Wild 0-dimensional sets and the fundamental group

by

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Extending a definition of Fox and Artin [5] we shall say that a compact 0-dimensional subset A of n-dimensional spherical space S^n $(n > 1)$ is *tamely imbedded* if there is a homeomorphism of S^n on itself which maps A into a segment of a straight line; if no such homeomorphism exists we shall say that A is *wildly imbedded*. A well-known example of a wild 0-dimensional subset of 3-dimensional spherical space S is due to Antoine [1].

It seems that all the known examples of wild 0-dimensional sets have non-trivial fundamental groups of the complement. In spite of this situation

There is in a three-dimensional spherical space S a wild 0-dimensional subset A whose complement is simply connected i. e. has a trivial fundamental group (1).

The construction of the set A will depend largely on examples of wild cells due to Fox and Artin [5]; hence we shall refer to their paper as FA and use its notation.

1. A wild arc. We need some modification (2) of example 1.1 FA obtained by replacing the three arcs K_-, K_0, K_+ by the arcs $K_-^x = r_-s_-$, $K_0^x = t_-r_+$, $K_+^x = s_+t_+$ situated in cylinder C as shown in figure 1. Proceeding as in 1 FA we construct the simple arc

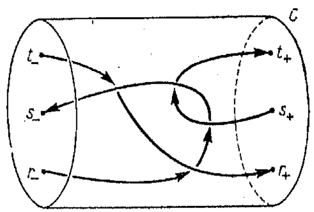

Fig. 1

$$(1) \qquad X^x = p \cup \bigcup_{n=-\infty}^{\infty} f_n(K^x) \cup q$$

where $K^x = K_-^x \cup K_0^x \cup K_+^x$. X^x is contained in the ellipsoid of revolution defined by $x^2 + 4y^2 + 4z^2 \leqslant 4$, with whose boundary it has only the points $p = (-2, 0, 0)$ and $q = (2, 0, 0)$ in common. Its projection in the xz-plane is shown in figure 2. The fundamental group $\pi(S - X^x)$ is generated by the elements a_n, b_n, c_n and d_n $(-\infty < n < \infty)$ indicated in figure 2. The set of defining relations is

$$\begin{aligned} d_n c_n^{-1} a_n &= 1\,, \\ d_n &= c_n^{-1} a_{n+1} c_n\,, \\ c_{n+1} &= c_n d_{n+1} c_n^{-1} \quad (-\infty < n < \infty)\,, \\ a_n &= a_{n+1}^{-1} b_n a_{n+1}\,, \\ b_n &= c_{n+1} c_n c_{n+1}^{-1}\,. \end{aligned}$$

Elimination of a_n, b_n and d_n leads to the single set of relations

$$(2) \qquad c_n c_{n-1}^{-1} c_n^{-1} c_{n-1} = c_n^{-1} c_{n+1} c_n c_{n+1}^{-1} c_n \qquad (-\infty < n < \infty)$$

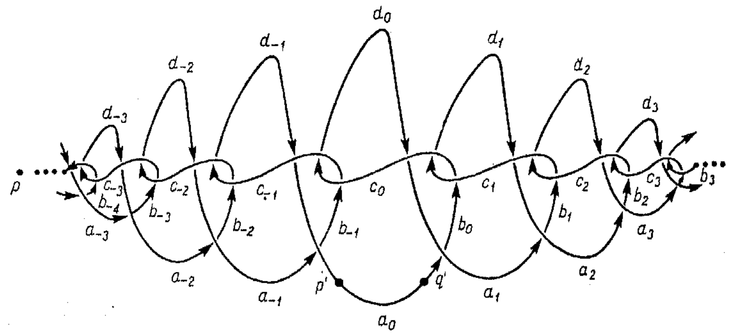

Fig. 2

in the generating set ..., c_{-1}, c_0, c_1, ... Using this presentation we shall prove that

(3) *$\pi(S - X^x)$ is a non-trivial locally infinite* (3) *group.*

Proof: Let

$$\mathfrak{G}_n = \{c_n, c_{n+1}\} \qquad (-\infty < n < \infty)$$

be a free group generated by the elements c_n and c_{n+1}. Let

$$\begin{aligned} \mathfrak{U}_n^- &= \{(c_n c_{n-1}^{-1} c_n^{-1} c_{n-1}), c_n\} \\ \mathfrak{U}_n^+ &= \{c_n, (c_n^{-1} c_{n+1} c_n c_{n+1}^{-1} c_n)\} \end{aligned} \qquad (-\infty < n < \infty)\,,$$

(1) This disproves a statement of Choquet [3], theorem 2.

(2) Added in proof: According to a recent result of C. D. Papakyriakopoulos [7], corollaries (31.8) and (31.9), no modification is necessary for the proof of (3) is granted.

(3) *I. e.* has no element of finite order except the neutral one.

and let $\chi_n\colon \mathfrak{U}_n^- \to \mathfrak{U}_n^+$ be an isomorphism of the group $\mathfrak{U}_n^-$ on the group $\mathfrak{U}_n^+$ defined by relations (2) and identities $c_n = c_n$ $(-\infty < n < \infty)$. $\mathfrak{U}_n^-$ is a subgroup of $\mathfrak{G}_{n-1}$ and $\mathfrak{U}_n^+$ is a subgroup of $\mathfrak{G}_n$. Define

$$\mathfrak{A}_1 = \mathfrak{G}_{-1} \underset{\mathfrak{U}_0^+}{*} [\mathfrak{G}_0 \underset{\mathfrak{U}_1^+}{*} \mathfrak{G}_1]$$

and for $n \geqslant 1$

$$\mathfrak{A}_{n+1} = \mathfrak{G}_{-n-1} \underset{\mathfrak{U}_{-n}^+}{*} [\mathfrak{A}_n \underset{\mathfrak{U}_{n+1}^+}{*} \mathfrak{G}_{n+1}]$$

where $\mathfrak{A} \underset{\mathfrak{U}}{*} \mathfrak{B}$ denotes Schreier's free product of groups $\mathfrak{A}$ and $\mathfrak{B}$ with the amalgamated subgroup $\mathfrak{U}$ [8]. Let $\mathfrak{A}$ be the direct limit of groups and injections $\mathfrak{A}_n \to \mathfrak{A}_{n+1}$. Obviously $\mathfrak{A} = \pi(S - X^x)$. A theorem of B. H. Neumann [6] says that

The free product of locally infinite groups with one amalgamated subgroup is a locally infinite group.

This implies (3) by easy induction.

Observe that

The arc X^x is rectifiable.

By 1 FA the set $f_n(C) = D_n$ is the section of ellipsoid $x^2 + 4y^2 + 4z^2 \leqslant 4$ defined by inequalities $2 - 2^{2-n} < x < 2 - 2^{1-n}$, whence D_n is contained in a cube whose edges are parallel to the axes of coordinates and have lengths equal to $4 \cdot 2^{-n/2}$. Therefore the diameter $\delta(D_n) < 8 \cdot 2^{-n/2}$. The set $f_n(K^x) \subset D_n$ can be assumed to have the total length less than a fixed multiple of $\delta(D_n)$ for $n = 1, 2, \ldots$ Hence the length of X^x is by (1) less than $2 \cdot \sum_{n=1}^{\infty} \lambda \cdot 2^{-n/2} < \infty$, where λ is a fixed positive number.

2. A couple of wild arcs. Let E be the ellipsoid of revolution $x^2 + 4y^2 + 4z^2 \leqslant 16$ and denote its vertices by $(-4, 0, 0) = p^x$ and $(4, 0, 0) = q^x$. On the arc $f_0(K_0^x) \cup f_1(K_-^x) \subset X^x$ choose two different points p', q' whose projections on the xz-plane are marked on the edge a_0 in figure 2. Join the points p' to p^x and q' to q^x by two disjoint polygonal arcs $p'p^x$ and q^xq' having only the end-points in common with the boundary of E and with the arc X^x. Denote the arcs by

$$pp' \cup p'p^x = Y_1, \qquad q^xq' \cup q'q = Y_2 .$$

Hence $Y_1 \cap Y_2 = 0$. The arcs Y_1 and Y_2 are arranged in E as shown in figure 3. Following closely 1.2 FA it can be proved that

(5) *Both Y_1 and Y_2 are wild though the complement of either of them is an open 3-cell.*

Clearly

(6) $$\pi[E - (Y_1 \cup Y_2)] \approx \pi(S - X^x) .$$

It is easy to notice that

(7) *There is such an isotopy h_t $(0 \leqslant t \leqslant 1)$ homeomorphically mapping S on itself that $h_1(Y_1)$ and $h_1(Y_2)$ are contained in disjoint solid spheres.*

For the isotopy h_t may be conceived so that in the course of it the subarc of Y_1 consisting of the arc $p'p^x$, the part of the edge labelled a_0

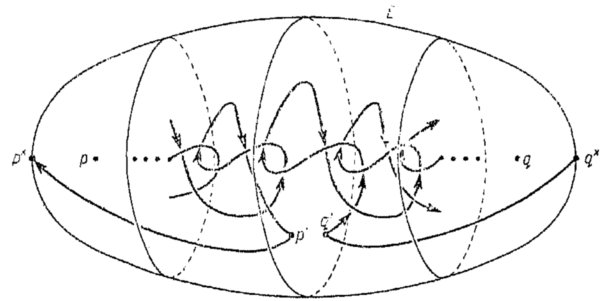

Fig. 3

in figure 2 and the whole edges labelled d_{-1}, c_{-1}, b_{-1} and a_{-1} shrink to the edge a_{-1}, while Y_2 remains fixed. The arcs $h_1(Y_1)$ and $h_1(Y_2)$ already possess the desired property.

3. Some more wild arcs. Let Y_i' and Y_j'' be two different copies (*i. e.* isometric images) of arcs Y_i and Y_j $(i, j = 1, 2)$. Denote by e_i' and e_j'' the non-singular end-points of Y_i' and Y_j'' respectively. Suppose that $Y_i' \subset Q_i'$ and $Y_j'' \subset Q_j''$ where Q_i', Q_j'' are disjoint solid spheres and their boundaries

$$\dot{Q}_i' \cap Y_i' = e_i', \qquad \dot{Q}_j'' \cap Y_j'' = e_j'' .$$

Denote by J_{ij} a polygonal arc having only the points e_i', e_j'' in common with Q_i' and Q_j''. Assume that

$$Y_{ij} = Y_i' \cup J_{ij} \cup Y_j'' \qquad (i, j = 1, 2) .$$

The arcs Y_{ij} are closely related to that of 1.3 FA and using the methods of 1.3 FA we can easily see that

(8) *$\pi(S - Y_{ij})$ is trivial for $i, j = 1, 2$.*

4. Entangling operation η. Let J be a segment or an infinite polygonal arc which is locally finite at every interior point. Assume $O = \{o_1, o_2, \ldots, o_k\}$ to be a set of points interior to some edges of J. Choose $\varepsilon > 0$ so that

$$V_i = \{u \in J \mid \varrho(o_i, u) \leqslant \varepsilon\} \qquad (i = 1, 2, \ldots, k)$$

are segments interior to some edges of J and $V_i \cap V_j = 0$ for $i \neq j$. Let E_i $(i = 1, 2, \ldots, k)$ be such an ellipsoid of revolution that V_i is its axis of rotation, $E_i \cap (J - V_i) = 0$ and $E_i \cap E_j = 0$ if $i \neq j$. Denote by φ_i a linear homeomorphism of the ellipsoid E (see 2) on E_i mapping segment $p^x q^x$ on V_i and define

$$\eta(J, O, \varepsilon) = (J - \bigcup_{i=1}^{k} V_i) \cup \bigcup_{i=1}^{k} \varphi_i(Y_1 \cup Y_2)\,.$$

(i) If L is a union of disjoint arcs $J_1, J_2, \ldots, J_m$, $O = \bigcup_{i=1}^{m} O_i$ where O_i is a finite set of points interior to some edges of J_i and ε is a positive number small enough, define

$$\eta(L, O, \varepsilon) = \bigcup_{i=1}^{m} \eta(J_i, O_i, \varepsilon)$$

so that $\eta(J_i, O_i, \varepsilon) \cap \eta(J_j, O_j, \varepsilon) = 0$ if $i \neq j$.

(ii) All components of $\eta(L, O, \varepsilon)$ are rectifiable polygonal arcs locally finite at every interior point if so are all components of L.

Now let us prove

(9) LEMMA ON η-OPERATION. *$\mathcal{E}$ being a 3-cell the group $\pi[\mathcal{E} - \eta(L, O, \varepsilon)]$ is not trivial if $\pi(\mathcal{E} - L)$ is not trivial.*

Proof. It will be sufficient to consider the simple case when the set O has one element only. But in such a case the proof follows easily from (3) and the Main Lemma of the next section.

5. Exchange of obstruction.

MAIN LEMMA [4]. Hypothesis. (i) *Let P be a tame n-dimensional curved polyhedron in S^n. Denote by G the interior and by $\dot{P}$ — the boundary of P.*

(ii) *Suppose that Z is such a closed subset of S^n that the kernels of injections $\pi(\dot{P} - Z) \to \pi(S^n - (Z \cup G))$ and $\pi(\dot{P} - Z) \to \pi(P - Z)$ are trivial.*

(iii) *Let $Z^* = (Z - P) \cup M$ where M satisfies the following conditions:*

(iii.a) *$M \subset P$ and $M \cap \dot{P} = Z \cap \dot{P}$,*

(iii.b) *the kernel of injection $\pi(\dot{P} - M) \to \pi(P - M)$ is trivial.*

THEOREM. *The kernel of injection $\pi[S^n - (Z \cup G)] \to \pi(S^n - Z)$ contains the kernel of injection $\pi[S^n - (Z \cup G)] \to \pi(S^n - Z^*)$.*

Remark: This lemma still holds when the groups involved are not defined. Its proof is based on the following

[4] The author owes the idea of this lemma to Professor K. Borsuk.

LEMMA h. *Let Q^2 be a 2-disk, P a polyhedron in a metric space R and ε a positive number. If g: $Q^2 \to R$ there is such a homotopy g_t: $Q^2 \to R$ $(0 \leqslant t \leqslant 1)$ that*

(i) *$g_0 = g$ and $\varrho(g_0, g_t) < \varepsilon$,*

(ii) *$g_1^{-1}(P)$ is an elementary figur* [2].

Proof of Lemma h. It will be sufficient to prove this lemma in the special case when Q^2 is a 2-simplex and g is a simplicial mapping. But in such a case the proof follows easily from 7 of [2].

Proof of the Main Lemma. Let g belong to the kernel of injection $\pi[S^n - (Z \cup G)] \to \pi(S^n - Z^*)$. Hence there is such a mapping g^*: $Q^2 \to S^n - Z^*$ that $g^*|_{\dot{Q}^2} = g$. By Lemma h it can be assumed that $g^{*-1}(P)$ is an elementary figur. Suppose that $g^{*-1}(P)$ is not void and denote by $\Omega_1, \Omega_2, \ldots, \Omega_k$ disjoint simple closed curves whose union is the boundary of $g^{*-1}(P)$. Let Γ_i denote the open 2-cell bounded by Ω_i, for $i = 1, 2, \ldots, k$.

There is such a number j $(1 \leqslant j \leqslant k)$ that either $\Gamma_j \cap g^{*-1}(P) = 0$ or Γ_j contains only such cells Γ_i's that $g^*(\Gamma_i) \subset P - Z^*$. By hypothesis (ii) and (iii.b) of Main Lemma the mapping g^* can be deformed so in $S^n - Z^*$ that the curve Ω_j would disappear and the number k would be reduced by one at least. This implies the Main Lemma by finite induction.

6. Disentanglement. Let L be defined as in (i) of 4. Suppose that there is such a polyhedral 3-cell $\mathcal{E}$ that $\dot{\mathcal{E}}$ cuts L only at one point o, which is interior to some edge of J_{i_0} $(1 \leqslant i_0 \leqslant k)$. Assuming the arcs $J_1^*, J_2^*, \ldots, J_{k+1}^*$ to be components of the set $L^* = \eta(L, o, \varepsilon)$ let $L^* = L' \cup L''$ so that

(i) L' is the union of those J_i^*'s which either lie in $\mathcal{E}$ or have with J_{i_0} that end-point in common which belongs to $\mathcal{E}$.

(ii) $L'' = L^* - L'$.

Obviously neither L' nor L'' are void. By means of (7) it is easy to prove that

(10) *There is such an isotopy h_t $(0 \leqslant t \leqslant 1)$ homeomorphically mapping S on itself that $h_1(L')$ and $h_1(L'')$ are subsets of two disjoint solid spheres.*

Proof. Let E denote the ellipsoid of revolution by means of which the operation η is defined at the point o. Choose $\varepsilon > 0$ so small that $\overline{\mathcal{E} - E} = \mathcal{E}'$ is a (tame) 3-cell whose boundary $\dot{\mathcal{E}}'$ cuts L' only at the vertex of E, say e. Let E' be such a small, closed, spherical neighbourhood of E that E' and E meet only the same edge of L. The isotopy h_t may be carried through in three steps:

First, let us perform an isotopy which shrinks $\mathcal{E}'$ to a 3-cell $\mathcal{E}'' \subset E'$ which lies in a sufficiently small neighbourhood of the vertex e and does not change L''.

Secondly, let us denote by Y_1 that component of $\eta(L, o, \varepsilon) \cap E$ which contains the vertex e, and by Y_2 — the other one. Now apply an isotopy that leaves $(S-E') \cup Y_2$ fixed and which, shrinking Y_1 and dragging $\mathcal{E}''$ behind, moves the set $\mathcal{E}'' \cup Y_1$ into a full sphere $Q \subset E$, $Y_2 \cap Q = 0$. This isotopy can be constructed by means of (7).

Thirdly, let us perform an isotopy which carries Q far away and leaves L'' fixed.

7. Thickening operation θ. Let J satisfy the conditions of 4. If J is a segment let $\theta(J, \varkappa)$ be the ellipsoid of revolution whose axis of rotation is J while $2\varkappa$ is the length of each of the other two axis.

In the opposite case $J = \bigcup_{i \in N} I_i$, where N is the set of all integers or of all positive integers, I_i's are the edges of J and $I_i \cap I_j \neq 0$ only when $|i-j| \leqslant 1$. Then let

$$\theta(J, \varkappa) = \bigcup_{i \in N} U_i$$

where U_i is either such

(a) a tabular neighbourhood [4] of I_i when J_i does not contain any end-point of J, or

(b) a pyramid in which I_i joins the vertex, which is an end-point of J, with an interior point of the base,

that the following conditions are satisfied:

(i) $\theta(J, \varkappa)$ is a polygonal 3-cell locally finite almost everywhere, the only singular points being one or both end-points of J,

(ii) $\varkappa$ is the Hausdorff distance between J and the boundary of $\theta(J, \varkappa)$.

There is a homeomorphism which maps $\theta(J, \varkappa)$ on a solid sphere and J on its diameter. Hence

(11) *If $\mathcal{E}$ is an open or closed 3-cell the injection $\pi[\mathcal{E} - \theta(J, \varkappa)] \to \pi(\mathcal{E} - J)$ is an isomorphism into provided the involved groups exist.*

Assuming L to be the union of disjoint arcs $J_1, J_2, \ldots, J_k$ and $\varkappa$ to be a positive number less than a half of the least span between any two components of L, define

$$\theta(L, \varkappa) = \bigcup_{i=1}^{k} \theta(J_i, \varkappa).$$

(12) Remark. Let J denote any of arcs Y_i, Y_{ij} $(i, j = 1, 2)$ and let u be an interior point, e an end-point of J. Using θ-operation we can construct such a polyhedral 3-cell $\mathcal{E}$ that $e \in \mathcal{E}$ and $J \cap \dot{\mathcal{E}} = u$.

8. 0-dimensional set A. Now let Q be a solid sphere in S and let the segment J_0 be a diameter of Q. Denote by $\varkappa_0$ any positive number less than $\frac{1}{2}\delta(Q)$. Define two sequences of sets $\{L_n\}$ and $\{A_n\}$ by induction:

$$L_0 = J_0, \quad A_0 = \theta(J_0, \varkappa_0).$$

Let o_0 be the centre of Q and let ε_1 be a positive number less than $\frac{1}{2}\varkappa_0$. Assume that

$$L_1 = \eta(L_0, o_0, \varepsilon_1), \quad A_1 = \theta(L_1, \varkappa_1)$$

where the positive number $\varkappa_1$ is so small that $\theta(L_1, \varkappa_1) \subset A_0$ holds.

Suppose the sets L_n and A_n to be already defined for some $n \geqslant 1$, L_n having a finite number of components which are rectifiable polygonal arcs, locally finite at every interior point (4, (ii)); denote by δ_n, δ_n' the greatest of the diameters which have the components of sets L_n and A_n respectively and choose a finite set O_n of points interior to some edges of L_n so that the diameter of any component of $L_n - O_n$ is less than $\frac{1}{3}\delta_n$. Then the positive numbers ε_{n+1} and $\varkappa_{n+1}$ can be chosen so small that the sets defined by

$$L_{n+1} = \eta(L_n, O_n, \varepsilon_{n+1}), \quad A_{n+1} = \theta(L_{n+1}, \varkappa_{n+1})$$

satisfy conditions: $A_{n+1} \subset A_n$ and $\delta_{n+1}' < \frac{1}{2}\delta_n'$.

Now let

$$A = \bigcap_{n=0}^{\infty} A_n.$$

It is easy to see that the Menger-Urysohn dimension of A is zero. Using (11) and (9) we can easily prove by induction that the injection $\pi(Q-J_0) \to \pi(Q-A_n)$ is an isomorphism into and therefore $\pi(Q-A)$ is not trivial. Hence the set A *is wild.*

In order to prove that $\pi(S-A)$ *is trivial* let us observe that

(i) $\pi(S-A)$ is a direct limit of groups and injections $\pi(S-A_n) \to \pi(S-A_{n+1})$ $(n = 0, 1, 2, \ldots)$;

(ii) by (11) $\pi(S-A_n) \to \pi(S-L_n)$ is isomorphism into;

(iii) any component of L_n $(n \geqslant 1)$ is an arc which is isotopic in S with one of the arcs Y_i or Y_{ij} $(i, j = 1, 2)$;

(iv) following the final remark of 7 it can be proved by induction that for any interior point u of L_n there is such a polyhedral 3-cell $\mathcal{E}$ that $\dot{\mathcal{E}}$ cuts L_n only at the point u and $\mathcal{E}$ contains only one end-point of J_0, fixed beforehand;

(v) using (iii), (iv), (12) and (10) we can prove by induction that for any given $n \geqslant 1$ there is such an isotopy h_t $(0 \leqslant t \leqslant 1)$ mapping homeomorphically S on itself that every component of $h_1(L_n)$ is a subset of a different solid sphere, any two of them being disjoint;

(vi) $\pi(S-L_n)$ is a free product of all groups $\pi(S-J_i^n)$ where J_i^n denotes a component of L_n. This follows by theorem 1 of [9];

(vii) by (iii) and (8) all groups $\pi(S-J_i^n)$ are trivial.

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On the convergence of nets of sets

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The topological convergence of net [1] of subsets of a topological space X may be defined in the same manner as the topological convergence of a sequence of sets: if $\{A_n, n \in D\}$ is a net of subsets of X, then $\mathrm{Li}\,A_n$ ($\mathrm{Ls}\,A_n$) is defined as the set of all $x \in X$ such that every neighbourhood of x intersects A_n for almost all (arbitrarily large) n [2]. A net $\{A_n, n \in D\}$ is said to be topologically convergent (to a set A) if $\mathrm{Ls}\,A_n = \mathrm{Li}\,A_n$ ($= A$) and in this case the set A will be denoted by $\mathrm{Lim}\,A_n$.

Hausdorff ([2], p. 145) has shown that if X is a compact metric space, then in the space 2^X consisting of all closed non-empty subsets of X a metric may be defined such that the convergence of sequences of sets induced by this metric [3] coincides with topological convergence. This result has been generalized by Watson [7] who has shown that if X is a locally compact separable metric space then another metric may be defined in 2^X which induces topological convergence. Watson has also shown that if X is not locally compact, then the space 2^X considered as a L^*-space (see [4], p. 89 and p. 274) topological convergence is not a topological space.

The present paper is devoted to generalizations of the above results. It will be shown that:

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(1) *A net* is a function defined on a directed set (a partially ordered set D is called *directed* if for every $n_1, n_2 \in D$ an element $n \in D$ may be found such that $n_1 \prec n$, $n_2 \prec n$, where $\prec$ is the relation which partially orders the set D). If a net defined on D assigns to an element $n \in D$ an element x_n, then it will be denoted by $\{x_n, n \in D\}$ (see [3], p. 65).

(2) We say that a statement T on elements of a directed set D is fulfiled for *almost all* $n \in D$ if an element $n_0 \in D$ may be found such that T is fulfiled for every $n \succ n_0$;
arbitrarily large $n \in D$ if the set of all $n \in D$ for which T is fulfiled is cofinal with D.

(3) We say that a metric ϱ (a topology $\mathcal{T}$) for a set X *induces* a certain convergence of nets in X of some sort if each net in X of that sort is convergent with respect to this convergence if and only if it is convergent with respect to the metric ϱ (the topology $\mathcal{T}$).