mapping F.

Obviously, if F is single-valued, then r is a homeomorphism and $F = sr^{-1}$. Hence the homomorphism defined above is a generalization of the notion of a homomorphism induced by a continuous single-valued mapping.

THEOREM 1. *Let X be a compact space and let the multi-valued continuous n-acyclic mapping $F\colon X \to Y$ admit a cross-section $f\colon X \to Y$. Then $\bar{F}_k = \bar{f}_k$ for every $k \leqslant n$.*

Proof. Let W be the graph of F and V — the graph of f. Let $r\colon W\to X$, $s\colon W\to Y$ be the projections defined by (1). Let $i\colon V\to W$ be the injection mapping and let $g=r|V$, $h=s|V$. Hence g is a homeomorphism such that

(3) $$hg^{-1}=f$$

and

(4) $$h=si\,.$$

Let $\varphi\colon W\to V$ be the mapping defined by $\varphi(x,y)=\big(x,f(x)\big)$. Then φ is a retraction of W onto V and $\varphi(x,y)=\big(x,f(x)\big)=g^{-1}(x)=g^{-1}r(x,y)$, for every $(x,y)\in W$. Hence

(5) $$\varphi=g^{-1}r\,.$$

Moreover, for every $v=\big(x,f(x)\big)\in V$, the set $\varphi^{-1}(v)$ is homeomorphic to $F(x)$, whence n-acyclic. By the corollary of Nr 2, $\bar i_k\bar\varphi_k$ is an identity, for every $k\leqslant n$. Therefore by (2), (5), (4) and (3) we have

$$\bar F_k=\bar s_k\bar r_k^{-1}=\bar s_k\bar i_k\bar\varphi_k\bar r_k^{-1}=\bar s_k\bar i_k\bar g_k^{-1}\bar r_k\bar r_k^{-1}=\bar s_k\bar i_k\bar g_k^{-1}=\bar h_k\bar g_k^{-1}=\bar f_k\,.$$

COROLLARY. *If $F\colon X\to Y$ is a constant multi-valued n-acyclic mapping, i. e., for every $x\in X$, $F(x)$ is a fixed closed n-acyclic subset of Y, then $\bar F_k=0$ for every $k\leqslant n$.*

For such a mapping admits a constant cross-section.

Let X, Y, Z be compact spaces. Let $f\colon X\to Y$ be a single-valued continuous mapping and $F\colon Y\to Z$ — a multi-valued continuous mapping. Then the mapping $G\colon X\to Z$ defined by $G(x)=F\big(f(x)\big)$ is continuous; it will be denoted by Ff. Obviously, if F is n-acyclic, then so is G.

THEOREM 2. *Let X, Y, Z be compact spaces. Let $f\colon X\to Y$ be a single-valued continuous mapping, and $F\colon Y\to Z$ — a multi-valued continuous n-acyclic mapping, and let $G=Ff\colon X\to Z$. Then $\bar G_k=\bar F_k\bar f_k$* for every $k\leqslant n$.

Proof. Let W be the graph of F and V — the graph of G. Let $r\colon W\to Y$, $s\colon W\to Z$ be the projections defined by

(6) $$r(y,z)=y\,,\qquad s(y,z)=z$$

and $t\colon V\to X$, $u\colon V\to Z$ — the projections defined by

(7) $$t(x,z)=x\,,\qquad u(x,z)=z\,.$$

Hence $\bar r_k$ and $\bar t_k$ are isomorphisms for $i\leqslant n$, and

(8) $$\bar F_k=\bar s_k\bar r_k^{-1}\,,$$

(9) $$\bar G_k=\bar u_k\bar t_k^{-1}\,.$$

The (single-valued) mapping g defined by $g(x,z)=\big(f(x),z\big)$ for every $(x,z)\in V$, is continuous and maps V into W. Moreover, by (6) and (7), $ft(x,z)=f(x)=rg(x,z)$, and $sg(x,z)=s\big(f(x),z\big)=z=u(x,z)$ for every $(x,z)\in V$. Hence $ft=rg$ and $u=sg$. It follows that $\bar f_k=\bar r_k\bar g_k\bar t_k^{-1}$ and $\bar u_k=\bar s_k\bar g_k$. Therefore, by (8) and (9),

$$\bar F_k\bar f_k=\bar s_k\bar r_k^{-1}\bar r_k\bar g_k\bar t_k^{-1}=\bar u_k\bar t_k^{-1}=\bar G_k\,.$$

4. Let F, G be two multi-valued continuous n-acyclic mappings from X into Y and let I be the interval $0\leqslant\tau\leqslant 1$. The mappings F and G are said to be *n-acyclically homotopic* if there exists a multi-valued continuous n-acyclic mapping $\Phi\colon X\times I\to Y$, called a (multi-valued) *homotopy*, such that $\Phi(x,0)=F(x)$ and $\Phi(x,1)=G(x)$ for every $x\in X$; if the homotopy Φ is acyclic, then F and G are said to be *acyclically homotopic.*

THEOREM 3. *Let X and Y be compact spaces and let F and G be two multi-valued continuous n-acyclic mappings of X into Y. If F and G are n-acyclically homotopic, then $\bar F_k=\bar G_k$ for every $k\leqslant n$.*

Proof. Let $\Phi\colon X\times I\to Y$ be an n-acyclic homotopy joining F and G. Let us consider the homomorphism $\bar\Phi_k\colon H_k(X\times I)\to H_k(Y)$ induced by Φ. Let $i,j\colon X\to X\times I$ be the injection mappings $i(x)=(x,0)$, $j(x)=(x,1)$. Thus we have

(10) $$\bar i_k=\bar j_k\,.$$

Moreover, $\Phi i=F$ and $\Phi j=G$. Hence by theorem 2 and by (10), $\bar F_k=\bar\Phi_k\bar i_k=\bar\Phi_k\bar j_k=\bar G_k$.

5. Let X and W be compact spaces and let $r\colon W\to X$, $s\colon W\to X$ be two continuous mappings. Let us assume that the set $r^{-1}(x)$ is acyclic for every $x\in X$. Then, by the Vietoris theorem, $\bar r_k\colon H_k(W)\to H_k(X)$ is an isomorphism onto and hence the endomorphism $\bar s_k\bar r_k^{-1}\colon H_k(X)\to H_k(X)$ is defined for every k.

Now let X be an ANR-space (*i. e.*, an absolute neighbourhood retract). Then the groups $H_k(X)$ have a finite system of generators and are trivial for sufficiently large k. It follows that the trace of $(\bar s_k\bar r_k^{-1})$ is defined. The number

$$\Lambda(s,r)=1+\sum_{k=0}^{\infty}(-1)^k\operatorname{trace}(\bar s_k\bar r_k^{-1})$$

is defined in [3] as the Lefschetz number of (s,r). S. Eilenberg and D. Montgomery have proved in [3] the following generalization of the fixed point theorem:

(I) *Let X be an ANR, W — a compact space and $r\colon W\to X$, $s\colon W\to X$ — two continuous mappings such that $r^{-1}(x)$ is acyclic for every $x\in X$. If $\Lambda(s,r)\neq 0$, then there exists a point $w\in W$ such that $r(w)=s(w)$.*

Let F be a multi-valued continuous acyclic mapping of an ANR-space X into itself. Let W be the graph of F and let $r\colon W\to X$, $s\colon W\to X$ be the projections defined by (1). Let $\Lambda(F)$ denote the Lefschetz number $\Lambda(s,r)$. Since $\overline{F}_k=\bar{s}_k\bar{r}_k^{-1}$, then

$$\Lambda(F)=1+\sum_{k=0}^{\infty}(-1)^k\operatorname{trace}(\overline{F}_k).$$

If $\Lambda(F)\neq 0$, then, by theorem (I), there exists a point $(x_0,y_0)\in W$ such that $x_0=y_0$. Hence $x_0\in F(x_0)$. A point like this is called a *fixed point* of the multi-valued mapping F. Hence we have

(II) *Let F be a multi-valued continuous acyclic mapping of a compact* ANR-*space X into itself. If $\Lambda(F)\neq 0$, then F has a fixed point* (see [3], p. 217).

Theorem 3 of No 4 implies

COROLLARY 1. *If two multi-valued continuous acyclic mappings F, G of a compact* ANR-*space X are acyclically homotopic, then $\Lambda(F)=\Lambda(G)$.*

If F is a multi-valued acyclic constant mapping, then $\Lambda(F)=1$. Hence we have

COROLLARY 2. *Every multi-valued continuous acyclic mapping of an* ANR-*space into itself which is acyclically homotopic to a (multi-valued) constant mapping has a fixed point.*

The Lefschetz number of the identity mapping is equal to the Euler-Poincaré characteristic

$$\chi(X)=1+\sum_{k=0}^{\infty}(-1)^k p_k(X)$$

($p_k(X)$ denotes the kth Betti number of X). Hence theorem 1 yields

COROLLARY 3. *If the Euler-Poincaré characteristic of a compact* ANR-*space X is different from zero, then every multi-valued continuous acyclic mapping of X into itself which is acyclically homotopic to a multi-valued identity has a fixed point.*

6. Let S_n denote the n-dimensional unit sphere in the $(n+1)$-dimensional Euclidean space E_{n+1}. We shall denote by $\alpha(x)$ the antipode of a point $x\in S_n$. The following lemma may be proved in the same way as the corresponding one concerning single-valued mappings into the sphere:

LEMMA. *Let $F\colon X\to S_n$ be a multi-valued continuous acyclic mapping of a compact space X into S_n and let $f\colon X\to S_n$ be a single-valued continuous mapping. If, for every $x\in X$, the set $F(x)$ does not contain the antipode of $f(x)$, then F is acyclically homotopic to f.*

Proof. Let $x\in X$ and $0\leqslant\tau\leqslant 1$. Since, for every $y\in F(x)$, $\alpha(f(x))\neq y$, then the points $f(x)$ and y determine uniquely the smaller arc $\widehat{f(x)y}$ with end-points $f(x)$ and y of the great circle passing through these points. Let us denote by $\Phi(x,\tau)$ the set of points $z\in S_n$ which divide the arc $\widehat{f(x)y}$, with $y\in F(x)$, at the ratio $(1-\tau)\colon\tau$. Then, for every $\tau<1$, the set $\Phi(x,\tau)$ is homeomorphic to $F(x)$, and $\Phi(x,1)=f(x)$ for every $x\in X$. Hence the sets $\Phi(x,\tau)$ are acyclic. It is evident that Φ is a continuous multi-valued mapping of $X\times I$ into S_n. Therefore Φ is a multi-valued acyclic homotopy joining F and f.

Since, for even n, $\chi(S_n)=2$, this lemma and corollary 3 of No 5 yield

THEOREM 4. *Let n be even and let $F\colon S_n\to S_n$ be a multi-valued continuous acyclic mapping such that the set $F(x)$ does not contain the antipode of x for every $x\in S_n$. Then F has a fixed point.*

Remark. This theorem may also be deduced immediately from the theorem of Eilenberg and Montgomery of No 5, without using the notion of homotopy of multi-valued mappings, as follows:

Let W be the graph of F and let $r\colon W\to S_n$, $s\colon W\to S_n$ be the projections defined by (1). Since, for every $w\in W$, we have $\alpha(r(w))\neq s(w)$, then the single-valued mappings r and s are homotopic (in the usual sense). Hence the homomorphism $\overline{F}_k=\bar{s}_k\bar{r}_k^{-1}$ is an identity. It follows that $\Lambda(F)=\chi(S_n)=2$, and, consequently, F has a fixed point.

7. The following theorem is a generalization of the theorem of Poincaré-Brouwer:

THEOREM 5. *Let n be even. Then there exists no multi-valued continuous acyclic function F which assigns to every point $x\in S_n$ a set $F(x)$ of unit vectors tangent to S_n at x.*

Proof. Let $F(x)$ denote the set of end-points of the vectors belonging to $F(x)$ and $P(x)$ — the n-dimensional hyperplane passing through the centre c of S_n and orthogonal to the vector cx. Let $G(x)$ be the orthogonal projection of the set $F(x)$ on $P(x)$. Then the set $G(x)$ is acyclic and

$$G(x)\subset S_n-\{x\}-\{\alpha(x)\}. \tag{11}$$

Hence G is a multi-valued acyclic mapping and (11) yields a contradiction to theorem 4.

8. Let P_n be an n-dimensional projective space considered as the set of all unordered pairs $\{p,\alpha(p)\}$, with $p\in S_n$. The distance between two points $q_1=\{p_1,\alpha(p_1)\}$ and $q_2=\{p_2,\alpha(p_2)\}$ of P_n we define by

$$\varrho(q_1,q_2)=\min\{\varrho(p_1,p_2),\ \varrho(p_1,\alpha(p_2))\}. \tag{12}$$

Let π be the mapping of S_n onto P_n defined by $\pi(p) = \{p, \alpha(p)\}$ for every $p \in S_n$. Let us observe that, by (12), π maps 1-chains in S_n onto 1-chains in P_n and the cycles 1-homologous to zero in S_n onto cycles 1-homologous to zero in P_n.

The following theorem will be used in the sequel:

THEOREM 6 ([2]). *Let m be an integer $\geqslant 2$, X — a continuum such that $H_1(X, J_m) = 0$ and f — a continuous mapping of X into P_n. Then there exists a continuous mapping g of X into S_n such that $f = \pi g$.*

The proof of this theorem is based on the following

LEMMA. *Let $\varkappa$ be a 1-dimensional 1-chain modulo m in S_n such that $\partial\varkappa = (p) - (\alpha(p))$ for some $p \in S_n$. Then the 1-cycle $\pi(\varkappa)$ is not 1-homologous to zero in P_n.*

Proof. Let λ denote the 1-dimensional chain on S_n obtained by dividing a great circle arc joining the points p and $\alpha(p)$ on arcs of diameter <1, such that $\partial\lambda = (p) - (\alpha(p))$. Hence $\pi(\lambda)$ is an 1-cycle in P_n. It is known that

(13) $\pi(\lambda)$ is not 1-homologous to zero in P_n.

In fact, $\pi(\lambda)$ represents a generator of $H_1(P_n, J_m)$.

In the case $n = 1$, it is evident that $\pi(\varkappa) \underset{1}{\sim} \pm c \cdot \pi(\lambda)$ in P_1, where c is odd.

If $n > 1$, then $\varkappa - \lambda$ is an 1-dimensional 1-cycle in S_n, and, consequently, $\varkappa - \lambda \underset{1}{\sim} 0$ in S_n. It follows that $\pi(\varkappa - \lambda) = \pi(\varkappa) - \pi(\lambda) \underset{1}{\sim} 0$ in P_n. Hence, in this case, $\pi(\varkappa) \underset{1}{\sim} \pi(\lambda)$ in P_n.

We conclude by (13) that in both cases $\pi(\varkappa)$ is not 1-homologous to zero in P_n.

Proof of the theorem. Let a be a point of X. Then $f(a) = \{p_0, \alpha(p_0)\}$ where $p_0 \in S_n$. Let $g(a)$ be an arbitrary point of $\{p_0, \alpha(p_0)\}$.

Let ε be a positive number such that

(14) $\varrho(x', x'') < \varepsilon$ implies $\varrho(f(x'), f(x'')) < 1$ for every $x \in X$.

It follows from the assumptions of the theorems that there exists a positive number $\eta < \varepsilon$ such that

(15) every 1-dimensional η-cycle in X is ε-homologous to zero in X.

Indeed, if (15) does not hold, then putting $\eta = 1/j$ $(j = 1, 2, \ldots)$, we obtain a sequence $\{\gamma_j\}$ of 1-dimensional $(1/j)$-cycles in X such that γ_j is not ε-homologous to zero in X. Since $m \geqslant 2$, the sequence $\{\gamma_j\}$

([2]) A similar theorem (for arcwise connected X and $m = 0$) has been proved by K. Moszyński and is published in [5] without proof.

contains a convergent cycle (see [1], p. 180), which is not homologous to zero in X. However, this is impossible, since $H_1(X, J_m) = 0$.

Let $x \in X$. Since X is connected, there exists a finite sequence of points of X

$$a = x_0, x_1, \ldots, x_k = x$$

such that

$$\varrho(x_{i-1}, x_i) < \eta, \quad \text{for} \quad i = 1, 2, \ldots, k. \tag{16}$$

We shall construct a sequence of points $p_i \in S_n$

$$g(a) = p_0, p_1, \ldots, p_k = p$$

such that

$$\begin{aligned} &\pi(p_i) = f(x_i), \quad \text{for} \quad i = 0, 1, \ldots, k, \\ &\varrho(p_{i-1}, p_i) < 1, \quad \text{for} \quad i = 1, 2, \ldots, k. \end{aligned} \tag{17}$$

It follows that

$$\varrho(\pi(p_{i-1}), \pi(p_i)) < 1, \quad \text{for} \quad i = 1, 2, \ldots, k.$$

Let us suppose that p_{i-1} is already defined. By (16) and (14), $\varrho(f(x_{i-1}), f(x_i)) < 1$. Therefore there exists a unique point in the set $f(x_i)$ (considered as a pair $\{p', \alpha(p')\}$) which is the nearest one to $p_{i-1} \in f(x_{i-1})$; we shall denote it by p_i.

Let

$$a = y_0, y_1, \ldots, y_l = x$$

be an another sequence of points of X such that $\varrho(y_{i-1}, y_i) < \eta$ and let

$$g(a) = q_0, q_1, \ldots, q_l = q$$

constructed as above. Then

$$\begin{aligned} &\pi(q_i) = f(y_i), \quad \text{for} \quad i = 0, 1, \ldots, 1, \\ &\varrho(q_{i-1}, q_i) < 1, \quad \text{for} \quad i = 1, 2, \ldots, 1. \end{aligned} \tag{18}$$

We assert that $p = q$.

Let us suppose that $p \neq q$. Then $q = \alpha(p)$. Let us consider the 1-dimensional η-cycle in X

$$\gamma = \sum_{i=1}^{k} (x_{i-1}, x_i) - \sum_{i=1}^{l} (y_{i-1}, y_i)$$

and the 1-dimensional 1-chain in S_n

$$\varkappa = \sum_{i=1}^{k} (p_{i-1}, p_i) - \sum_{i=1}^{l} (q_{i-1}, q_i).$$

By (15), $\gamma \underset{\varepsilon}{\sim} 0$ in X, whence by (14), $f(\gamma) \underset{1}{\sim} 0$ in P_n. Since $\partial\varkappa = (p) - (a(p))$, then, by lemma, $\pi(\varkappa)$ is not 1-homologous to zero in P_n. But, by (17) and (18), $\pi(\varkappa) = f(\gamma)$ and therefore we have got a contradiction.

Let us put $g(x) = p$. We have shown that the definition of $g(x)$ is independent of the choice of the sequence $x_0, x_1, \ldots, x_k$. Hence g is a mapping of X into P_n such that $f = \pi g$. Let us observe that, if $\varrho(x', x'') < \eta$, then $\varrho(g(x'), g(x'')) = \varrho(f(x'), f(x''))$. It follows that g is continuous. Hence theorem 6 is proved.

Example. Let $X(k)$ be the van Danzig solenoid in the 3-space E_3 obtained as the intersection of an infinite sequence $\{T_i\}$ of solid tori, where T_i lies in the interior of T_{i-1}, runs k times around the interior and has a cross section of diameter $<1/i$. It is known that $H_1(X(k), \mathfrak{G})$ is isomorphic with a subgroup of $\mathfrak{G}$ composed of those elements of $\mathfrak{G}$ which are divisible by an arbitrary power of k (for $k = 2$, this is proved in [6]). Hence $H_1(X(2), J_2) = 0$ and $H_1(X(3), J_2) = J_2$.

Let $C(x)$ denote, for every $x \in T_1$, the meridian disk of T_1 containing x; it is defined as the intersection of T_1 with the plane passing through x and through the axis of T_1. Let φ be the mapping which maps every point x of T_1 into the centre of $C(x)$. Then φ maps T_1 onto the central circle S of T_1.

Let f be the mapping of $X(2)$ onto S defined by $f = \varphi | X(2)$. Since $H_1(X(2), J_2) = 0$, the assumptions of theorem 6 are fulfilled. It follows that there exists a mapping $g\colon X(2) \to S$ such that $f = \pi g$. Let us observe that the mapping g may be defined immediately: obviously, there exists a homeomorphism h of T_2 onto T_1 such that, for every $x \in T_2$, h maps two disks of the intersection of T_2 with $C(x)$ onto two meridian disks C_1, C_2 of T_1 which are symmetric with respect to the axis of T_1. Then we may put $g = fh | X(2)$.

Now let f be the mapping of $X(3)$ onto S defined by $f = \varphi | X(3)$. In this case the assumptions of theorem 6 are not fulfilled. We shall show that there exists no mapping g of $X(3)$ satisfying the statement of the theorem. Let γ_i be a 1-dimensional $(1/i)$-cycle which represents a basis of the group $H_1(T, J_2)$. Then $\gamma_i \underset{\eta_i}{\sim} 3\gamma_{i-1} = \gamma_{i-1}$ in T_{i-1}. By a displacement of every vertex of γ_i to the nearest point of $X(3)$ we obtain a convergent cycle $\{\gamma_i'\}$ in $X(3)$ which represents a basis of the group $H_1(X(3), J_2)$; hence it is not homologous to zero in $X(3)$. It is evident that f induces an isomorphism. But π_1 maps $H_1(S, J_2)$ into zero, whence the factorization $f = \pi g$ is impossible.

9. It is known that the theorem of Poincaré-Brouwer may be generalized so that the vector fields on the sphere may be replaced in it by fields of straight lines tangent to the sphere. We shall prove a similar theorem for multi-valued fields of lines.

Theorem 7. *Let n be even and m — an arbitrary integer $\geqslant 2$. Then there exists no multi-valued continuous acyclic function F which assigns to every point $x \in S_n$ a set $F(x)$ of straight lines tangent to S_n at x and such that $H_1(F(x), J_m) = 0$ for every $x \in S_n$.*

Proof. Let us assign to every $x \in S_n$ the set $G(x)$ of antipodal pairs of points in which the straight lines passing through the centre of S_n and parallel to the lines of $F(x)$ intersect S_n. Thus we obtain a multi-valued continuous acyclic mapping G of S_n into P_n. Let W be the graph of G and let $r\colon W \to S_n$, $s\colon W \to P_n$ be the projections defined by (1). Let us observe that

(19) $$\text{if } p \in S_n,\ w = (x, y) \in W \text{ and } \pi(p) = s(w), \text{ then } \varrho(r(w), p) = \sqrt{2}.$$

Since $r^{-1}(x)$ is homeomorphic to $F(x)$, then $H_i(r^{-1}(x), J_m) = 0$ for $i = -1, 0, 1$. Hence by the Vietoris theorem, $\bar{r}_1\colon H_1(W, J_m) \to H_1(S_n, J_m)$ is an isomorphism onto, and since $n > 1$, then $H_1(W, J_m) = 0$. Then theorem 6 provides a continuous mapping $g\colon W \to S_n$ such that

(20) $$s = \pi g\,.$$

By (19) and (20)

(21) $$\varrho(r(w), g(w)) = \sqrt{2}, \quad \text{for every} \quad w \in W\,.$$

By (21), the mappings r and g are homotopic. Hence $\bar{r}_k = \bar{g}_k$, for every k. Since $r^{-1}(x)$ is acyclic for every $x \in S_n$, then by the Vietoris theorem, $\bar{r}_k\colon H_k(W) \approx H_k(S_n)$. It follows that $\bar{g}_k \bar{r}_k^{-1}$ is the identity automorphism of $H_k(S_n)$ and therefore $\Lambda(g, r) = \chi(S_n) = 2$ (see No 5). Consequently, the theorem of Eilenberg and Montgomery implies that there exists a point $w_0 \in W$ such that $r(w_0) = g(w_0)$. However, this is impossible, by (21). Thus theorem 7 is proved.

Let $\mathfrak{B}$ be a set of straight lines in E_{n+1}, such that every line $L \in \mathfrak{B}$ is tangent to S_n. Let us denote by $F(x)$ the set of all straight lines $L \in \mathfrak{B}$ which are tangent to S_n at the point $x \in S_n$. Then we observe that the multi-valued function F is continuous if and only if the set $\mathfrak{B}$ is closed. It follows that the theorem 7 may be formulated as follows:

Theorem 7'. *Let n be even and m — an integer $\geqslant 2$. Then there exists in E_{n+1} no closed set $\mathfrak{B}$ of straight lines tangent to S_n such that, for every $x \in S_n$, the set $F(x)$ of lines $L \in \mathfrak{B}$ tangent to S_n at x is acyclic and $H_1(F(x), J_m) = 0$.*

For $n = 2$, this theorem may be formulated as follows:

Corollary. *Let $\mathfrak{B}$ be a closed set of straight lines in E_3 tangent to S_2. Let $F(x)$ denote the set of lines $L \in \mathfrak{B}$ tangent to S_2 at $x \in S_2$. If $F(x)$ is non-*

empty and connected for every $x \in S_n$, then there exists a point $x_0 \in S_n$ such that $F(x_0)$ contains all straight lines in E_3 passing through x_0 and tangent to S_n.

Remark. The questions whether theorem 6 and 7 remain true for $m = 0$ is open.

References

[1] P. Alexandroff, *Dimensionstheorie*, Math. Ann. 106 (1932), p. 160-238.

[2] E. G. Begle, *The Vietoris mapping theorem for bicompact spaces*, Annals of Math. 51 (1950), p. 534-543.

[3] S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformations*, Amer. Journ. of Math. 58 (1946), p. 214-222.

[4] J. W. Jaworowski, *Theorem on Antipodes for Multi-Valued Mappings and a Fixed Point Theorem*, Bull. Acad. Polon. Sci., Cl. III, 4 (1956), p. 187-192.

[5] — and K. Moszyński, *A Theorem on Mappings of the Sphere into the Projective Space*, Bull. Acad. Polon. Sci., Cl. III, 4 (1956), p. 75-77.

[6] N. E. Steenrod, *Regular cycles of compact spaces*, Lectures in Topology, University of Michigan Press, 1940, p. 43-45.

[7] W. L. Strother, *Multi-homotopy*, Duke Math. Journal 22 (1955), p. 281-285.

[8] L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse zusammenhangstreuen Abbildungen*, Math. Ann. 97 (1927), p. 454-472.

Reçu par la Rédaction le 18. 6. 1957

On function spaces

by

S. Mrówka (Warszawa)

In the present paper we are concerned with the study of the properties of topologies in function spaces; in particular, we shall consider the so-called k-topology (1). The following problems will be treated:

1° conditions regarding the spaces X and Y under which the space Y^X with a k-topology is of the character $\leqslant \mathfrak{m}$ (2) in particular, conditions under which the space Y^X is $\mathfrak{m}$-almost-metrizable (3);

2° conditions regarding X and Y under which there exists a topology for Y^X which induces the continuous convergence of nets of functions (see [3], p. 241);

3° let f be a function defined on the product $X \times T$ of topological spaces X and T with values from a topological space Y and let $\mathfrak{f}$ be the function defined on T whose value at a point t_0 is the function f_{t_0} defined by the equality $f_{t_0}(x) = f(x, t_0)$. Clearly, the continuity of f depends only on topologies in X, Y, T, and the continuity of $\mathfrak{f}$ depends on topologies in T and Y^X. Is there a topology for Y^X such that the continuity of $\mathfrak{f}$ with respect to this topology is equivalent to the continuity of f?

Known results relating to these problems may be listed as follows:

(1) A basis of k-topology for a function space Y^X (= space of all continuous functions on X to Y) consists of all sets of the form $W(C_1, \ldots, C_k;\ U_1, \ldots, U_k)$, where C_i are compact subsets of X, U_i are open subsets of Y, and

$$W(C_1, \ldots, C_k;\ U_1, \ldots, U_k) = \{f \in Y^X:\ f(C_i) \subset U_i;\ i = 1, \ldots, k\}.$$

Clearly, sets of the form $W(C; U)$ form a subbasis for k-topology.

(2) The character of a point x (in symbols: $\chi(x)$) is the least cardinal $\mathfrak{m}$ for which there is a basis of x of the power $\mathfrak{m}$. The character of a space X (in symbols: $\chi(X)$) is, by definition, the number $\sup_{x \in X} \chi(x)$.

(3) A space X is said to be $\mathfrak{m}$-almost-metrizable if there is a family $P = \{\varrho_\xi\}_{\xi \in \Xi}$ ($\overline{\overline{\Xi}} = \mathfrak{m}$) of pseudometrics on X such that $\bar{A} = \{x \in X:\ \varrho_\xi(x, A) = 0$ for each ξ in $\Xi\}$ for each $A \subset X$. It may always be assumed that $\max\{\varrho_{\xi'}, \varrho_{\xi''}\} \in P$ for each $\varrho_{\xi'}, \varrho_{\xi''}$ in P. An $\mathfrak{m}$-almost-metrizable space is of the character $\leqslant \mathfrak{m}$ (see [4]).