On the spaces of ideals of semirings

by

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1. L. Gillman (see [2]) has proved that if a structural set $\mathcal{J}$ of ideals of a ring is a Hausdorff space under Stone topology, then every prime ideal which contains the intersection of ideals in $\mathcal{J}$ is contained in at most one ideal of $\mathcal{J}$. It is easy to generalize this theorem to the case when R is a semiring (theorem 3.9). The principal result of this paper is the proof of a converse theorem for semirings R which are c-regular (1) (this class contains in particular distributive lattices, commutative rings and biregular rings) and for sets consisting exclusively of prime ideals of R. Moreover we give a few theorems on some topologies of families of sets having the finite character as well as some applications of those theorems to problems concerning spaces of ideals.

2. Let B be the set formed only of integers 0 and 1. Let B^1 be the set B with the following definition of topology: open subsets of B are $\emptyset$ (2), $\{0\}$ and $\{0,1\}$. Let B^2 be the set B with the Hausdorff topology.

We shall consider an arbitrary but fixed non-empty set R and a set $\mathcal{J}$ of subsets of R. It is known that we can treat $\mathcal{J}$ as a subset of $\underset{a\in R}{P} B_a$ where $B_a = B$ for every $a \in R$ (we assign the characteristic function $\chi_{\mathfrak{i}} \in \underset{a\in R}{P} B_a$ to each $\mathfrak{i} \in \mathcal{J}$). Let $\mathcal{J}^*$ denote the subset of $\underset{a\in R}{P} B_a$ such that $x \in \mathcal{J}^* \equiv \sum_{\mathfrak{i}\in\mathcal{J}} (x = \chi_{\mathfrak{i}})$.

Let $\mathcal{J}^1$ and $\mathcal{J}^2$ denote respectively the set $\mathcal{J}^*$ with the following definitions of topology:

1. a subset $\mathcal{I} \subset \mathcal{J}^*$ is open if and only if there exists an open subset $\mathcal{I}_1$ of $\underset{a\in R}{P} B_a^1$ (where $B_a^1 = B^1$ for every $a \in R$) such that $\mathcal{I}_1 \cap \mathcal{J}^* = \mathcal{I}$;

2. a subset $\mathcal{I} \subset \mathcal{J}^*$ is open if and only if there exists an open subset $\mathcal{I}_1$ of $\underset{a\in R}{P} B_a^2$ (where $B_a^2 = B^2$ for every $a \in R$) such that $\mathcal{I}_1 \cap \mathcal{J}^* = \mathcal{I}$.

(1) This notion will be defined later.

(2) $\emptyset$ denotes here the empty set.

The following definition is due to Gillman (see [2]):

A set $\mathcal{J}$ of subsets of R is said to be *structural* if $\mathfrak{i}_1 \cap \mathfrak{i}_2 \subset \mathfrak{i}_3$ implies $\mathfrak{i}_1 \subset \mathfrak{i}_3$ or $\mathfrak{i}_2 \subset \mathfrak{i}_3$ for every $\mathfrak{i}_1, \mathfrak{i}_2, \mathfrak{i}_3 \in \mathcal{J}$.

It is known that if a set $\mathcal{J}$ is structural, then it admits the Stone topology. More exactly, if we define for every subset $\mathcal{A}$ of the structural set $\mathcal{J}$ the closure operator

$$\overline{\mathcal{A}} = \mathop{E}_{\mathfrak{i} \in \mathcal{J}} (\mathfrak{i} \supset \bigcap_{\mathfrak{j} \in \mathcal{A}} \mathfrak{j})$$

then this operation satisfies the well-known Kuratowski axioms of general topology (*i. e.*, $\overline{\mathcal{A}+\mathcal{B}} = \overline{\mathcal{A}} + \overline{\mathcal{B}}$, $\overline{\overline{\mathcal{A}}} = \overline{\mathcal{A}}$, $\overline{\emptyset} = \emptyset$, $\mathcal{A} \subset \overline{\mathcal{A}}$ for every $\mathcal{A}, \mathcal{B} \subset \mathcal{J}$).

It is easy to verify that:

2.1. *If $\mathcal{J}$ is a structural set, then the set $\mathcal{J}$ with the Stone topology is homeomorphic with $\mathcal{J}^1$.*

Following Birkhoff and Frink (see [1]) we say that:

A property Φ of the subset $\mathfrak{i}$ of a set R is of finite character if and only if, for a set $\mathcal{F}$ of finite subsets $\mathfrak{f}$ of R and a set $\mathcal{G}$ of ordered pairs $(\mathfrak{g}, \mathfrak{f})$ such that $\mathfrak{g} \subset \mathfrak{f}$ and $\mathfrak{f} \in \mathcal{F}$, it is true that $\mathfrak{j} \in \Phi$ if and only if $(\mathfrak{j} \cap \mathfrak{f}, \mathfrak{f}) \in \mathcal{G}$ for all $\mathfrak{f} \in \mathcal{F}$.

We shall say that a set $\mathcal{J}$ is of finite character if the property $\mathfrak{i} \in \mathcal{J}$ is of finite character.

Let $\mathcal{S}(R)$ denote the set of all subsets of R. For every set $\mathcal{A} \subset \mathop{P}_{a \in R} B_a$ $(\mathcal{B} \subset \mathcal{S}(R))$ and $a \in R$ let us denote by $\mathcal{A}(a)$ $(\mathcal{B}(a))$ the set $\mathop{E}_{f \in \mathcal{A}} (f(a) = 1)$ $(\mathop{E}_{\mathfrak{i} \in \mathcal{B}} (a \in \mathfrak{i}))$.

We shall prove the following:

2.2. *A set $\mathcal{J}$ of subsets of R is of finite character if and only if $\mathcal{J}^2$ is bicompact.*

First, let us suppose that $\mathcal{J}$ is a set of finite character. Hence there exists a set $\mathcal{F}$ of finite subsets $\mathfrak{f}$ of R and a set $\mathcal{G}$ of ordered pairs $(\mathfrak{g}, \mathfrak{f})$ such that $\mathfrak{i} \in \mathcal{J}$ if and only if $(\mathfrak{i} \cap \mathfrak{f}, \mathfrak{f}) \in \mathcal{G}$ for all $\mathfrak{f} \in \mathcal{F}$. It suffices to show that $\mathcal{J}^*$ is a closed subset of $\mathop{P}_{a \in R} B_a^2$ (where $B_a^2 = B^2$ for every $a \in R$). Let $\chi_\mathfrak{i} \in \overline{\mathcal{J}}^*$ and $\mathfrak{f} \in \mathcal{F}$. We consider the set $\mathcal{N}_\mathfrak{f} = \bigcap_{a \in \mathfrak{i} \cap \mathfrak{f}} \mathcal{S}(R)^*(a) \cap \bigcap_{a \in \mathfrak{f} - \mathfrak{i}} (-\mathcal{S}(R)^*(a))$. It is a neighbourhood of $\chi_\mathfrak{i}$. Hence there exists an element $\chi_{\mathfrak{i}_1} \in \mathcal{J}^*$ such that $\chi_{\mathfrak{i}_1} \in \mathcal{N}_\mathfrak{f}$. Thus $(\mathfrak{i}_1 \cap \mathfrak{f}, \mathfrak{f}) \in \mathcal{G}$ and $\mathfrak{i} \cap \mathfrak{f} = \mathfrak{i}_1 \cap \mathfrak{f}$. Consequently $(\mathfrak{i} \cap \mathfrak{f}, \mathfrak{f}) \in \mathcal{G}$. Hence $\chi_\mathfrak{i} \in \mathcal{J}^*$.

We now suppose that $\mathcal{J}^2$ is bicompact. Hence $\mathcal{J}^*$ is a closed subset of $\mathop{P}_{a \in R} B_a^2$. Let $\mathcal{F}$ be the set of all finite subsets of R and $\mathcal{G}$ the set of all pairs $(\mathfrak{i} \cap \mathfrak{f}, \mathfrak{f})$ where $\mathfrak{f} \in \mathcal{F}$ and $\mathfrak{i} \in \mathcal{J}$. It is easy to verify that $\mathfrak{i} \in \mathcal{J}$ if and only if $(\mathfrak{i} \cap \mathfrak{f}, \mathfrak{f}) \in \mathcal{G}$ for all $\mathfrak{f} \in \mathcal{F}$. Hence $\mathcal{J}$ is a set of finite character.

As an immediate corollary to 2.2 we obtain

2.3. *If $\mathcal{J}$ is a set of finite character, then $\mathcal{J}^1$ is bicompact.*

2.4. *Let $\mathcal{J}$ be a set of finite character. The following conditions are equivalent:*

(a) *$\mathcal{J}^1$ is a Hausdorff space;*

(b) *$\mathcal{J}^*(a)$ is an open subset of $\mathcal{J}^1$ for every $a \in R$;*

(c) *$\mathcal{J}^1$ is homeomorphic with $\mathcal{J}^2$.*

First we suppose (a) and prove (b) and (c). Since $\mathcal{J}^1$ and $\mathcal{J}^2$ contain exactly the same elements, are bicompact Hausdorff spaces and are such that sets which are open in $\mathcal{J}^1$ are open in $\mathcal{J}^2$, we infer that $\mathcal{J}^1$ and $\mathcal{J}^2$ are homeomorphic. Hence every set which is open in $\mathcal{J}^2$ is open in $\mathcal{J}^1$. Thus $\mathcal{J}^*(a)$ is an open subset of $\mathcal{J}^1$.

The implications (c)→(b) and (b)→(a) are obvious.

2.5. *Let $\mathcal{J}$ be a set of finite character. The following conditions are equivalent:*

(a) *$(\mathcal{J} - \{R\})^1$ is a Hausdorff space;*

(b) *$\mathcal{J}^* - \mathcal{J}^*(a)$ is a closed subset of $\mathcal{J}^1$ for every $a \in R$;*

(c) *$(\mathcal{J} - \{R\})^*(a)$ is an open subset of $(\mathcal{J} - \{R\})^1$ for every $a \in R$;*

(d) *$(\mathcal{J} - \{R\})^1$ is homeomorphic with $(\mathcal{J} - \{R\})^2$.*

It is clear that if $\mathcal{J}$ is a set of finite character then $(\mathcal{J} - \mathcal{J}(a))^2$ and $(\mathcal{J} - \mathcal{J}(a))^1$ are bicompact. Thus (a) implies (b). The implications (b)→(c), (c)→(d), (d)→(a) are obvious.

3. A *semiring* is a set R of elements which are closed under two binary operations: addition $+$ and multiplication $\cdot$, with the following properties:

1. both the addition and the multiplication are associative;
2. the addition is commutative;
3. the addition is distributive under the multiplication: $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for every $a, b, c \in R$;
4. there exists an element 0 in R such that for every $a \in R$ we have $a + 0 = a = 0 + a$; $0a = 0 = a0$.

A subset $\mathfrak{i}$ of R is said to be an *ideal of R* provided that:

1. if $a \in \mathfrak{i}$ and $b \in \mathfrak{i}$, then $a + b \in \mathfrak{i}$;
2. if $a \in \mathfrak{i}$ and $x \in R$, then $ax \in \mathfrak{i}$ and $xa \in \mathfrak{i}$;
3. if $a + b = 0$ and $a \in \mathfrak{i}$, then $b \in \mathfrak{i}$;
4. $0 \in \mathfrak{i}$.

If $\mathfrak{a}, \mathfrak{b}$ are subsets of R, then $\mathfrak{a}\mathfrak{b}$ denotes the set $\mathop{E}_{x} (\sum_{a \in \mathfrak{a}} \sum_{b \in \mathfrak{b}} x = ab)$.

For every set $\mathfrak{b} \subset R$ let us denote by $[\mathfrak{b}]$ the intersection of all the ideals $\mathfrak{i} \subset R$ such that $\mathfrak{b} \subset \mathfrak{i}$. Clearly $[\mathfrak{b}]$ is an ideal for every $\mathfrak{b} \subset R$.

It is easy to verify that

3.1. *If* $\mathfrak{i}_1$ *and* $\mathfrak{i}_2$ *are ideals in* R, *then* $[\mathfrak{i}_1 \cup \mathfrak{i}_2] = \underset{x \in R}{E}\big(\sum_{a \in \mathfrak{i}_1} \sum_{b \in \mathfrak{i}_2} (x = a+b)\big)$.

In the sequel, the following lemma will be useful:

3.2. *If* $\mathfrak{i}$ *is an ideal in* R *and* a, b *are elements of* R *such that* $aRb \subset \mathfrak{i}$, *then* $[\{a\}]R[\{b\}] \subset \mathfrak{i}$.

Indeed it is an easy consequence of the definition of ideal that

a) if $x_1Rb \subset \mathfrak{i}$ and $x_2Rb \subset \mathfrak{i}$, then $(x_1+x_2)Rb \subset \mathfrak{i}$;

b) if $xRb \subset \mathfrak{i}$ and $y \in R$, then $xyRb \subset \mathfrak{i}$ and $yxRb \subset \mathfrak{i}$;

c) if $x_1Rb \subset \mathfrak{i}$ and $x_1+x_2 = 0$, then $x_2Rb \subset \mathfrak{i}$;

d) $0Rb \subset \mathfrak{i}$.

Hence, if $aRb \subset \mathfrak{i}$, then $[\{a\}]Rb \subset \mathfrak{i}$. A similar argument will show that if $cRb \subset \mathfrak{i}$ then $cR[\{b\}] \subset \mathfrak{i}$ for every $c \in R$. Thus $[\{a\}]R[\{b\}] \subset \mathfrak{i}$.

An ideal $\mathfrak{i} \underline{\subset} R$ is said to be prime (³) if $aRb \subset \mathfrak{i}$ implies $a \in \mathfrak{i}$ or $b \in \mathfrak{i}$.

A set $\mathfrak{m}$ of elements of R is an m-system (⁴) if and only if $0 \notin \mathfrak{m}$ and $c \in \mathfrak{m}$, $d \in \mathfrak{m}$ imply that there exists an element x of R such that $cxd \in \mathfrak{m}$.

The importance of this concept lies in the fact that an ideal $\mathfrak{i}$ in R is prime if and only if its complement in R is an m-system.

3.3. *If* $\mathfrak{i} \neq R$ *is an ideal in a semiring* R, *then the following condition are equivalent*:

(a) $\mathfrak{i}$ *is a prime ideal*;

(b) *if* $\mathfrak{i}_1$ *and* $\mathfrak{i}_2$ *are ideals in* R *such that* $\mathfrak{i}_1\mathfrak{i}_2 \subset \mathfrak{i}$, *then* $\mathfrak{i}_1 \subset \mathfrak{i}$ *or* $\mathfrak{i}_2 \subset \mathfrak{i}$.

At first let us assume (a) and suppose that $\mathfrak{i}_1$ and $\mathfrak{i}_2$ are ideals such that $\mathfrak{i}_1\mathfrak{i}_2 \subset \mathfrak{i}$ with $\mathfrak{i}_1 \not\subset \mathfrak{i}$. Let a_1 be an element of $\mathfrak{i}_1$ not in $\mathfrak{i}$. Then for every element $a_2 \in \mathfrak{i}_2$ we have $a_1Ra_2 \subset \mathfrak{i}_1\mathfrak{i}_2 \subset \mathfrak{i}$. Hence by (a) we have $a_2 \in \mathfrak{i}$. Thus $\mathfrak{i}_2 \subset \mathfrak{i}$ and we have therefore shown that (a) implies (b).

We now assume (b) and prove (a). Suppose that $aRb \subset \mathfrak{i}$; from 3.2 it follows that $[\{a\}]R[\{b\}] \subset \mathfrak{i}$, and thus $\big[[\{a\}]R\big][\{b\}] \subset \mathfrak{i}$ (the proof is similar to that of 3.2). Hence (b) implies $[\{a\}] \subset \mathfrak{i}$ or $[\{b\}] \subset \mathfrak{i}$.

Let $Q(R)$ denote the set of all prime ideals of a semiring R.

(³) This definition is an extension (without any essential change) of the well-known definition of a prime ideal of a ring; cf. [3]. It is also easy to verify that if R is a distributive lattice, then the definition given here coincides with the usual one.

(⁴) The notion of an m-system was introduced by McCoy [3] for the case of rings. Our definition is an extension of McCoy's definition to semirings.

3.4. $Q(R)$ *is a structural set.*

This statement follows immediately from 3.3.

It follows that $Q(R)$ with the Stone topology is homeomorphic with $\big(Q(R)\big)^1$.

Now we shall prove the following:

3.5 (⁵). *Let* $\mathfrak{m}$ *be an* m-*system in* R, $\mathfrak{i}_0$ *an ideal which does not meet* $\mathfrak{m}$. *Then* $\mathfrak{i}_0$ *is contained in an ideal* $\mathfrak{i}_1$ *which is maximal in the class of ideals which do not meet* $\mathfrak{m}$. *The ideal* $\mathfrak{i}$ *is necessarily a prime ideal.*

The existence of $\mathfrak{i}_1$ follows at once from Zorn's Lemma. We now show that $\mathfrak{i}_1$ is a prime ideal. Suppose that $a_1 \notin \mathfrak{i}_1$ and $a_2 \notin \mathfrak{i}_1$. Then the maximal property of $\mathfrak{i}_1$ implies that $\big[i_1 \cup [\{a\}]\big] \cap \mathfrak{m} \neq \emptyset \neq \big[i_1 \cup [\{a\}]\big] \cap \mathfrak{m}$. Thus by 3.1 there exist elements

$$m_1 = i_1 + b_1 \in \big[\mathfrak{i}_1 \cup [\{a_1\}]\big] \cap \mathfrak{m} \quad \text{and} \quad m_2 = i_2 + b_2 \in \big[\mathfrak{i}_2 \cup [\{a_2\}]\big] \cap \mathfrak{m}$$

where $i_1, i_2 \in \mathfrak{i}_1$, $b_1 \in [\{a_1\}]$, $b_2 \in [\{a_2\}]$.

Since $\mathfrak{m}$ is an m-system, there is an element x of R such that $m_1xm_2 \in \mathfrak{m}$; consequently $m_1xm_2 \notin \mathfrak{i}_1$. But $m_1xm_2 = (i_1+b_1)x(i_2+b_2) = (i_1xi_2 + i_1xb_2 + b_1xi_2) + b_1xb_2$, whence $b_1xb_2 \notin \mathfrak{i}_1$. Therefore $b_1Rb_2 \not\subset \mathfrak{i}_1$, whence by 3.2 $a_1Ra_2 \not\subset \mathfrak{i}_1$.

Let $C(R)$ denote the centre of R.

A semiring R is said to be *c-regular* if every principal ideal in R can be generated by a central element.

It follows from this definition that commutative rings, biregular rings, distributive lattices are c-regular semirings.

Let R be a c-regular semiring. Let us denote for every element $a \in R$ by c_a an element of R such that $[\{a\}] = [\{c_a\}]$ and $c_a \in C(R)$.

It is easy to verify that:

3.6. *If* $\mathfrak{i}$ *is an ideal in the arbitrary* c-*regular semiring* R, *then the following conditions are equivalent*:

(a) $\mathfrak{i}$ *is a prime ideal*;

(b) $\mathfrak{i} \neq R$ *and if* $c_1 \in C(R)$, $c_2 \in C(R)$, $c_1c_2 \in \mathfrak{i}$, *then* $c_1 \in \mathfrak{i}$ *or* $c_2 \in \mathfrak{i}$.

As an immediate consequence of 3.6 we find that:

3.7. *If* R *is a* c-*regular semiring, then the set* $\{R\} \cup Q(R)$ *is of finite character.*

It follows from 2.5. that

3.8 (⁶). *If* R *is a* c-*regular semiring, then* $\big(Q(R)\big)^1$ *is a Hausdorff space if and only if* $\big(Q(R)\big)(a)$ *is open for every* $a \in R$.

(⁵) Proofs of theorems 3.3 and 3.5 are modelled on proofs of theorems of [3].

(⁶) This theorem represents a generalization of a result given in [2], p. 8.

3.9 (7). *Let R be a semiring. Let $\mathcal{J}$ be a set of ideals of R. If $\mathcal{J}^1$ is a Hausdorff space, then each prime ideal $\mathfrak{i}_0$ such that $\mathfrak{i}_0 \supset \bigcap_{\mathfrak{j}\in\mathcal{J}} \mathfrak{j}$ is contained in at most one $\mathfrak{j} \in \mathcal{J}$.*

Let $\mathfrak{i}_1, \mathfrak{i}_2$ be different elements of $\mathcal{J}$. Let $\mathfrak{i}_0$ be an element of $\mathcal{Q}(R)$ such that $\bigcap_{\mathfrak{j}\in\mathcal{J}} \mathfrak{j} \subset \mathfrak{i}_0 \subset \mathfrak{i}_1$. If $\mathcal{J}^1$ is a Hausdorff space, then there exist two sets $\{a_1, \ldots, a_k\} \in R$, $\{b_1, \ldots, b_l\} \in R$ such that

$$\mathfrak{i}_1 \in -\mathcal{J}(a_1) \cap -\mathcal{J}(a_2) \cap \ldots \cap -\mathcal{J}(a_k), \quad \mathfrak{i}_2 \in -\mathcal{J}(b_1) \cap -\mathcal{J}(b_2) \cap \ldots \cap -\mathcal{J}(b_l),$$
$$-\mathcal{J}(a_1) \cap \ldots \cap -\mathcal{J}(a_k) \cap -\mathcal{J}(b_1) \cap \ldots \cap -\mathcal{J}(b_l) = \emptyset .$$

Hence $a_1, a_2, \ldots, a_k \notin \mathfrak{i}_1, b_1, \ldots, b_l \notin \mathfrak{i}_2$ and $\mathcal{J}(a_1) \cup \ldots \cup \mathcal{J}(a_k) \cup \mathcal{J}(b_1) \cup \ldots \cup \mathcal{J}(b_l) = \mathcal{J}$. In consequence $a_1, \ldots, a_k \notin \mathfrak{i}_0$ and $[\{a_1\}] \ldots [\{a_k\}][\{b_1\}] \ldots [\{b_l\}] \subset \bigcap_{\mathfrak{j}\in\mathcal{J}} \mathfrak{j} \subset \mathfrak{i}_0$. Since $\mathfrak{i}_0$ is prime, it follows that $[\{b_i\}] \subset \mathfrak{i}_0$ for some $1 \leqslant i \leqslant l$. Thus $\mathfrak{i}_0 \not\subset \mathfrak{i}_2$ and the theorem is therefore established.

3.10. *Let R be a c-regular semiring. If $\mathcal{J}$ is a subset of $\mathcal{Q}(R)$, then the following conditions are equivalent:*

(a) *$\mathcal{J}^1$ is a Hausdorff space;*

(b) *if $\mathfrak{i}_0$ is a prime ideal in R such that $\mathfrak{i}_0 \supset \bigcap_{\mathfrak{j}\in\mathcal{J}} \mathfrak{j}$, then there exists at most one ideal $\mathfrak{j} \in \mathcal{J}$ such that $\mathfrak{i}_0 \subset \mathfrak{j}$.*

The implication (a)→(b) follows from 3.9.

We now assume (b) and prove (a). Let $\mathfrak{i}_1$ and $\mathfrak{i}_2$ be different elements of $\mathcal{J}$. It is easy to verify that if $0 \notin (C(R)-\mathfrak{i}_1)(C(R)-\mathfrak{i}_2)$ the set

$$\mathfrak{m} = (C(R)-\mathfrak{i}_1)(C(R)-\mathfrak{i}_2) \cup (C(R)-\mathfrak{i}_1) \cup (C(R)-\mathfrak{i}_2)$$

is an m-system. If $\mathfrak{m} \cap \bigcap_{\mathfrak{j}\in\mathcal{J}} \mathfrak{j} = \emptyset$, then there exists a prime ideal $\mathfrak{i}_3$ such that $\bigcap_{\mathfrak{j}\in\mathcal{J}} \mathfrak{j} \subset \mathfrak{i}_3$ and $\mathfrak{i}_3 \cap \mathfrak{m} = \emptyset$. Hence $\mathfrak{i}_3 \cap (R-\mathfrak{i}_1) = \emptyset = \mathfrak{i}_3 \cap (R-\mathfrak{i}_2)$ (indeed, if $a \in \mathfrak{i}_3 \cap (R-\mathfrak{i}_i)$ then $c_a \in \mathfrak{i}_3 \cap (R-\mathfrak{i}_i)$ and $c_a \in \mathfrak{i}_3 \cap (C(R)-\mathfrak{i}_i)$ for $i = 1, 2$), and in consequence $\mathfrak{i}_3 \subset \mathfrak{i}_1 \cap \mathfrak{i}_2$. But there exists at most one ideal $\mathfrak{i} \in \mathcal{J}$ such that $\mathfrak{i}_3 \subset \mathfrak{i}$. Thus $\mathfrak{i}_1 = \mathfrak{i}_2$. This contradiction shows that we must have $\mathfrak{m} \cap \bigcap_{\mathfrak{j}\in\mathcal{J}} \mathfrak{j} \neq \emptyset$. Thus there exist elements a, b such that $a \in C(R)-\mathfrak{i}_1$, $b \in C(R)-\mathfrak{i}_2$ and $ab \in \bigcap_{\mathfrak{j}\in\mathcal{J}} \mathfrak{j}$. It follows by 3.6 that $a \in \mathfrak{i}$ or $b \in \mathfrak{i}$ for every $\mathfrak{i} \in \mathcal{J}$. Hence $-\mathcal{J}(a) \cap -\mathcal{J}(b) = \emptyset$. Moreover, since $a \notin \mathfrak{i}_1$, $b \notin \mathfrak{i}_2$, we infer that $\mathfrak{i}_1 \in -\mathcal{J}(a)$ and $\mathfrak{i}_2 \in -\mathcal{J}(b)$. Thus the proof is complete.

(7) This theorem represents a generalization of a result given in [2], p. 6.

Let R be a c-regular semiring. We shall consider the set $\mathcal{Q}(R)$ with the Stone topology. It easily follows by 3.10 that the following conditions are equivalent:

(a) $(\mathcal{Q}(R))^1$ is T_1;

(b) $(\mathcal{Q}(R))^1$ is a Hausdorff space;

(c) $(\mathcal{Q}(R))(a)$ is open for every $a \in R$.

Finally we note the following result:

Let $\mathcal{Q}_0(R)$ be the set of all minimal prime ideals in a semiring R. It follows by 3.10 that if R is a c-regular semiring, then $(\mathcal{Q}_0(R))^1$ is a Hausdorff space.

References

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