(vi) $\pi(S-L_n)$ is a free product of all groups $\pi(S-J_i^g)$ where $J_i^g$ denotes a component of $L_n$. This follows by theorem 1 of [9];
(vii) by (iii) and (8) all groups $\pi(S-J_i^g)$ are trivial.

References


On the convergence of nets of sets

by

S. Mrówka (Warszawa)

The topological convergence of net $(\mathcal{N})$ of subsets of a topological space $X$ may be defined in the same manner as the topological convergence of a sequence of sets: if $(A_n, n \in D)$ is a net of subsets of $X$, then $\lim A_n = (\lim A_n)$ is defined as the set of all $x \in X$ such that every neighbourhood of $x$ intersects $A_n$ for almost all (arbitrarily large) $n$. A net $(A_n, n \in D)$ is said to be topologically convergent (to a set $A$) if $\lim A_n = \lim A_n$ for each $x \in X$ and in this case the set $A$ will be denoted by $\lim A_n$.

Hausdorff ([2], p. 145) has shown that if $X$ is a compact metric space, then in the space $2^X$ consisting of all closed non-empty subsets of $X$ a metric may be defined such that the convergence of sequences of sets induced by this metric $(2)$ coincides with topological convergence. This result has been generalized by Watson [7] who has shown that if $X$ is a locally compact separable metric space then another metric may be defined in $2^X$ which induces topological convergence. Watson has also shown that if $X$ is not locally compact, then the space $2^X$ considered as a $L^*$-space (see [4], p. 89 and 274) topological convergence is not a topological space.

The present paper is devoted to generalizations of the above results. It will be shown that:

(1) A set is a function defined on a directed set $(a, b)$ ordered set $D$ is called directed if for every $a, a' \in D$ an element $a \times D$ may be found such that $a \times D$ is the relation which partially orders the set $D$. If a set defined on $D$ assigns to an element $a \times D$ an element $a$, then it will be denoted by $(a, a \times D)$ (see [3], p. 61).
(2) We say that a statement $T$ is an element of a directed set is fulfilled for every $a \in D$ if an element $a \times D$ may be found such that $T$ is fulfilled for every $a \times D$.
(3) We say that an arbitrary large $a \times D$ if the set of all $a \in D$ for which $T$ is fulfilled is cofinal with $D$.
(4) We say that a metric $\varphi$ (a topology $\mathcal{T}$) for a set $X$ induces a certain convergence of nets in $X$ of the sort if each net in $X$ of that sort is convergent with respect to this convergence if and only if it is convergent with respect to the metric $\varphi$ (the topology $\mathcal{T}$).
1. If \( X \) is a Hausdorff bicomplect space, then the Vietoris topology in \( 2^X \) (the basis of the Vietoris topology in \( 2^X \)) consists of all sets of the form \( \{ U_i, \ldots, U_n \} \), where \( U_i \) are aribtrary open sets in \( X \) and \( \bigcap_{i=1}^n U_i \neq \emptyset \) (see [6] and [5], p. 113) induces the topological convergence of nets of sets and that this is not the case if \( X \) is locally bicomplect without being bicomplect.

2. If \( X \) is a locally bicomplect space, then another topology may be defined in \( 2^X \) which induces the topological convergence of nets of sets.

3. If \( X \) is not locally bicomplect, then there exists no topology in \( 2^X \) which induces the topological convergence of nets of sets.

I. Some properties of topological limits.

1. \( \{ A_n \} \subset \{ B_n \} \) implies \( \lim A_n = \emptyset \).

This property is obvious.

2. If a net \( \{ A_n \} \) topologically converges to \( A \) and \( E \) is a cofinal subset of \( D \), then the net \( \{ A_n \cap E \} \) is also topologically convergent and \( \lim A_n = A \).

Proof. If \( x \in A \) then every neighbourhood \( U \) of \( x \) intersects \( A_n \) for almost all \( n \in D \); thus \( U \) intersects \( A_n \) for almost all \( n \in E \) and \( x \in \lim A_n \), whence \( A \subset \lim A_n \). If \( x \notin A \) then there exists a neighbourhood \( U_n \) of \( x \) that is disjoint with \( A_n \) for almost all \( n \in D \); thus \( U_n \) is disjoint with \( A_n \) for almost all \( n \in E \) and \( x \notin \lim A_n \), whence \( A \subset \lim A_n \). From \( A \subset \lim A_n \) and \( \lim A_n \subset A \) it follows that the net \( \{ A_n \cap E \} \) is topologically convergent and \( \lim A_n = A \).

3. If \( F \) is closed and \( A_n \cap F \) then \( \lim A_n \subset F \).

In fact, if \( x \in \lim A_n \), then every neighbourhood \( U \) of \( x \) intersects some \( A_n \); thus \( U \) contains points of \( F \) and \( x \in F \).

4. \( \lim A_n = \bigcap_{n \in D} \bigcup_{m \geq n} A_m \).

Proof. If \( x \in \lim A_n \), then for each \( m \in D \), and each neighbourhood \( U \) of \( x \) there is an \( n \geq m \) such that \( U \) intersects \( A_n \), whence \( U \) intersects \( \bigcup_{n \geq m} A_n \); thus \( x \in \bigcup_{n \geq m} A_n \) for each \( m \) and it follows that \( x \in \bigcap_{n \in D} \bigcup_{m \geq n} A_m \).

(*) Obviously a cofinal subset of a directed set is also a directed set; thus \( \{ A_n \} \) is, in fact, a net.

5. If \( A_n \subset \bigcap_{n \in D} \lim A_n \subset \bigcap_{n \in D} A_n \), then there is a neighbourhood \( U \) of \( x \) and \( m \geq n \) such that \( U \) intersects \( A_n \) with \( n \geq m \), whence \( x \in \bigcup_{n \geq m} A_m \).

6. If \( A_n \subset \bigcap_{n \in D} \lim A_n \) and \( x \in \lim A_n \), then there is a neighbourhood \( U \) of \( x \) and an index \( n \) such that \( U \cap A_n = 0 \) for \( n \geq n \).

If \( D \) is an arbitrary neighbourhood of \( x \), then \( U \cap A_n = 0 \) for \( n \geq n \) implies that \( U \) intersects \( A_n \) for arbitrarily large \( n \); thus \( x \in \lim A_n \) and \( \lim A_n \subset \bigcap_{n \in D} A_n \).

7. If \( A_n \subset \bigcap_{n \in D} \lim A_n \) and \( x \notin \lim A_n \), then there is a neighbourhood \( V \) of \( x \) which is disjoint with \( A_n \) for almost all \( n \) and it follows that \( x \notin \lim A_n \) and \( x \notin \lim B_n \), whence \( \lim A_n \subset \bigcap_{n \in D} \lim A_n \).

8. If \( \{ A_n \} \subset \{ B_n \} \) and \( \{ B_n \} \subset \{ C_n \} \) are topologically convergent, then the net \( \{ A_n \} \subset \{ C_n \} \) is topologically convergent and \( \lim A_n = \lim C_n \).

Proof. This is an immediate consequence of formulae 6 and 7.

II. The co-topology in \( 2^X \). In this section \( X \) is supposed to be locally bicomplect, and \( 2^X \) consists of all closed non-empty subsets of \( X \).

The basis of the co-topology in \( 2^X \) of all sets of the form \( \{ U_1 \} \) is the collection of all arbitrary open sets in \( X \) with a bicomplect closure and

\[
\{ U_1, \ldots, U_n \} = \bigcap_{i=1}^n (A \cap U_i) \neq \emptyset ; \quad \bigcap_{i=1}^n V_i = \emptyset ; \quad i = 1, \ldots, n ; \quad n = 1, \ldots, k.
\]

A basis of \( \Sigma \) is called bicomplect if for every \( U \in \Sigma \), \( \overline{U} \) is bicomplect.

Theorem 1. If \( \Sigma \) is a bicomplect basis, then the family of all \( \{ U_1, \ldots, U_n \} \), \( V_1, \ldots, V_k \) where \( U_i, V_j \in \Sigma \) is a basis for the co-topology in \( 2^X \).

Proof. Let \( A \in \{ U_1, \ldots, U_n \}; V_1, \ldots, V_k \) where \( U_i \) and \( V_j \) are arbitrary open sets with a bicomplect closure. It must be shown that there
exist \( U_{1}, \ldots, U_{n}, V_{1}, \ldots, V_{n} \in \mathcal{B} \) such that \( A \in \{ U_{1}, \ldots, U_{n}, V_{1}, \ldots, V_{n} \} \) and \( U_{i} \cap U_{j} = \emptyset \) for every \( i \neq j \). Let us take \( U_{i} \in \mathcal{B} \) such that \( U_{i} \cap U_{j} \neq \emptyset \) and \( A \cap U_{i} \neq \emptyset \). Since \( A \cap U_{i} \neq \emptyset \), there exists \( V_{i} \in \mathcal{B} \) such that \( x \in V_{i} \) and \( V_{i} \cap U_{i} \neq \emptyset \). Since \( V_{i} \cap U_{i} \neq \emptyset \), there exists a finite system \( V_{1}, \ldots, V_{n} \) such that \( V_{i} \cap U_{i} \neq \emptyset \). Obviously,

\[
A \in \{ U_{1}, \ldots, U_{n}; V_{1}, \ldots, V_{n} \} \subseteq \{ U_{1}, \ldots, U_{n}; V_{1}, \ldots, V_{n} \}.
\]

**Theorem 3.** If \( X \) is a bicomponent space, then the Vietoris topology in \( 2^{X} \) coincides with the leb-topology.

**Proof.** Let \( A \in \mathcal{C}(X) \). The set \( C \in \mathcal{C}(X) \) is bicomponent and disjoint with \( A \), whence there exists a finite system \( V_{1}, \ldots, V_{n} \) of open sets that have closures disjoint with \( A \) such that \( C \subseteq V_{1}, \ldots, V_{n} \). Obviously, \( A \in \mathcal{C}(X) \).

Conversely, let \( A \in \mathcal{C}(X) \). Setting \( U_{i} = V \setminus (V_{1} \cup \ldots \cup V_{n}) \) for all \( i = 1, \ldots, n \) and \( U_{n+1} = \mathcal{U}(V_{1} \cup \ldots \cup V_{n}) \), we have \( A \subseteq U_{1}, \ldots, U_{n}, U_{n+1} \) and \( A \in \mathcal{C}(X) \).

**Theorem 4.** The Vietoris topology in \( 2^{X} \) induces the topological convergence of nets of sets.

**Proof.** Let us agree to write \( A_{n} \rightarrow A \) if a net \( (A_{n}, n \in D) \) is convergent to \( A \) in the sense of leb-topology.

Let \( (A_{n}, n \in D) \) be a net of closed non-empty subsets of \( X \) and \( A_{n} \rightarrow A \). If \( x \in A \), then there exists a neighbourhood \( V \) of \( x \) with a bicomponent closure and disjoint with \( A \). Let \( U \) be an arbitrary neighbourhood with a bicomponent closure which intersects \( A \). Since \( A \in \mathcal{C}(X) \), \( A_{n} = \mathcal{C}(U) \) for almost all \( n \in D \) and \( A_{n} \cap V_{n} = \emptyset \) for almost all \( n \in D \) and \( x \in D \). Hence \( A_{n} \cap U_{n} \neq \emptyset \) for almost all \( n \in D \) and \( x \in D \). Thus \( A \subseteq \mathcal{C}(A_{n}) \). From \( A \subseteq A_{n} \cap D \) and \( A \subseteq \mathcal{C}(A_{n}) \), it follows that \( \lim_{n \to D} A_{n} = A \).

Let \( A_{n} \subseteq A \subseteq \mathcal{C}(A_{n}) \) and let \( \{ U_{1}, \ldots, U_{i}, V_{1}, \ldots, V_{k} \} \) be an arbitrary neighbourhood of \( A \) in \( 2^{X} \). Suppose \( A_{n} \cap U_{i} = \emptyset \) for some \( i \) (\( i = 1, \ldots, k \)) and for arbitrarily large \( n \in D \) and let \( E = \mathcal{E}(A_{n} \cap U_{i} = \emptyset) \). The set \( E \) is cofinal with \( D \), whence by the property (3), \( \lim_{n \to D} A_{n} = A \).

But \( A_{n} \subseteq A \) for all \( n \in E \) and the set \( A \) is closed, whence it follows, by the property (3), that \( A \subseteq \mathcal{C}(X) \). Hence \( A \subseteq \mathcal{C}(X) \).

**Theorem 5.** If \( X \) is a bicomponent space, then the Vietoris topology in \( 2^{X} \) coincides with the topological convergence of nets of sets.

**Proof.** Let \( X = X \cup (a) \) be the one-point bicomponentification of \( X \). Since \( X \) is not bicomponent, the point \( a \) is not isolated in \( X \), hence there
exists a net \((x_n, n \in D)\) of points of \(X\) that converges to \(a\). Let \(x_0\) be an arbitrary point of \(X\) and let \(A_n = (x_n, x_0)\). The net \((A_n, n \in D)\) is topologically convergent to \((x_0)\), but it does not converge to \((x_0)\) in the sense of the Vietoris topology.

III. Application to the theory of continua. A completely regular space is said to be a continuum if it is biocompact and connected.

**Theorem 7.** The set of all continua of a biocompact space \(X\) is closed in \(2^X\).

**Proof.** Let \(A\) be a closed non-connected subset of \(X\) and let \(A = H_1 \cup H_2\), where \(H_1, H_2\) are closed and \(H_1 \cap H_2 = \emptyset\). Then there are open sets \(U_1, U_2 \subset X\) with \(H_1 \subseteq U_1, H_2 \subseteq U_2\); \(U_1 \cap U_2 = \emptyset\). Clearly \(\langle U_1, U_2 \rangle\) is a neighbourhood of \(A\) and for each \(B \in \langle U_1, U_2 \rangle\), \(B\) is non-connected.

An immediate consequence of the preceding theorem and theorem 6 is the following:

**Theorem 8.** If \((A_n, n \in D)\) is a topologically convergent net of continua of a biocompact space, then \(\text{lim}_{n \to \infty} A_n\) is a continuum.

**Theorem 9.** If \((A_n, n \in D)\) is a monotonous family of continua (i.e., \(A \subseteq A_n\) for each \(n \in D\)), then \(\bigcap_{n \in D} A_n\) is a continuum.

**Proof.** Let us agree that \(n \leq m\) if and only if \(A_n \subseteq A_m\). Then the set \(J\) is directed. Let \(i_0\) be an arbitrary element of \(J\) and let \(I_{i_0} = \bigcap_{i \leq i_0} A_i\). By formula 5, \(\text{lim}_{n \to \infty} A_n = \bigcap_{n \in D} A_n\). On the other hand, \(A_i \subseteq A_{i_0}\) for each \(i \leq i_0\) and \(A_{i_0}\) is biocompact, whence, by theorem 7, \(\text{lim}_{n \to \infty} A_n\) is a continuum.

IV. Topological convergence of nets of sets of a non-locally biocompact space. In this section we shall prove the following theorem:

**Theorem 10.** If \(X\) is a non-locally biocompact space, then there exists no topology in \(2^X\) which induces the topological convergence of nets of sets \((X\) is supposed to be regular).

By the product \(P\) of directed sets \(D_n\), we understand the set of all functions \(f\) defined on \(I\) such that \(f(i) \in D_i\) for every \(i \in I\) partly ordered by the relation \(f \preceq g\) in \(P \times D\), if and only if \(f(i) \preceq g(i)\) in \(D_i\) for each \(i \in I\) (we write \(n \preceq n'\) in \(D\)) for elements \(n, n'\) belonging to the directed set \(D\) and only if \(n = n'\) or the element \(n\) precedes the element \(n'\) in the sense of the relation which partly orders the set \(D\).

The product of two directed sets \(D_1\) and \(D_2\) will be denoted by \(D_1 \times D_2\).

It is known that a convergence of nets of elements of a set \(X\) induced by some topology in \(X\) satisfies the following condition:

(W) Let \(D\) be a directed set and suppose that for every \(n \in D\) a directed set \(E_n\) is given, and let \(E = \bigcap_{n \in D} E_n\). Suppose that to every pair \(\langle u, v \rangle\), where \(u, v \in D\), an element \(x_{u,v} \in X\) is assigned. If, for every \(n \in D\), the net \((x_{u,v}, u, v \in D)\) is convergent to \(x_n\) and the net \((x_{u,v}, n \in D)\) is convergent to \(x\), then the net \((x_{u,v}, (u, v) \in D \times E)\) where \(y_{u,v} = x_{u,v} \cap E\) is convergent to \(x\) (see [3], p. 69).

To prove Theorem 10 we shall show that the topological convergence of closed subsets of \(X\) does not satisfy the condition (W).

**Lemma 1.** For every two directed sets \(D_1\) and \(D_2\) there exist a directed set \(D\) and functions \(q_1\) and \(q_2\) such that \(q_i\) maps \(D\) onto \(D_i\) and \(q_i(n) \preceq q_i(n')\) in \(D_i\) for every \(n \subseteq n'\) in \(D\) (i = 1, 2).

**Proof.** Let \(D = D_1 \times D_2\) and \(q_i(n) = n_i\) for \(n = \langle n_1, n_2 \rangle\in D\) and \(i = 1, 2\).

**Lemma 2.** If \((y_n, n \in D)\) is a net of points of a topological space \(X\) which has no cluster point and \(q\) is a function which maps a directed set \(D\) onto \(D_i\) in such a way that \(q(n) \preceq q(n')\) in \(D_i\) for every \(n \subseteq n'\) in \(D\), then \(\text{lim}_{n \to \infty} q(n)\) in \(D\) where \(X_{\text{cl}}\) or \(\text{lim}_{n \to \infty} q(n)\) also has no cluster point.

**Proof.** Let \(x\) be an arbitrary point of \(X\). Since \(x\) is not a cluster point of \((y_n, n \in D)\), there exist a neighbourhood \(U\) of \(x\) and an element \(k_n \in D\) such that \(y_n \not\in U\) for every \(k \geq k_n \in D\). If \(n_0\) is an element of \(D\) such that \(q(n_0) = k_n\), then clearly \(x_n \in U\) for every \(n \geq n_0\) in \(D\); thus \(x\) is not a cluster point of \((x_n, n \in D)\).

Now we shall show that if \(X\) is a regular non-locally biocompact space, then the topological convergence of nets of closed subsets of \(X\) does not satisfy the condition (W).

Let \(x_0\) be a point of \(X\) which has no neighbourhood with a biocompact closure and let \(D_0\) be a basis of neighbourhoods of \(x_0\). Let us agree that \(U \subseteq U'\) in \(D_0\) if \(U \subseteq U'\). Then \(D_0\) is a directed set. Let \(x_0\) be an arbitrary point of \(X\) which is different from \(x_0\) and let \((y_n, n \in D_0)\) be a net of elements of \(X\) which has no cluster point. By Lemma 1 there exist a directed set \(D\) and functions \(q_1\) and \(q_2\) such that \(q_1(n) \subseteq D_1\) and \(q_2(n) \subseteq D_2\) for every \(n \subseteq n'\) in \(D\) (i = 1, 2). Let us set \(U_n = q_1(n)\) and \(U_n = q_2(n)\) for every \(n \in D\). Then \(U_n \subseteq U_n'\) for every \(n \subseteq n'\) in \(D\).

By Lemma 2, the net \((x_n, n \in D_0)\) has no cluster point. Since \(U_n\) is not biocompact, there exists a net \((z_{n}, n \in E_0)\) of elements of \(U_n\) which has no cluster point. Let us set \(x_{n} = x_0 = x_{n} \circ (z_{n})\) for every \(n \in D_0\). The net \((x_{n}, n \in E_0)\) is topologically convergent to \(x_0\) for every \(n \in D_0\) (since the net \((z_{n}), n \in E_0)\) has no cluster point) and
the net \((A_n, \tau \in D)\) is topologically convergent to \(A = (a)\) (since the net \((x_n, y \in D)\) has no cluster point). Let \(E = P \cap D\), and let us consider the net \((A_{x_0}, \tau, y \in D \times \{x\})\), where \(A_{x_0, y} = A_{x_0, y_0}\). If \(U\) is an arbitrary neighborhood of \(a\) and \(U \cap U \neq \emptyset\), then \(A_{x_0, y} \cap U \neq \emptyset\), \(\tau \in D \times \{x\}\) for every \(\tau, y \in D \times \{x\}\). Then \(A_{x_0, y} \cap U \neq \emptyset\), \(\tau \in D \times \{x\}\) for every \(\tau, y \in D \times \{x\}\). Hence, \(x_0 \in D \times \{x\}\) and the net \((A_{x_0, y}, \tau, y \in D \times \{x\})\) is not topologically convergent to \(A\).

Remark. It is interesting to compare Theorem 8 with a result of Watson, who states that if \(X\) is a separable metric non-locally biconnected space, then the space \(2^X\) considered as a \(L^*\)-space with the topological convergence of sequences of sets is not a topological space. Let us consider the following examples:

**Example 1.** Let \(X\) be the unit (closed) interval \((0, 1)\) with a discrete topology. Then \(2^X\) consists of all non-empty subsets of \(X\). By Theorem 5, the \(L^*\)-topology in \(2^X\) induces the topological convergence of sequences of sets (a set with a discrete topology is clearly a locally biconnected space), whence, in particular, the \(L^*\)-topology induces the topological convergence of sequences of sets. On the other hand, the space \(2^X\) considered as a \(L^*\)-space is not a topological space. In fact, the topological upper and lower limits of sequences of sets coincide respectively with the upper and lower limits in the sense of the set-theory (i.e., \(\lim A_n = \bigcup_{n=1}^{\infty} A_n\), \(\lim A_n = \bigcap_{n=1}^{\infty} A_n\)). Let \(X\) be a subset of \(2^X\) consisting of all closed non-empty subsets of \(X\) with the ordinary topology and let \(A\) be a \(G_\delta\)-set which is not an \(F_\sigma\)-set. There exists a sequence \(G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots\) such that \(A = \bigcap_{n=1}^{\infty} G_n\) and for every \(n\) there exists a sequence \(A_n \subset C \subset A_2 \subset \ldots \subset A_n \subset \ldots\) of sets of \(X\) such that \(G_n = \bigcup_{m=1}^{\infty} A_m\).

We have \(G_n = \lim A_n\) and \(A = \lim G_n\), i.e., \(A \in [X_\{\}]\). \([X_\{\}]\) denotes the limit-closure of a set \(X \subset 2^X\). But \(A \in [X_\{\}]\). In fact, \(A \in [X_\{\}]\) implies the existence of a sequence \((B_n)\) of sets of \(X\) such that \(A = \lim B_n\), i.e., \(A = \bigcup_{n=1}^{\infty} B_n\). But the set \(C_n = \bigcap_{n=1}^{\infty} B_n\) belongs to \(X\) and it follows that \(A\) is an \(F_\sigma\)-set. Thus the space \(2^X\) considered as a \(L^*\)-space is not a topological space but there exists a topology in \(2^X\) which induces the topological convergence of arbitrary nets of sets, in particular there exists a topology in \(2^X\) which induces the topological convergence of sequence of sets.

**Example 2.** Let \(X\) be a set of the power of the continuum and a point which does not belong to \(X\). Let us introduce a topology in the set \(X = X \cup \{a\}\) taking as a basis of neighborhoods the family consisting of all the one-point sets \(\{x\}\) where \(x \in X\) and all the sets of the form \(X \setminus S\) where \(S\) is an arbitrary enumerable subset of \(X\). Then, as may easily be verified, the equalities \(L_{A_n} = L_{A_n}\) and \(L_{A_n} = L_{A_n}\) hold for every infinite sequence \((A_n)\) of closed subsets of \(X\) \((L_{A_n}\) and \(L_{A_n}\) denote, respectively, the upper and the lower limits of the sequence \((A_n)\) in the sense of the set-theory). Let \(A = \bigcup_{n=1}^{\infty} A_n\). Then \(L_{A_n} = L_{A_n}\) and \(L_{A_n} = L_{A_n}\) hold for every sequence of subsets of \(Y\). But a discrete space is locally biconnected and it follows that the convergence of sequences of sets induced by the \(L^*\)-topology in \(2^X\) coincides with convergence in the sense of the set-theory. But \(X\) is not locally biconnected and it follows by Theorem 10 that there exists a topology in \(2^X\) which induces the topological convergence of arbitrary nets of sets. One may also shown (in the same way as in Example 1) that the space \(2^X\) considered as a \(L^*\)-space is not a topological space.

The following problem arises:

**Problem.** Suppose that \(2^X\) considered as a \(L^*\)-space is a topological space. Does the sequential topology in \(2^X\) induce the topological convergence of arbitrary nets of sets? It is true if \(X\) is separable metric. In fact, in this case, if \(2^X\) considered as a \(L^*\)-space is a topological space, then \(X\) is locally biconnected, whence the \(L^*\)-topology in \(2^X\) induces the topological convergence of nets of sets. But \(X\) being separable metric implies that \(2^X\) with a \(L^*\)-topology is first countable, and it follows that the \(L^*\)-topology coincides with the sequential topology, whence the sequential topology in \(2^X\) induces the topological convergence of arbitrary nets of sets.

**References**


Fundamenta Mathematicae, T. XLV.
On the spaces of ideals of semirings

by

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1. L. Gillman (see [2]) has proved that if a structural set \( \mathcal{J} \) of ideals of a ring is a Hausdorff space under Stone topology, then every prime ideal which contains the intersection of ideals in \( \mathcal{J} \) is contained in at most one ideal of \( \mathcal{J} \). It is easy to generalize this theorem to the case when \( R \) is a semiring (theorem 3.9). The principal result of this paper is the proof of a converse theorem for semirings \( R \) which are \( c \)-regular (1) (this class contains in particular distributive lattices, commutative rings and biregular rings) and for sets consisting exclusively of prime ideals of \( R \). Moreover we give a few theorems on some topologies of families of sets having the finite character as well as some applications of those theorems to problems concerning spaces of ideals.

2. Let \( B \) be the set formed only of integers 0 and 1. Let \( B^* \) be the set \( B \) with the following definition of topology: open subsets of \( B \) are \( \emptyset \) (1), \( \{0\} \) and \( \{0, 1\} \). Let \( B^* \) be the set \( B \) with the Hausdorff topology.

We shall consider an arbitrary but fixed non-empty set \( R \) and a set \( \mathcal{J} \) of subsets of \( R \). It is known that we can treat \( \mathcal{J} \) as a subset of \( \mathcal{P} B_\alpha \) where \( B_\alpha = B \) for every \( \alpha \in R \) (we assign the characteristic function \( \chi_\alpha \in \mathcal{P} B_\alpha \) to each \( \alpha \in R \)). Let \( \mathcal{J}^* \) denote the subset of \( \mathcal{P} B_\alpha \) such that \( x \in \mathcal{J}^* \) if and only if \( x = \sum_{\alpha \in R} (s = \chi_\alpha) \).

Let \( \mathcal{J}^\mathcal{F} \) and \( \mathcal{J}^\mathcal{F}^* \) denote respectively the set \( \mathcal{J}^* \) with the following definitions of topology:

1. a subset \( \mathcal{J} \subset \mathcal{J}^* \) is open if and only if there exists an open subset \( \mathcal{J}_1 \) of \( \mathcal{P} B_\alpha \) (where \( B_\alpha = B^* \) for every \( \alpha \in R \)) such that \( \mathcal{J}_1 \subset \mathcal{J} = \mathcal{J}^\mathcal{F} \); 
2. a subset \( \mathcal{J} \subset \mathcal{J}^* \) is open if and only if there exists an open subset \( \mathcal{J}_1 \) of \( \mathcal{P} B_\alpha \) (where \( B_\alpha = B^* \) for every \( \alpha \in R \)) such that \( \mathcal{J}_1 \subset \mathcal{J}^* = \mathcal{J}^\mathcal{F}^* \).

(1) This notion will be defined later.
(2) \( \emptyset \) denotes here the empty set.