On the convergence of nets of sets

by

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The topological convergence of net [1] of subsets of a topological space X may be defined in the same manner as the topological convergence of a sequence of sets: if $\{A_n, n \in D\}$ is a net of subsets of X, then $\mathrm{Li}\,A_n$ ($\mathrm{Ls}\,A_n$) is defined as the set of all $x \in X$ such that every neighbourhood of x intersects A_n for almost all (arbitrarily large) n [2]. A net $\{A_n, n \in D\}$ is said to be topologically convergent (to a set A) if $\mathrm{Ls}\,A_n = \mathrm{Li}\,A_n$ ($= A$) and in this case the set A will be denoted by $\mathrm{Lim}\,A_n$.

Hausdorff ([2], p. 145) has shown that if X is a compact metric space, then in the space 2^X consisting of all closed non-empty subsets of X a metric may be defined such that the convergence of sequences of sets induced by this metric [3] coincides with topological convergence. This result has been generalized by Watson [7] who has shown that if X is a locally compact separable metric space then another metric may be defined in 2^X which induces topological convergence. Watson has also shown that if X is not locally compact, then the space 2^X considered as a L^*-space (see [4], p. 89 and p. 274) topological convergence is not a topological space.

The present paper is devoted to generalizations of the above results. It will be shown that:

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[1] *A net* is a function defined on a directed set (a partially ordered set D is called *directed* if for every $n_1, n_2 \in D$ an element $n \in D$ may be found such that $n_1 \prec n$, $n_2 \prec n$, where $\prec$ is the relation which partially orders the set D). If a net defined on D assigns to an element $n \in D$ an element x_n, then it will be denoted by $\{x_n, n \in D\}$ (see [3], p. 65).

[2] We say that a statement T on elements of a directed set D is fulfiled for *almost all* $n \in D$ if an element $n_0 \in D$ may be found such that T is fulfiled for every $n \succ n_0$;

arbitrarily large $n \in D$ if the set of all $n \in D$ for which T is fulfiled is cofinal with D.

[3] We say that a metric ϱ (a topology $\mathcal{T}$) for a set X *induces* a certain convergence of nets in X of some sort if each net in X of that sort is convergent with respect to this convergence if and only if it is convergent with respect to the metric ϱ (the topology $\mathcal{T}$).

1. If X is a Hausdorff bicompact space, then the Vietoris topology in 2^X (the basis of the Vietoris topology in 2^X consists of all sets of the form $\langle U_1, \dots, U_n\rangle$, where U_i are arbitrary open sets in X and $\langle U_1, \dots, U_n\rangle = \underset{A\in 2^X}{E}(A \subset \bigcup_{i=1}^{n} U_i;\ A \cap U_i \neq 0;\ i = 1, \dots, n)$ (see [6] and [5], p. 153) induces the topological convergence of nets of sets and that this is not the case if X is locally bicompact without being bicompact.

2. If X is a locally bicompact space, then another topology may be defined in 2^X which induces the topological convergence of nets of sets.

3. If X is not locally bicompact, then there exists no topology in 2^X which induces the topological convergence of nets of sets.

I. Some properties of topological limits.

1. $\underset{n\in D}{\mathrm{Li}}\, A_n \subset \underset{n\in D}{\mathrm{Ls}}\, A_n$.

This property is obvious.

2. *If a net $\{A_n, n \in D\}$ topologically converges to A and E is a cofinal subset of D, then the net $\{A_n, n \in E\}$ (⁴) is also topologically convergent and* $\underset{n\in E}{\mathrm{Lim}}\, A_n = A$.

Proof. If $x \in A$ then every neighbourhood U of x intersects A_n for almost all $n \in D$; thus U intersects A_n for almost all $n \in E$ and $x \in \underset{n\in E}{\mathrm{Li}}\, A_n$, whence $A \subset \underset{n\in E}{\mathrm{Li}}\, A_n$. If $x \notin A$ then there exists a neighbourhood U_0 of x that is disjoint with A_n for almost all $n \in D$; thus U_0 is disjoint with A_n for almost all $n \in E$ and $x_0 \notin \underset{n\in E}{\mathrm{Ls}}\, A_n$, whence $\underset{n\in E}{\mathrm{Ls}}\, A_n \subset A$. From $A \subset \underset{n\in E}{\mathrm{Li}}\, A_n$ and $\underset{n\in E}{\mathrm{Ls}}\, A_n \subset A$ it follows that the net $\{A_n, n \in E\}$ is topologically convergent and $\underset{n\in E}{\mathrm{Lim}}\, A_n = A$.

3. *If F is closed and $A_n \subset F$ for all $n \in D$, then $\underset{n\in D}{\mathrm{Ls}}\, A_n \subset F$.*

In fact, if $x \in \mathrm{Ls}\, A_n$, then every neighbourhood U of x intersects some A_n; thus U contains points of F and $x \in F$.

4. $\underset{n\in D}{\mathrm{Ls}}\, A_n = \bigcap_{m\in D} \overline{\bigcup_{n\geqslant m} A_n}$.

Proof. If $x \in \underset{n\in D}{\mathrm{Ls}}\, A_n$, then for each $m \in D$ and each neighbourhood U of x there is an $n \geqslant m$ such that U intersects A_n, whence U intersects $\bigcup_{n\geqslant m} A_n$; thus $x \in \overline{\bigcup_{n\geqslant m} A_n}$ for each m and it follows that $x \in \bigcap_{m\in D} \overline{\bigcup_{n\geqslant m} A_n}$. If $x \notin \mathrm{Ls}\, A_n$, then there is a neighbourhood U_0 of x and $m_0 \in D$ such that U_0 intersects no A_n with $n \geqslant m_0$, whence $x \notin \overline{\bigcup_{n\geqslant m_0} A_n}$.

(⁴) Obviously a cofinal subset of a directed set is also a directed set; thus $\{A_n, n \in E\}$ is, in fact, a net.

5. *If $A_n \subset A_{n'}$ for $n, n' \in D$; $n \leqslant n'$, then the net $\{A_n, n \in D\}$ is topologically convergent and* $\underset{n\in D}{\mathrm{Lim}}\, A_n = \bigcap_{n\in D} \overline{A}_n$.

Proof. Clearly, $\bigcap_{n\in D} \overline{A}_n \subset \underset{n\in D}{\mathrm{Li}}\, A_n$. On the other hand, from formula 4 follows $\underset{n\in D}{\mathrm{Ls}}\, A_n \bigcup_{m\in D} = \overline{A}_m$.

6. $\underset{n\in D}{\mathrm{Ls}}\,(A_n \cup B_n) = \underset{n\in D}{\mathrm{Ls}}\, A_n \cup \underset{n\in D}{\mathrm{Ls}}\, B_n$.

Proof. If $x \in \underset{n\in D}{\mathrm{Ls}}\,(A_n \cup B_n)$ and $x \notin \underset{n\in D}{\mathrm{Ls}}\, A_n$, then there are a neighbourhood U_0 of x and an index n_0 such that $U_0 \cap A_n = 0$ for $n \geqslant n_0$. If U is an arbitrary neighbourhood of x, then $U_0 \cap U$ intersects $A_n \cup B_n$ for arbitrarily large n. But $U_0 \cap A_n = 0$ for $n \geqslant n_0$ implies that U intersects B_n for arbitrarily large n; thus $x \in \underset{n\in D}{\mathrm{Ls}}\, B_n$ and $\underset{n\in D}{\mathrm{Ls}}\,(A_n \cup B_n) \subset \underset{n\in D}{\mathrm{Ls}}\, A_n \cup \underset{n\in D}{\mathrm{Ls}}\, B_n$. If $x \notin \underset{n\in D}{\mathrm{Ls}}\,(A_n \cup B_n)$, then there is a neighbourhood V of x which is disjoint with $A_n \cup B_n$ for almost all n and it follows that $x \notin \underset{n\in D}{\mathrm{Ls}}\, A_n$ and $x \notin \underset{n\in D}{\mathrm{Ls}}\, B_n$, whence $\underset{n\in D}{\mathrm{Ls}}\,(A_n \cup B_n) \supset \underset{n\in D}{\mathrm{Ls}}\, A_n \cup \underset{n\in D}{\mathrm{Ls}}\, B_n$.

7. $\underset{n\in D}{\mathrm{Li}}\,(A_n \cup B_n) \subset \underset{n\in D}{\mathrm{Li}}\, A_n \cup \underset{n\in D}{\mathrm{Li}}\, B_n$.

Proof. $A_n \subset A_n \cup B_n$ implies $\underset{n\in D}{\mathrm{Li}}\, A_n \subset \underset{n\in D}{\mathrm{Li}}\,(A_n \cup B_n)$.

8. *If nets $\{A_n, n \in D\}$ and $\{B_n, n \in D\}$ are topologically convergent, then the net $\{A_n \cup B_n, n \in D\}$ is topologically convergent and*

$$\underset{n\in D}{\mathrm{Lim}}\,(A_n \cup B_n) = \underset{n\in D}{\mathrm{Lim}}\, A_n \cup \underset{n\in B}{\mathrm{Lim}}\, B_n.$$

Proof. This is an immediate consequence of formulae 6 and 7.

II. lbc-topology in 2^X.

In this section X is supposed to be locally bicompact, and 2^X consists of all closed non-empty subsets of X. The basis of the lbc-*topology* in 2^X of all sets of the form $[U_1, \dots, U_n; V_1, \dots, V_k]$ where U_i and V_j are arbitrary open sets in X with a *bicompact closure* and

$$[U_1, \dots, U_n;\ V_1, \dots, V_j] = \underset{A\in 2^X}{E}(A \cap U_i \neq 0;\ A \cap \overline{V}_j = 0;\ i = 1, \dots, n;\ j = 1, \dots, k),$$

A basis $\mathfrak{B}$ of X is called *bicompact* if for every $U \in \mathfrak{B}$, $\overline{U}$ is bicompact.

THEOREM 1. *If $\mathfrak{B}$ is a bicompact basis, then the family of all $[U_1, \dots, U_n; V_1, \dots, V_k]$ where $U_i, V_j \in \mathfrak{B}$ is a basis for the* lbc-*topology in* 2^X.

Proof. Let $A \in [U_1, \dots, U_n; V_1, \dots, V_k]$ where U_i and V_j are arbitrary open sets with a *bicompact closure*. It must be shown that there exist $U_1^*, \ldots, U_n^*, V_1^*, \ldots, V_s^* \in \mathfrak{B}$ such that $A \in [U_1^*, \ldots, U_n^*; V_1^*, \ldots, V_s^*] \subset [U_1, \ldots, U_n; V_1, \ldots, V_k]$. Let us take $U_i^* \in \mathfrak{B}$ such that $U_i^* \subset U_i$ and $A \cap U_i^* \neq 0$ $(i = 1, \ldots, n)$. Since $A \cap \overline{V}_j = 0$, for every $x \in \overline{V}_j$ there exists $V_j^x \in \mathfrak{B}$ such that $x \in V_j^x$ and $\overline{V_j^x}$ is disjoint with A. Since $\overline{V}_j \subset \bigcup_{x \in \overline{V}_j} V_j^x$ and $\overline{V}_j$ is bicompact, there exists a finite system $V_j^{x_1}, \ldots, V_j^{x_{r_j}}$ such that $\overline{V}_j \subset V_j^{x_1} \cup \ldots \cup V_j^{x_{r_j}}$. Obviously

$$A \in [U_1^*, \ldots, U_n^*; V_1^{x_1}, \ldots, V_1^{x_{r_1}}, V_2^{x_1}, \ldots, V_2^{x_{r_2}}, \ldots, V_k^{x_1}, \ldots, V_k^{x_{r_k}}] \subset [U_1, \ldots, U_n; V_1, \ldots, V_k].$$

THEOREM 2. *If X is a bicompact space, then the Vietoris topology in 2^X coincides with the* lbc-*topology.*

Proof. Let $A \in \langle U_1, \ldots, U_n \rangle$. The set $C = X \setminus (U_1 \cup \ldots \cup U_n)$ is bicompact and disjoint with A, whence there exists a finite system $V_1, \ldots, V_k$ of open sets that have closures disjoint with A such that $C \subset V_1 \cup \ldots \cup V_k$. Obviously $A \in [U_1, \ldots, U_n; V_1, \ldots, V_k] \subset \langle U_1, \ldots, U_n \rangle$.

Conversely, let $A \in [U_1, \ldots, U_n; V_1, \ldots, V_k]$. Setting $U_i^* = U_i \setminus (\overline{V}_1 \cup \ldots \cup \overline{V}_k)$ $(i = 1, \ldots, n)$ and $U_{n+1}^* = X \setminus (\overline{V}_1 \cup \ldots \cup \overline{V}_k)$ we have $A \in \langle U_1^*, \ldots, U_n^*, U_{n+1}^* \rangle \subset [U_1, \ldots, U_n; V_1, \ldots, V_k]$.

THEOREM 3. *The space 2^X with a* lbc-*topology is locally bicompact.*

Proof. Let $X_1 = X \cup \{a\}$ be the Aleksandroff one-point bicompactification of the space X (see [1] and [3], p. 150). The space $\mathfrak{X}_1 = 2^{X_1}$ with the Vietoris topology is bicompact (see [5], p. 161). Let $\mathfrak{X}'$ be the set of all $A \in \mathfrak{X}_1$ such that $a \in A$ and $\mathfrak{X}_2 = \mathfrak{X}' \setminus \{\{a\}\}$. The set $\mathfrak{X}'$ is closed in $\mathfrak{X}_1$, whence it is bicompact and it follows that $\mathfrak{X}_2$ is locally bicompact. We shall show that 2^X (with a lbc-topology) is homeomorphic to $\mathfrak{X}_2$. Let h be the mapping that assigns to a set $A \in 2^X$ the set $A \cup \{a\}$. Obviously, h is a one-to-one mapping and $h(2^X) = \mathfrak{X}_2$ (if A is a closed subset of X, then $A \cup \{a\}$ is a closed subset of X_1). Let $\langle U_1, \ldots, U_n \rangle$ be an arbitrary neighbourhood of $A \cup \{a\}$ in $\mathfrak{X}_2$ and let $U_{i_1}, \ldots, U_{i_s}$ denote all the sets among $U_1, \ldots, U_n$ that do not contain a. For every i_r $(r = 1, \ldots, s)$ there exists an open set U_r^* of X with a bicompact closure in X such that $A \cap U_r^* \neq 0$ and $U_r^* \subset U_{i_r}$. Let $C = X_1 \setminus (U_1 \cup \ldots \cup U_n)$. We have $a \notin C$, whence $C \subset X$. Since C is bicompact and $A \cap C = 0$, there exists a finite system $V_1, \ldots, V_k$ of open sets of X having bicompact closures in X such that $C \subset V_1 \cup \ldots \cup V_k$ and $A \cap \overline{V}_j = 0$ $(j = 1, \ldots, k)$. It follows that $U = [U_1^*, \ldots, U_s^*; V_1, \ldots, V_k]$ is a neighbourhood of A in 2^X and $h(\mathfrak{U}) \subset \langle U_1, \ldots, U_n \rangle$. Conversely, if $\mathfrak{U} = [U_1, \ldots, U_n; V_1, \ldots, V_k]$ is an arbitrary neighbourhood of A in 2^X, then setting $U_i^* = U_i \setminus (\overline{V}_1 \cup \ldots \cup \overline{V}_k)$ $(i = 1, \ldots, n)$ and $U_{n+1}^* = X_1 \setminus (\overline{V}_1 \cup \ldots \cup \overline{V}_k)$ and $\mathfrak{U}^* = \langle U_1^*, \ldots, U_n^*, U_{n+1}^* \rangle$ we see that U^* is a neighbourhood of $A \cup \{a\}$ in $\mathfrak{X}_2$ and $h^{-1}(\mathfrak{U}^*) \subset \mathfrak{U}$.

THEOREM 4. lbc-*topology in 2^X induces the topological convergence of nets of sets.*

Proof. Let us agree to write $A_n \to A$ if a net $\{A_n, n \in D\}$ is convergent to A in the sense of lbc-topology.

Let $\{A_n, n \in D\}$ be a net of closed non-empty subsets of X and let $A_n \to A$. If $x \notin A$ then there exists a neighbourhood V_1 of x with a bicompact closure and disjoint with A. Let U_1 be an arbitrary neighbourhood with a bicompact closure which intersects A. Since $A \in [U_1; V_1]$, $A_n \in [U_1; V_1]$ for almost all $n \in D$ and $A_n \cap \overline{V}_1 = 0$ for almost all $n \in D$ and $x \notin \operatorname{Ls}_{n \in D} A_n$. Hence $\operatorname{Ls}_{n \in D} A_n \subset A$. Let $x \in A$ and let U be an arbitrary neighbourhood of x. There exists a neighbourhood U_1 of x with a bicompact closure that is contained in U. Since $A \in [U_1;]$, $A_n \in [U_1;]$ for almost all $n \in D$ and $A_n \cap U_1 \neq 0$, whence $A_n \cap U \neq 0$ for almost all $n \in D$ and $x \in \operatorname{Li}_{n \in D} A_n$. Thus $A \subset \operatorname{Li}_{n \in D} A_n$. From $\operatorname{Ls}_{n \in D} A_n \subset A$ and $A \subset \operatorname{Li}_{n \in D} A_n$ it follows that $\operatorname{Lim}_{n \in D} A_n = A$.

Let $\operatorname{Ls}_{n \in D} A_n \subset A \subset \operatorname{Li}_{n \in D} A_n$ and let $[U_1, \ldots, U_s; V_1, \ldots, V_k]$ be an arbitrary neighbourhood of A in 2^X. Suppose $A_n \cap U_i = 0$ for some i $(i = 1, \ldots, s)$ and for arbitrarily large $n \in D$ and let $E = \underset{n \in D}{E}(A_n \cap U_i = 0)$. The set E is cofinal with D, whence, by the property 2, $\operatorname{Lim}_{n \in E} A_n = A$. But $A_n \subset X \setminus U_i$ for all $n \in E$ and the set $X \setminus U_i$ is closed, whence it follows, by the property 3, that $A \subset X \setminus U_i$, and this contradicts $A \cap U_i \neq 0$. Hence $A_n \cap U_i \neq 0$ for all i $(i = 1, \ldots, s)$ and for almost all $n \in D$. Suppose $A \cap \overline{V}_j \neq 0$ for some j $(j = 1, \ldots, k)$ and for arbitrarily large $n \in D$. Let $E_1 = \underset{n \in D}{E}(A_n \cap \overline{V}_j \neq 0)$. The set E_1 is cofinal with D, whence $\operatorname{Lim}_{n \in E_1} A_n = A$. For every $n \in E_1$, there exists a point x_n that belongs to $A_n \cap \overline{V}_j$ and for $\overline{V}_j$ is bicompact; there exists a point x_0 that is a cluster point (see [3], p. 71) of the net $\{x_n, n \in E_1\}$. Obviously every neighbourhood of x_0 intersects A_n for arbitrarily large $n \in E_1$, whence $x_0 \in \operatorname{Ls}_{n \in E_1} A_n = \operatorname{Lim}_{n \in E_1} A_n = A$. On the other hand, $x_0 \in \overline{V}_j$, which contradicts $A \cap \overline{V}_j = 0$. Thus $A_n \cap \overline{V}_j = 0$ for all j $(j = 1, \ldots, k)$ and for almost all $n \in D$. Finally, $A_n \in [U_1, \ldots, U_s; V_1, \ldots, V_k]$ for almost all $n \in D$ and $A_n \to A$.

From Theorems 2 and 4 follows

THEOREM 5. *If X is a bicompact space, then the Vietoris topology in 2^X induces the topological convergence of nets of sets.*

THEOREM 6. *If X is a locally bicompact space without being bicompact, then the Vietoris topology in 2^X does not induce the topological convergence of nets of sets.*

Proof. Let $X_1 = X \cup \{a\}$ be the one-point bicompactification of X. Since X is not bicompact, the point a is not isolated in X_1, whence there exists a net $\{x_n,\ n \in D\}$ of points of X that converges to a. Let x_0 be an arbitrary point of X and let $A_n = \{x_0, x_n\}$. The net $\{A_n,\ n \in D\}$ is topologically convergent to $\{x_0\}$, but it does not converge to $\{x_0\}$ in the sense of the Vietoris topology.

III. Application to the theory of continua. A completely regular space is said to be *a continuum* if it is bicompact and connected.

THEOREM 7. *The set of all continua of a bicompact space X is closed in 2^X.*

Proof. Let A be a closed non-connected subset of X and let $A = H_1 \cup H_2$, where H_1, H_2 are closed and $H_1 \cap H_2 = 0$. Then there are open sets $U_1, U_2 \subset X_X$ with $H_1 \subset U_1$, $H_2 \subset U_2$; $U_1 \cap U_2 = 0$. Clearly $\langle U_1, U_2\rangle$ is a neighbourhood of A and for each $B \in \langle U_1; U_2\rangle$, B is non-connected.

An immediate consequence of the preceding theorem and theorem 6 is the following:

THEOREM 8. *If $\{A_n,\ n \in D\}$ is a topologically convergent net of continua of a bicompact space, then $\operatorname{Lim}_{n \in D} A_n$ is a continuum.*

THEOREM 9. *If $\{A_\iota\}_{\iota \in I}$ is a monotone family of continua (i. e., $A_\iota \subset A_{\iota'}$ or $A \supset A_{\iota'}$ for each $\iota, \iota' \in I$), then $\bigcap_{\iota \in I} A_\iota$ is a continuum.*

Proof. Let us agree that $\iota \leqslant \iota'$ in I if and only if $A_\iota \supset A_{\iota'}$. Then the set I is directed. Let ι_0 be an arbitrary element of I and let $I_0 = \underset{\iota \in I}{E}(\iota \geqslant \iota_0)$. By formula 5, $\operatorname{Lim}_{\iota \in I_0} A_\iota = \bigcap_{\iota \in I} A_\iota = \bigcap_{\iota \in I_0} A_\iota$. On the other hand, $A_\iota \subset A_{\iota_0}$ for each $\iota \in I_0$, and A_{ι_0} is bicompact, whence, by theorem 7, $\operatorname{Lim}_{\iota \in I_0} A_\iota$ is a continuum.

IV. Topological convergence of nets of sets of a non-locally bicompact space. In this section we shall prove the following theorem:

THEOREM 10. *If X is a non-locally bicompact space, then there exists no topology in 2^X which induces the topological convergence of nets of sets (X is supposed to be regular).*

By the product $\underset{\iota \in I}{P} D_\iota$ of directed sets D_ι we understand the set of all functions f defined on I such that $f(\iota) \in D_\iota$ for every $\iota \in I$ partly ordered by the relation "$f \leqslant g$ in $\underset{\iota \in I}{P} D_\iota$ if and only if $f(\iota) \leqslant g(\iota)$ in D_ι for each $\iota \in I$" (we write "$n \leqslant n'$ in D" for elements n, n' belonging to the directed set D if and only if $n = n'$ or the element n precedes the element n' in the sense of the relation which partly orders the set D). The product of two directed sets D_1 and D_2 will be denoted by $D_1 \times D_2$.

It is known that a convergence of nets of elements of a set $\mathfrak{X}$ induced by some topology in $\mathfrak{X}$ satisfies the following condition:

(W) *Let D be a directed set and suppose that for every $n \in D$ a directed set E_n is given, and let $E = \underset{n \in D}{P} E_n$. Suppose that to every pair $\langle n, m\rangle$, where $n \in D$, $m \in E_n$, an element $x_{nm} \in \mathfrak{X}$ is assigned. If, for every $n \in D$, the net $\{x_{nm},\ m \in D\}$ is convergent to x_n and the net $\{x_n,\ n \in D\}$ is convergent to x, then the net $\{y_{\langle n,f\rangle},\ \langle n, f\rangle \in D \times E\}$ where $y_{\langle n,f\rangle} = x_{n,f(n)}$ is convergent to x* (see [3], p. 69).

To prove Theorem 10 we shall show that the topological convergence of closed subsets of X does not satisfy the condition (W).

LEMMA 1. *For every two directed sets D_1 and D_2 there exist a directed set D and functions φ_1 and φ_2 such that φ_i maps D onto D_i and $\varphi_i(n) \leqslant \varphi_i(n')$ in D_i for every $n \leqslant n'$ in D $(i = 1, 2)$.*

Proof. Let $D = D_1 \times D_2$ and $\varphi_i(n) = n_i$ for $n = \langle n_1, n_2\rangle \in D$ and $i = 1, 2$.

LEMMA 2. *If $\{y_k,\ k \in D_1\}$ is a net of points of a topological space X which has no cluster point and φ is a function which maps a directed set D onto D_1 in such a way that $\varphi(n) \leqslant \varphi(n')$ in D_1 for every $n \leqslant n'$ in D, then the net $\{x_n,\ n \in D\}$ where $x_n = y_{\varphi(n)}$ also has no cluster point.*

Proof. Let x be an arbitrary point of X. Since x is not a cluster point of $\{y_k,\ k \in D_1\}$, there exist a neighbourhood U of x and an element $k_0 \in D_1$ such that $y_k \in X \setminus U$ for every $k \geqslant k_0$ in D_1. If n_0 is an element of D such that $\varphi(n_0) = k_0$, then clearly $x_n \in X \setminus U$ for every $n \geqslant n_0$ in D; thus x is not a cluster point of $\{x_n,\ n \in D\}$.

Now we shall show that if X is a regular non-locally bicompact space, then the topological convergence of nets of closed subsets of X does not satisfy the condition (W).

Let x_0 be a point of X which has no neighbourhood with a bicompact closure and let D_1 be a basis of neighbourhoods of x_0. Let us agree that $U \leqslant U'$ in D_1 if $U \supset U'$. Then D_1 is a directed set. Let x_1 be an arbitrary point of X which is different from x_0 and let $\{y_k,\ k \in D_2\}$ be a net of elements of X which has no cluster point. By Lemma 1 there exist a directed set D and functions φ_1 and φ_2 such that φ_i maps D onto D_i and $\varphi_i(n) \leqslant \varphi_i(n')$ in D_i for every $n \leqslant n'$ in D $(i = 1, 2)$. Let us set $U_n = \varphi_1(n)$ and $x_n = y_{\varphi_2(n)}$ for every $n \in D$. Then $U_n \supset U_{n'}$ for $n \leqslant n'$ in D. By Lemma 2, the net $\{x_n,\ n \in D\}$ has no cluster point. Since $\overline{U}_n$ is not bicompact, there exists a net $\{x_m^{(n)},\ m \in E_n\}$ of elements of $\overline{U}_n$ which has no cluster point. Let us set $A_{nm} = \{x_1\} \cup \{x_n\} \cup \{x_m^{(n)}\}$ for $n \in D$, $m \in E_n$. The net $\{A_{nm},\ m \in E_n\}$ is topologically convergent to $A_n = \{x_1\} \cup \{x_n\}$ for every $n \in D$ (since the net $\{x_m^{(n)},\ m \in E_n\}$ has no cluster point) and the net $\{A_n,\ n \in D\}$ is topologically convergent to $A = \{x_1\}$ (since the net $\{x_n,\ n \in D\}$ has no cluster point). Let $E = \underset{n\in D}{P} E_n$ and let us consider the net $\{A_{\langle n,f\rangle},\ \langle n,f\rangle \in D\times E\}$, where $A_{\langle n,f\rangle} = A_{n,f(n)}$. If U is an arbitrary neighbourhood of x_0 and $\overline{U}_{n_0} \subset U$ and f_0 is an arbitrary element of E, then $A_{\langle n,f\rangle} \cap U \neq 0$ for every $\langle n,f\rangle \geqslant \langle n_0,f_0\rangle$ in $D\times E$, whence x_0 belongs to $\underset{\langle n,f\rangle\in D\times E}{\mathrm{Li}} A_{\langle n,f\rangle}$ and the net $\{A_{\langle n,f\rangle},\ \langle n,f\rangle \in D\times E\}$ is not topologically convergent to A.

Remark. It is interesting to compare Theorem 8 with a result of Watson, who states that if X is a separable metric non-locally bicompact space, then the space 2^X considered as a L^*-space with the topological convergence of sequences of sets is not a topological space. Let us consider the following examples:

EXAMPLE 1. Let X be the unit (closed) interval $\langle 0,1\rangle$ with a discrete topology. Then 2^X consists of all non-empty subsets of X. By Theorem 5, the lbc-topology in 2^X induces the topological convergence of nets of sets (a set with a discrete topology is clearly a locally bicompact space), whence, in particular, the lbc-topology induces the topological convergence of sequences of sets. On the other hand, the space 2^X considered as a L^*-space is not a topological space. In fact, the topological upper and lower limits of sequences of sets coincide respectively with the upper and lower limits in the sense of the set-theory (*i. e.*, $\underset{n}{\mathrm{Li}} A_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n$, $\underset{n}{\mathrm{Ls}} A_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n$). Let $\mathfrak{X}_0$ be a subset of 2^X consisting of all closed non-empty subsets of X (with the ordinary topology) and let A be a G_δ-set which is not an F_σ-set. There exists a sequence $G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots$ such that $A = \bigcap_{n=1}^{\infty} G_n$ and for every n there exists a sequence $A_{n1} \subset A_{n2} \subset \ldots \subset A_{nm} \subset \ldots$ of sets of $\mathfrak{X}_0$ such that $G_n = \bigcup_{m=1}^{\infty} A_{nm}$. We have $G_n = \underset{m}{\mathrm{Lim}} A_{nm}$ and $A = \underset{n}{\mathrm{Lim}} G_n$, *i. e.*, $A \in [[\mathfrak{X}_0]_L]_L$ ($[\mathfrak{X}]_L$ denotes the limit-closure of a set $\mathfrak{X} \subset 2^X$). But $A \notin [\mathfrak{X}_0]_L$. In fact, $A \in [\mathfrak{X}_0]_L$ implies the existence of a sequence (B_n) of sets of $\mathfrak{X}_0$ such that $A = \underset{n}{\mathrm{Lim}} B_n$, *i. e.*, $A = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} B_n$. But the set $C_k = \bigcap_{n=k}^{\infty} B_n$ belongs to $\mathfrak{X}_0$ and it follows that A is an F_σ-set. Thus the space 2^X considered as a L^*-space is not a topological space but there exists a topology in 2^X which induces the topological convergence of arbitrary nets of sets, in particular there exists a topology in 2^X which induces the topological convergence of sequence of sets.

EXAMPLE 2. Let X_1 be a set of the power of the continuum and a a point which does not belong to X_1. Let us introduce a topology in the set $X = X_1 \cup \{a\}$ taking as a basis of neighbourhoods the family consisting of all the one-point sets $\{x\}$ where $x \in X_1$ and all the sets of the form $X \setminus S$ where S is an arbitrary enumerable subset of X_1. Then, as may easily be verified, the equalities $\underset{n}{\mathrm{Ls}} A_n = \underset{n}{\mathrm{ls}} A_n$ and $\underset{n}{\mathrm{Li}} A_n = \underset{n}{\mathrm{li}} A_n$ hold for every infinite sequence (A_n) of closed subsets of X ($\underset{n}{\mathrm{ls}} A_n$ and $\underset{n}{\mathrm{li}} A_n$ denote, respectively, the upper and the lower limits of the sequence (A_n) in the sense of the set-theory, *i. e.*, $\underset{n}{\mathrm{ls}} A_n = \bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n$, $\underset{n}{\mathrm{li}} A_n = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n$) and it follows that there exists a topology in 2^X which induces the topological convergence of sequences of sets. (In fact, if Y is an arbitrary set considered as a discrete topological space, then $\underset{n}{\mathrm{Ls}} A_n = \underset{n}{\mathrm{ls}} A_n$, $\underset{n}{\mathrm{Li}} A_n = \underset{n}{\mathrm{li}} A_n$ for every sequence of subsets of Y. But a discrete space is locally bicompact and it follows that the convergence of sequences of sets induced by the lbc-topology in 2^X coincides with convergence in the sense of the set-theory.) But X is not locally bicompact and it follows by Theorem 10 that there exists no topology in 2^X which induces the topological convergence of arbitrary nets of sets. One may also shown (in the same way as in Example 1) that the space 2^X considered as a L^*-space is not a topological space.

The following problem arises:

PROBLEM. Suppose that 2^X considered as a L^*-space is a topological space. Does the sequential topology in 2^X induce the topological convergence of arbitrary nets of sets? (It is true if X is separable metric. In fact, in this case, if 2^X considered as a L^*-space is a topological space, then X is locally bicompact, whence the lbc-topology in 2^X induces the topological convergence of nets of sets. But X being separable metric implies that 2^X with a lbc-topology is first countable, and it follows that the lbc-topology coincides with the sequential topology, whence the sequential topology in 2^X induces the topological convergence of arbitrary nets of sets.)

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On the spaces of ideals of semirings

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1. L. Gillman (see [2]) has proved that if a structural set $\mathfrak{J}$ of ideals of a ring is a Hausdorff space under Stone topology, then every prime ideal which contains the intersection of ideals in $\mathfrak{J}$ is contained in at most one ideal of $\mathfrak{J}$. It is easy to generalize this theorem to the case when R is a semiring (theorem 3.9). The principal result of this paper is the proof of a converse theorem for semirings R which are c-regular ([1]) (this class contains in particular distributive lattices, commutative rings and biregular rings) and for sets consisting exclusively of prime ideals of R. Moreover we give a few theorems on some topologies of families of sets having the finite character as well as some applications of those theorems to problems concerning spaces of ideals.

2. Let B be the set formed only of integers 0 and 1. Let B^1 be the set B with the following definition of topology: open subsets of B are $\emptyset$ ([2]), $\{0\}$ and $\{0,1\}$. Let B^2 be the set B with the Hausdorff topology.

We shall consider an arbitrary but fixed non-empty set R and a set $\mathfrak{J}$ of subsets of R. It is known that we can treat $\mathfrak{J}$ as a subset of $\underset{a\in R}{P} B_a$ where $B_a = B$ for every $a \in R$ (we assign the characteristic function $\chi_{\mathfrak{i}} \in \underset{a\in R}{P} B_a$ to each $\mathfrak{i} \in \mathfrak{J}$). Let $\mathfrak{J}^*$ denote the subset of $\underset{a\in R}{P} B_a$ such that $x \in \mathfrak{J}^* \equiv \sum_{\mathfrak{i}\in\mathfrak{J}} (x = \chi_{\mathfrak{i}})$.

Let $\mathfrak{J}^1$ and $\mathfrak{J}^2$ denote respectively the set $\mathfrak{J}^*$ with the following definitions of topology:

1. a subset $\mathcal{J} \subset \mathfrak{J}^*$ is open if and only if there exists an open subset $\mathcal{J}_1$ of $\underset{a\in R}{P} B_a^1$ (where $B_a^1 = B^1$ for every $a \in R$) such that $\mathcal{J}_1 \cap \mathfrak{J}^* = \mathcal{J}$;

2. a subset $\mathcal{J} \subset \mathfrak{J}^*$ is open if and only if there exists an open subset $\mathcal{J}_1$ of $\underset{a\in R}{P} B_a^2$ (where $B_a^2 = B^2$ for every $a \in R$) such that $\mathcal{J}_1 \cap \mathfrak{J}^* = \mathcal{J}$.

([1]) This notion will be defined later.
([2]) $\emptyset$ denotes here the empty set.