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On a closed mapping between ANR's

by

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1. Let X and Y be topological spaces. A continuous mapping f of X into Y is called a *closed mapping* if and only if, whenever A is a closed subset of X, $f(A)$ is a closed subset of Y. In the present note, we shall show that if X and Y are ANR's for metric spaces and there exists a closed continuous mapping f of X onto Y satisfying a certain condition, then there exist some intimate relations between combinatorial invariants of X and Y. In more detail, if Y is a finite dimensional ANR for metric spaces and for each point y of Y $f^{-1}(y)$ is an ANR for metric spaces having a certain acyclic property, we shall prove that there exists a continuous mapping g of Y into X such that $fg \simeq 1$. Moreover, if X is a finite dimensional ANR for metric spaces, we shall prove that X has the same homotopy type as Y.

In 2, several notations and lemmas which we shall need later on are given. We shall prove our main theorems in 3. In 4, we shall prove some theorems strengthening the main theorems.

2. A topological space X is called an AR (resp. ANR) *for metric spaces* if and only if, whenever X is a closed subset of a metric space Y, there exists a retraction [1] of Y (resp. some neighborhood of X in Y) onto X. (Cf. [1] or [13], Definition 2.2, p. 790.) A topological space X is called an NES *for metric spaces* if and only if, whenever Y is a metric space and B is a closed subset of Y, any continuous mapping of B into X can be extended to a continuous mapping of some neighborhood of B in Y into X. (Cf. [13], Definition 2.1, p. 790.) A metric space X is called an LC^n *space* if and only if for each point x of X and for each neighborhood U of x there exists a neighborhood V of x such that any continuous mapping g from an i-sphere S^i to V is extended to a continuous mapping $\tilde{g}$ from an $(i+1)$-element E^{i+1} with the boundary S^i into U, $i = 0, 1, \ldots, n$. (Cf. [12], p. 79.) A metric space X is called a C^n *space* if and only if any continuous mapping g from an i-sphere S^i to X is

(1) By a *retraction* h of Y into X we mean a continuous mapping from Y onto X such that $h(x) = x$ for each point x of X.

extended to a continuous mapping $\tilde{g}$ from an $(i+1)$-element E^{i+1} with the boundary S^i into X, $i=0,1,2,\dots,n$. (Cf. [12], p. 78.) If $\mathfrak{U}=\{U_\alpha\}$ and $\mathfrak{V}=\{V_\beta\}$ are open coverings of a topological space X, a covering $\mathfrak{U}$ is called a *star refinement of a covering* $\mathfrak{V}$ if for each element U_α of $\mathfrak{U}$ there exists an element V_β of a covering $\mathfrak{V}$ such that $\mathrm{St}(U_\alpha,\mathfrak{U})\subset V_\beta$, where $\mathrm{St}(U_\alpha,\mathfrak{U})$ means the union of all elements $U_{\alpha'}$ of $\mathfrak{U}$ such that $U_\alpha\cap U_{\alpha'}\neq\emptyset$. An open covering $\mathfrak{U}$ of X is called a *locally finite covering of* X if for each point x of X there exists a neighborhood of x which meets only a finite number of elements of $\mathfrak{U}$. By the *order* of an open covering $\mathfrak{U}$ of X we mean the largest number n such that there exist n elements of $\mathfrak{U}$ with non-void intersection. The *dimension* of a normal space X is the smallest number n such that for every locally finite open covering $\mathfrak{V}$ of X there exists a locally finite refinement $\mathfrak{U}$ of $\mathfrak{V}$ whose order $\leqslant n+1$. (Cf. [14], Theorem 2.1, p. 18, and [15], p. 351.)

LEMMA 1. *Let X be an* ANR *for metric spaces and let A be a closed subset of X which is an* ANR *for metric spaces. Let U be an open neighborhood of A in X. Then there exist an open neighborhood V of A contained in U and a deformation homotopy $H\colon V\times I\to U$ such that*

$$H(a,t)=a \quad \text{for} \quad a\in A \text{ and } t\in I,$$
$$H(x,0)=x \quad \text{for} \quad x\in V,$$
$$H(x,1)\in A \quad \text{for} \quad x\in V,$$

where I is the closed interval $\langle 0,1\rangle$.

Remark. S. T. Hu ([8], p. 30) proved this lemma in the case of X and Y being ANR's for separable metric spaces. For completeness we shall prove this lemma.

Proof of Lemma 1. By Wojdysławski's theorem ([20], p. 186), we can assume that X is a closed subset of a convex subspace B of a normed vector space. Since X is an ANR for metric spaces, there exist an open neighborhood W_0 of X in B and a retraction $h\colon W_0\to X$. Put $W_1=h^{-1}(U)$. Then W_1 is an open set in W_0. Since A is an ANR for metric spaces and a closed subset of W_1, there exist an open neighborhood W_2 contained in W_1, of A in B and a retraction $g\colon W_2\to A$. For each point a of A, denote by $S_1(a)$ a spherical neighborhood in B with the centre a contained in W_2. Then we have $S_1(a)\cap X\subset W_1\cap X\subset U$. Since $g^{-1}(S_1(a)\cap A)$ is an open set in W_2 and contains a, there exists a spherical neighborhood $S_2(a)$ in W_2 with the centre a such that $S_2(a)\subset g^{-1}(S_1(a)\cap A)\cap S_1(a)$. Put $V=\bigcup\{S_2(a)\cap X \mid a\in A\}$. Define a mapping $H'\colon V\times I\to W_2$ as follows. Take a point x of V. There exists a spherical neighborhood $S_2(a)$ containing x. Then $g(x)\in g(S_2(a))\subset S_1(a)\cap A$. Therefore $x\cup g(x)\subset S_1(a)$. Let us map $x\times I$ proportionally onto the segment $\overline{xg(x)}$ in B. This is possible because $S_1(a)$ is a convex subset in B. Since B is a normed vector space and g is a continuous mapping, H' is a continuous mapping. Define $H\colon V\times I\to U$ by putting $H=ghH'$. Then H is the required deformation homotopy.

LEMMA 2. *If X is an* ANR *for metric spaces, there exists an open covering $\mathfrak{U}=\{U_\alpha \mid \alpha\in\Omega\}$ with the following property: Whenever Z is a metric space and f_0,f_1 are two continuous mapping of Z into X such that for a fixed point z_0 of Z $f_0(z_0)=f_1(z_0)=x_0$ and for each point z of Z there exists an element U_α of $\mathfrak{U}$ containing $f_0(z)$ and $f_1(z)$, then $f_0\simeq f_1\colon (Z,z_0)\to(X,x_0)$* (2).

This lemma is proved in the same way as [5], p. 363, or [9], p. 38, and we omit its proof.

3. THEOREM 1. *Let X be an* ANR *for metric spaces, let Y be an n-dimensional* ANR *for metric spaces and let x_0 and y_0 be points of X and Y respectively. Moreover, let f be a closed continuous mapping from (X,x_0) onto (Y,y_0) such that for each point y of Y $f^{-1}(y)$ is an* ANR *for metric spaces which is a C^S space, where S is any integer such that $n-1\leqslant S$. Then there exists a continuous mapping g of (Y,y_0) into (X,x_0) such that*

(i) *$fg\simeq 1$ (3): $(Y,y_0)\to(Y,y_0)$,*

(ii) *for each integer $i=0,1,2,\dots,S$, the induced homomorphism $g_*\colon \pi_i(Y,y_0)\to\pi_i(X,x_0)$ is an isomorphism onto and the induced homomorphism $g_*f_*\colon \pi_i(X,x_0)\to\pi_i(X,x_0)$ is the identity isomorphism, where $\pi_i(X,x_0)$ is the i-dimensional homotopy group of (X,x_0).*

Proof. Since Y is an ANR for metric spaces, there exists an open covering $\mathfrak{W}$ with the property stated in Lemma 2. Let $\mathfrak{U}_0$ be an open star refinement of $\mathfrak{W}$. Suppose that we have already constructed an open covering $\mathfrak{U}_j$ for $j=0,1,\dots,i$. We shall then construct an open covering $\mathfrak{U}_{i+1}$ as follows. For each point y of Y, take a fixed element $U_i(y)$ of $\mathfrak{U}_i$ containing y. Then $f^{-1}(U_i(y))$ is an open neighborhood of $A_y=f^{-1}(y)$. By Lemma 1, there exist an open neighborhood $V_i(y)$ of A_y in X such that $V_i(y)\subset f^{-1}(U_i(y))$, and a deformation homotopy $H_y^i\colon V_i(y)\times I\to f^{-1}(U_i(y))$ such that $H_y^i(V_i(y)\times 1)\subset A_y$ for each $y\in Y$. Denote the open covering $\{V_i(y) \mid y\in Y\}$ of X by $\mathfrak{V}_i$. Since f is a closed mapping, the set $Q_i(y)=Y-f(X-V_i(y))$ is an open set containing y and con-

(2) Let (X,A) and (Y,B) be two pairs of topological spaces and let f_0 and f_1 be two continuous mappings of (X,A) to (Y,B). By $f_0\simeq f_1\colon (X,A)\to(Y,B)$ we mean that there exists a homotopy $H\colon X\times I\to Y$ such that $H|X\times 0=f_0$, $H|X\times 1=f_1$ and $H(A\times I)\subset B$.

(3) We mean by "1" the identity mapping.

tained in $U_i(y)$. Denote the open covering $\{Q_i(y)|\ y \in Y\}$ of Y by $\mathfrak{Q}_i$. Let $\tilde{\mathfrak{U}}_i$ be a star-star open refinement (4) of $\mathfrak{Q}_i$. For each point y of Y, take a fixed element $\tilde{U}_i(y)$ of $\tilde{\mathfrak{U}}_i$ containing y. Then $f^{-1}(\tilde{U}_i(y))$ is an open neighborhood of A_y. By Lemma 1, there exist an open neighborhood $\tilde{V}_i(y)$ of A_y contained in $f^{-1}(\tilde{U}_i(y))$ and a deformation homotopy $\tilde{H}^i_y$: $\tilde{V}_i(y)\times I \to f^{-1}(\tilde{U}_i(y))$ such that $\tilde{H}^i_y(\tilde{V}_i(y)\times 1) \subset A_y$ for each point y of Y. Denote the open covering $\{\tilde{V}_i(y)|\ y \in Y\}$ of X by $\tilde{\mathfrak{B}}_i$. Since f is a closed mapping, the set $\tilde{Q}_i(y) = Y - f(X - \tilde{V}_i(y))$ is an open set containing y. Denote the open covering $\{\tilde{Q}_i(y)|\ y \in Y\}$ of Y by $\tilde{\mathfrak{Q}}_i$. Let $\mathfrak{U}_{i+1}$ be an open star refinement of $\tilde{\mathfrak{Q}}_i$. Thus by induction we can construct sequences $\{\mathfrak{U}_i, \mathfrak{Q}_i, \tilde{\mathfrak{Q}}_i\}$ and $\{\mathfrak{B}_i, \tilde{\mathfrak{B}}_i\}$ of open coverings of Y and X. Let $\mathfrak{W}_0$ be a locally finite open covering of Y ([16], Corollary 1, p. 979) which is a refinement of $\mathfrak{U}_{S+2}$ and whose order is $n+1$. Moreover, we can assume that $\mathfrak{W}_0$ is an irreducible (5) open covering such that the point y_0 is contained in only one element of $\mathfrak{W}_0$. Let K be the nerve of $\mathfrak{W}_0$ with Whitehead's topology ([18], p. 316) and let Φ be a canonical mapping ([3], p. 202) of (Y, y_0) into (K, k_0), where k_0 is the vertex of K such that $\Phi(y_0) = k_0$. (Cf. [4], Theorem 3, p. 576.) We shall construct a continuous mapping $\tilde{g}$ of (K, k_0) into (X, x_0). For each vertex v of K, take a fixed point x_v of the set $f^{-1}\Phi^{-1}(v)$. This is possible because $\mathfrak{W}_0$ is an irreducible covering of Y. Define $\tilde{g}$: $(K^0, k_0) \to (X, x_0)$ by putting $\tilde{g}(v) = x_v$ and $\tilde{g}(k_0) = x_0$, where K^i is the i-section of K. Let $\overline{v_1v_2}$ be a 1-simplex of K. If we denote the elements of $\mathfrak{W}_0$ corresponding to v_1 and v_2 by W_1 and W_2, we have $W_1 \cap W_2 \neq \emptyset$. Hence, there exists an element $\tilde{Q}_{S+1}(y)$ of $\tilde{\mathfrak{Q}}_{S+1}$ such that $W_1 \cup W_2 \subset \mathrm{St}(W_1, \mathfrak{W}_0) \subset \tilde{Q}_{S+1}(y)$. Therefore, we have $\tilde{g}(v_1) \cup \tilde{g}(v_2) \subset f^{-1}\Phi^{-1}(v_1) \cup f^{-1}\Phi^{-1}(v_2) \subset f^{-1}(W_1 \cup W_2) \subset f^{-1}(\tilde{Q}_{S+1}(y))$ and there exist elements $\tilde{V}_{S+1}(y')$ and $\tilde{U}_{S+1}(y')$ of $\tilde{\mathfrak{B}}_{S+1}$ and $\tilde{\mathfrak{U}}_{S+1}$ such that $\tilde{g}(v_1 \cup v_2) \subset \tilde{V}_{S+1}(y') \subset f^{-1}(\tilde{U}_{S+1}(y'))$, and the deformation homotopy $\tilde{H}^{S+1}_{y'}$: $\tilde{V}_{S+1}(y') \times I \to f^{-1}(\tilde{U}_{S+1}(y'))$ such that $\tilde{H}^{S+1}_{y'}(\tilde{V}_{S+1}(y')\times 1) \subset A_{y'}$. Consider the product space $\overline{v_1v_2}\times I$. The subspace $\tau = \{v_1 \cup v_2\}\times I \cup \overline{v_1v_2}\times 1$ of $\overline{v_1v_2}\times I$ is a retract of $\overline{v_1v_2}\times I$ whose retraction we denote by r. Define $\mu_{\overline{v_1v_2}}$: $\{v_1 \cup v_2\}\times I \to f^{-1}(\tilde{U}_{S+1}(y'))$ by putting $\mu_{\overline{v_1v_2}}(v_i, t) = \tilde{H}^{S+1}_{y'}(\tilde{g}(v_i), t)$ for $i = 1, 2$ and $t \in I$. Since $\mu_{\overline{v_1v_2}}(v_1, 1) \cup \mu_{\overline{v_1v_2}}(v_2, 1)$ are two points of $A_{y'}$, which is a C^S space, and $n-1 \leqslant S$, we have an extension $\tilde{\mu}_{\overline{v_1v_2}}$ of $\mu_{\overline{v_1v_2}}$ over τ such that $\tilde{\mu}_{\overline{v_1v_2}}(\overline{v_1v_2}\times 1) \subset A_{y'}$. Define $\tilde{g}$: $K' \to X$ by putting $\tilde{g}(k) = \tilde{\mu}_{\overline{v_1v_2}}r(k, 0)$ for $k \in \overline{v_1v_2}$. Since K is a Whitehead complex, $\tilde{g}$ is a continuous mapping by [19], p. 224, and for each 1-simplex $\overline{v_1v_2}$ of K we can find an element $\tilde{U}_{S+1}(y)$ of $\tilde{\mathfrak{U}}_{S+1}$ such that $\tilde{g}(\overline{v_1v_2}) \subset f^{-1}(\tilde{U}_{S+1}(y))$. Suppose we construct a continuous mapping $\tilde{g}$: $K^i \to X$ for $i < j$ such that for i-simplex s there exists an element $\tilde{U}_{S+2-i}(y_s)$ of $\tilde{\mathfrak{U}}_{S+2-i}$ containing $\tilde{g}(s)$. Take a j-simplex s of K. Consider the boundary $F(s)$ of the simplex s. Then $F(s) = \bigcup_{i=1}^{j+1} s_i$, where s_i's are the $(j-1)$-dimensional faces of s. For each s_i we can find an element $\tilde{U}_{S+2-(j-1)}(y_{s_i})$ of $\tilde{\mathfrak{U}}_{S+2-(j-1)}$ such that $\tilde{g}(s_i)$ is contained in $f^{-1}(\tilde{U}_{S+2-(j-1)}(y_{s_i}))$. Then $\tilde{U}_{S+2-(j-1)}(y_{s_i}) \neq \emptyset$. Hence, there exists an element $U_{S+2-(j-1)}$ of $\mathfrak{U}_{S+2-(j-1)}$ containing $\bigcup_{i=1}^{j+1} \tilde{U}_{S+2-(j-1)}(y_{s_i})$. Therefore we can find elements $\tilde{V}_{S+2-j}(y_s)$ and $\tilde{U}_{S+2-j}(y_s)$ of $\tilde{\mathfrak{B}}_{S+2-j}$ and $\tilde{\mathfrak{U}}_{S+2-j}$ such that $\tilde{g}(F(s)) \cup A_{y_s} \subset \tilde{V}_{S+2-j}(y_s) \subset f^{-1}(\tilde{U}_{S+2-j}(y_s))$ and the deformation homotopy $\tilde{H}^{S+2-j}_{y_s}$: $\tilde{V}_{S+2-j}(y_s)\times I \to f^{-1}(\tilde{U}_{S+2-j}(y_s))$ such that $\tilde{H}^{S+-}_{y_s}(\tilde{V}_{S+2-j}(y_s), 1) \subset A_{y_s}$. Consider the product space $s \times I$. The subset $F(s)\times I \cup s\times 1$ of $s\times I$ is a retract of $s \times I$ whose retraction we denote by r_s. Define μ_s: $F(s)\times I \to f^{-1}(\tilde{U}_{S+2-j}(y_s))$ by putting $\mu_s(k, t) = \tilde{H}^{S+2-j}_{y_s}(\tilde{g}(k), t)$ for $k \in F(s)$. Then $\mu_s(F(s), 1) \subset A_{y_s}$. Since A_{y_s} is a C^S space and $n-1 \leqslant S$, μ_s is extended to $\tilde{\mu}_s$ from $F(s)\times I \cup s \times 1$ to $f^{-1}(\tilde{U}_{S+2-j}(y_s))$ such that $\tilde{\mu}(s, 1) \subset A_{y_s}$. Define g: $K^j \to X$ by putting $\tilde{g}(k) = \tilde{\mu}_s r_s(k, 0)$ for $k \in s \subset K^j$. Obviously $\tilde{g}$ is a continuous mapping and for each j-simplex s of K we can find an element $\tilde{U}_{S+2-j}(y_s)$ of $\tilde{\mathfrak{U}}_{S+2-j}$ such that $\tilde{g}(s) \subset f^{-1}(\tilde{U}_{S+2-j}(y_s))$. Thus we have a continuous mapping $\tilde{g}$: $(K, k_0) \to (X, x_0)$ such that for each simplex s of K there exists an element $\tilde{U}_{S+2-n}$ of $\tilde{\mathfrak{U}}_{S+2-n}$ such that $f\tilde{g}(s) \subset \tilde{U}_{S+2-n}$. Define g: $(Y, y_0) \to (X, x_0)$ by putting $g(y) = \tilde{g}\Phi(y)$ for $y \in Y$. Since $\tilde{\mathfrak{U}}_i$ is a star refinement of $\mathfrak{W}$, for each point y of Y, we can find an element W of $\mathfrak{W}$ such that $y \cup fg(y) \subset W$. Thus we have $fg \simeq 1$: $(Y, y_0) \to (Y, y_0)$.

Next, we shall prove that the induced homomorphism g_*: $\pi_i(Y, y_0) \to \pi_i(X, x_0)$ is an isomorphism onto, $i \leqslant S$. Since X is an ANR for metric spaces, we can find a locally finite open covering $\mathfrak{U}$ [16] such that

(i) $\mathfrak{U}$ is a star refinement of $f^{-1}(\mathfrak{W}_0)$,

(ii) if N is the nerve of $\mathfrak{U}$ with Whitehead's weak topology and Φ_0 is a canonical mapping of (X, x_0) into (N, n_0), there exists a continuous mapping Ψ_0: $(N, n_0) \to (X, x_0)$ such that $\Psi_0\Phi_0 \simeq 1$: $(X, x_0) \to (X, x_0)$.

We shall prove that $\Psi_0|N^S \simeq gf\Psi_0|N^S$: $(N^S, n_0) \to (X, x_0)$. To prove this, we construct a homotopy H: $N^S \times I \to X$ such that $H(n, 0) = \Psi_0(n)$ and $H(n, 1) = gf\Psi_0(n)$ for $n \in N^S$. At first, we define H: $N^S \times (0 \cup 1) \cup \{n_0\} \times I \to X$ by putting $H(n, 0) = \Psi_0(n)$, $H(n, 1) = gf\Psi_0(n)$ for $n \in N^S$ and

(4) An open covering $\mathfrak{U} = \{U_\alpha\}$ of X is called a *star-star refinement of an open covering* $\mathfrak{V}$ *of* X if the open covering $\{\mathrm{St}(U_\alpha, \mathfrak{U})|\ U_\alpha \in \mathfrak{U}\}$ is a star refinement of $\mathfrak{V}$.

(5) An open covering $\mathfrak{W} = \{W_\alpha\}$ of X is called an *irreducible open covering* if for each element W_α of $\mathfrak{W}$ there exists a point x_α of X such that $x_\alpha \notin W_\beta$ for any W_β, $\beta \neq \alpha$.

$H(n_0, t) = x_0$ for $t \in I$. Take a vertex n_1 of N^S. Since the covering $\mathfrak{U}$ is a star refinement of $f^{-1}(\mathfrak{W}_0)$, we can find elements $\tilde{V}_{S+1}(y)$ and $\tilde{U}_{S+1}(y)$ of $\mathfrak{V}_{S+1}$ and $\mathfrak{U}_{S+1}$ such that $\Psi_0(n_1) \cup gf\Psi_0(n_1) \subset \tilde{V}_{S+1}(y) \subset f^{-1}(\tilde{U}_{S+1}(y))$, and the deformation homotopy $\tilde{H}_y^{S+1}\colon \tilde{V}_{S+1}(y)\times I \to f^{-1}(\tilde{U}_{S+1}(y))$ such that $\tilde{H}_y^{S+1}(\tilde{V}_{S+1}(y), 1) \subset A_y$. Consider the product space $\{n_1\}\times I\times J$, where J is the closed interval $\langle 0, 1\rangle$. The subset $\{n_1\}\times(0\cup 1)\times J \cup \{n_1\}\times I\times\{1\}$ of $\{n_1\}\times I\times J$ is a retract of $\{n_1\}\times I\times J$ whose retraction we denote by r_{n_1}. Define $\mu_{n_1}\colon \{n_1\}\times(0\cup 1)\times J\to f^{-1}(\tilde{U}_{S+1}(y))$ by putting $\mu_{n_1}(n_1, 0, t) = \tilde{H}_y^{S+1}(\Psi_0(n_1), t)$ and $\mu_{n_1}(n_1, 1, t) = \tilde{H}_y^{S+1}(gf\Psi_0(n_1), t)$ for $t\in J$. Since A_y is a C^S space and $n-1\leqslant s$, μ_{n_1} is extended to $\tilde{\mu}_{n_1}\colon \{n_1\}\times(0\cup 1)\times J\cup\{n_1\}\times I\times\{1\}\to f^{-1}(\tilde{U}_{S+1}(y))$ such that $\tilde{\mu}_{n_1}(n_1, I, 1)\subset A_y$. Define $H\colon N^0\times I\cup N^S\times(0\cup 1)\to X$ by putting $H(n_1, t) = \tilde{\mu}_{n_1} r_{n_1}(n_1, t)$ for $(n_1, t)\in N^0\times I$. Let n_1n_2 be a 1-simplex of N. There exists an element $\tilde{U}_{S+1}$ of $\mathfrak{U}_{S+1}$ such that $fH(\overline{n_1n_2}\times(0\cup 1)\cup(n_1\cup n_2)\times I)\subset \mathrm{St}\{\mathrm{St}(\tilde{U}_{S+1}, \mathfrak{U}_{S+1}), \mathfrak{U}_{S+1}\}$. Since $\mathfrak{U}_{S+1}$ is a star-star refinement $\mathfrak{Q}_{S+1}$, we can find an element $V_{S+1}(y)$ of $\mathfrak{V}_{S+1}$ such that $H(\overline{n_1n_2}\times(0\cup 1)\cup(n_1\cup n_2)\times I)\subset V_{S+1}(y)$. Suppose that $H\colon N^{i-1}\times I\cup N^S\times(0\cup 1)\to X$ is defined such that for each $(i-1)$-simplex s of N there exists an element $V_{S+2-i}(y_s)$ of $\mathfrak{V}_{S+2-i}$ containing $H(F(s\times I))$, where $F(s\times I) = F(s)\times I\cup s\times(0\cup 1)$. Then there exist the element $U_{S+2-i}(y_s)$ of $\mathfrak{U}_{S+2-i}$ containing $V_{S+2-i}(y_s)$ and the deformation homotopy $H_{y_s}^{S+2-i}\colon V_{S+2-i}(y_s)\times I\to f^{-1}(U_{S+2-i}(y_s))$ such that $H_{y_s}^{S+2-i}(V_{S+2-i}(y_s), 1)\subset A_{y_s}$. Take an i-simplex s of N. Consider the product space $s\times I\times J$. The set $F(s\times I)\times J\cup s\times I\times 1$ is a retract of $s\times I\times J$ whose retraction we denote by r_s. Define $\mu_s\colon F(s\times I)\times J\to f^{-1}(U_{S+2-i}(y_s))$ by putting $\mu_s(n, t, t') = H_{y_s}^{S+2-i}(H(n, t), t')$ for $(n, t, t')\in F(s\times I)\times J$. Since A_{y_s} is a C^S space, μ_s is extended to a continuous mapping $\tilde{\mu}_s$ from $F(s\times I)\times J\cup s\times I\times 1$ to $f^{-1}(U_{S+2-i}(y_s))$. Define $H\colon N^i\times I\cup N^S\times(0\cup 1)\to X$ by putting $H(n, t) = \tilde{\mu}_s r_s(n, t, 0)$ for $n\in s\subset N^S$. Since $N^i\times I$ is a Whitehead complex, $\tilde{\mu}_s$ is a continuous mapping, and for each $(i+1)$-simplex $\tilde{s}$, there exists an element $V_{S+2-(i+1)}(y_{\tilde{s}})$ of $\mathfrak{V}_{S+2-(i+1)}$ containing $H(F(\tilde{s}\times I))$. Thus by induction we can obtain the homotopy $H\colon N^S\times I\to X$ such that $H(n, 0) = \Psi_0(n)$ and $H(n, 1) = gf\Psi_0(n)$. To complete the proof of Theorem 1, it is sufficient that $g_*\colon \pi_i(Y, y_0)\to\pi_i(X, x_0)$ be a homomorphism onto for $i\leqslant S$, because $fg\simeq 1\colon (Y, y_0)\to(Y, y_0)$. Take an element α of $\pi_i(X, x_0)$. Let h be a continuous mapping of (S^i, p_0) into (X, x_0) defining α. We have $h\simeq\Psi_0\Phi_0 h\colon (S^i, p_0)\to(X, x_0)$. Let $\tilde{h}$ be a simplicial approximation to $\Phi_0 h$ such that $\tilde{h}(S^i)\subset N^i$. Obviously, the mapping $\Psi_0\tilde{h}\colon (S^i, p_0)\to(X, x_0)$ represents the element α of $\pi_i(X, x_0)$. Moreover, since $\tilde{h}(S^i)\subset N^i$ and $i\leqslant s$, $gf\Psi_0\tilde{h}\simeq\Psi_0\tilde{h}\colon (S^i, p_0)\to(X, x_0)$. Therefore, we have $gfh\simeq gf\Psi_0\tilde{h}\simeq\Psi_0\tilde{h}\simeq h\colon (S^i, p_0)\to(X, x_0)$. Let β be the element of $\pi_i(Y, y_0)$ represented by

the continuous mapping $fh\colon (S^i, p_0)\to(Y, y_0)$. Then the element $g_*\beta$ of $\pi_i(X, x_0)$ is represented by the mapping gfh. Therefore we have $\alpha = g_*\beta$. This completes the proof of Theorem 1.

THEOREM 2. *Let X and Y be finite dimensional* ANR's *for metric spaces. Let f be a closed continuous mapping from X onto Y such that $f(x_0) = y_0$ and that for each point y of Y $f^{-1}(y)$ is an* AR *for metric spaces. Then f has a homotopy inverse g, that is, there exists a continuous mapping g of Y into X such that $g(y_0) = x_0$ and that $gf\simeq 1\colon (X, x_0)\to(X, x_0)$ and $fg\simeq 1\colon (Y, y_0)\to(Y, y_0)$.*

Proof. Let $\dim X = S$. Since for each point y of Y $f^{-1}(y)$ is an AR for metric spaces, it is a C^S space. In the same way as in the proof of Theorem 1, we can construct an open covering $\mathfrak{U}$ of X and a continuous mapping $g\colon (Y, y_0)\to(X, x_0)$ such that

(i) $\mathfrak{U}$ is a locally finite covering and its order is $S+1$,

(ii) if we denote the nerve of $\mathfrak{U}$ with Whitehead's weak topology by N, there exists $\Psi_0\colon (N, n_0)\to(X, x_0)$ such that $\Psi_0\Phi_0\simeq 1\colon (X, x_0)\to(X, x_0)$, where Φ_0 is a canonical mapping of (X, x_0) into (N, n_0),

(iii) we have homotopies $fg\simeq 1\colon (Y, y_0)\to(Y, y_0)$ and $gf\Psi_0\simeq\Psi_0\colon (N, n_0)\to(X, x_0)$.

By (i), (ii) and (iii), we can prove that $gf\simeq 1\colon (X, x_0)\to(X, x_0)$ and $fg\simeq 1\colon (Y, y_0)\to(Y, y_0)$.

The following corollaries are consequences of Theorems 1 and 2, ([7], Theorem 3.5, p. 15, [6], Theorem 7.1, p. 195 and Theorem 5.1, p. 240 and [17], Theorem 7.4, p. 214).

COROLLARY 1. *Under the same assumptions as in Theorem 1, the homotopy groups, homology groups, cohomotopy groups and cohomology groups of X contain the corresponding groups of Y as a direct factor for each dimension. The homomorphism induced by f in an isomorphism for each dimension $i = 0, 1, 2, \ldots, S$.*

COROLLARY 2. *Under the same assumptions of Theorem 2, the homotopy groups, homology groups, cohomotopy groups and cohomology groups of X are isomorphic to the corresponding groups of Y for each dimension.*

4. In this section, we strengthen the theorems of 3 in the following form:

THEOREM 3. *Let (X, A) and (Y, B) be pairs of* ANR's *for metric spaces and let $\dim Y = n$. Let f be a closed continuous mapping from (X, A) onto (Y, B) such that $f^{-1}(B) = A$ and for each point y of Y $f^{-1}(y)$ is an* ANR *for metric spaces and a C^S space, where S is any integer such that*

$n-1 \simeq S$. *Then there exists a continuous mapping* g *of* (Y, B) *into* (X, A) *such that*

(i) $fg \simeq 1$: $(Y, B) \to (Y, B)$,

(ii) *for each integer* $i = 0, 1, \ldots, S$, *the induced homomorphism* g_*: $\pi_i(Y, B) \to \pi_i(X, A)$ *is an isomorphism onto and the induced homomorphism* $g_* f_*$: $\pi_i(X, A) \to \pi_i(X, A)$ *is the identity isomorphism.*

THEOREM 4. *Let* (X, A) *and* (Y, B) *be pairs of finite dimensional ANR's for metric spaces and let* f *be a closed continuous mapping from* (X, A) *onto* (Y, B) *such that* $f^{-1}(B) = A$ *and for each point* y *of* Y $f^{-1}(y)$ *is an AR for metric spaces. Then there exists a continuous mapping* g *of* (Y, B) *into* (X, A) *such that* $gf \simeq 1$: $(X, x_0) \to (X, A)$ *and* $fg \simeq 1$: $(Y, B) \to (Y, B)$.

These theorems are proved in the same way as Theorems 1 and 2, by virtue of the following lemmas:

LEMMA 3. *Let* (X, A) *be a pair of ANR's for metric spaces. Then there exists an open covering* $\mathfrak{U} = \{U_a\}$ *of* X *with the following property: Whenever* (Z, C) *is a pair of metric spaces and* f_0, f_1: $(Z, C) \to (X, A)$ *such that for each point* x *of* X *there exists an element* U_a *of* $\mathfrak{U}$ *containing* $f_0(x) \cup f_1(x)$, *we have* $f_0 \simeq f_1$: $(Z, C) \to (X, A)$.

LEMMA 4. *Let* (X, A) *be a pair of ANR's for metric spaces. Then, for any open covering* $\mathfrak{U} = \{U_a\}$ *of* X, *there exist a pair of Whitehead's complexes* (K, L) *and two mappings* Φ: $(X, A) \to (K, L)$, Ψ: $(K, L) \to (X, A)$ *such that for each point* x *of* X *there exists an element* U_a *of* $\mathfrak{U}$ *containing* $x \cup \Psi\Phi(x)$.

We prove only Lemmas 4 and 3, and we omit the proofs of Theorems 3 and 4.

Proof of Lemma 3. Since (X, A) is a pair of ANR's for metric spaces, by Lemma 1, we can find a neighborhood V of A in X and a deformation homotopy H: $V \times I \to X$ such that $H(a, t) = a$ for $(a, t) \in A \times I$ and $H(x, 1) \in A$ for $x \in V$. For each point a of A, take an open set $W(a)$ containing a and contained in V. For each point x of $X - A$, take an open set $W(x)$ containing x and contained in $X - A$. Denote by $\mathfrak{W}$ the open covering $\{W(x) \mid x \in X\}$ of X. We can find an open covering $\mathfrak{U} = \{U_a\}$ of X such that

(i) $\mathfrak{U}$ is an open refinement of $\mathfrak{W}$,

(ii) whenever Z is a metric space and g_0, g_1: $Z \to X$ such that for each point z of Z there exists an element U_a of $\mathfrak{U}$ containing $g_0(z) \cup g_1(z)$, there exists a homotopy H_0: $Z \times I \to X$ such that $H_0(z, 0) = g_0(z)$, $H_0(z, 1) = g_1(z)$ and for each point z of Z we can find an element $W(x)$ of $\mathfrak{W}$ containing $H_0(z \times I)$.

We shall show that the covering $\mathfrak{U}$ is the required one. Let f_0 and f_1 be two mappings of (Z, C) to (X, A) such that for each point z of Z there exists an element U_a of $\mathfrak{U}$ containing $f_0(z) \cup f_1(z)$. By (ii), we have a homotopy H_0: $Z \times I \to X$ such that $H_0(z, 0) = f_0(z)$, $H_0(z, 1) = f_1(z)$ and for each point z of Z there exists an element $W(x)$ of $\mathfrak{W}$ such that $H_0(z \times I) \subset W(x)$. Then we have $H_0(c \times I) \subset V$ for $c \in C$. Define H_1: $Z \times I \times 0 \cup Z \times (0 \cup 1) \times J \cup C \times I \times J \to X$ by putting

$$
\begin{aligned}
H_1(z, t, 0) &= H_0(z, t) && \text{for} \quad (z, t) \in Z \times I,\\
H_1(z, 0, t') &= f_0(z) && \text{for} \quad (z, 0, t') \in Z \times 0 \times J,\\
H_1(z, 1, t') &= f_1(z) && \text{for} \quad (z, 1, t') \in Z \times 1 \times J,\\
H_1(c, t, t') &= H(H_0(c, t), t') && \text{for} \quad (c, t, t') \in C \times I \times J,
\end{aligned}
$$

where J is the closed interval $\langle 0, 1 \rangle$. Then H_1 is a continuous mapping and we have $H_1(C \times I \times 1) \subset A$. Since X is an ANR for metric spaces, by [5] Theorem 7.1, p. 363, H_1 has an extension H_2 over a neighborhood W of $Z \times I \times 0 \cup Z \times (0 \cup 1) \times J \cup C \times I \times J$ in $Z \times I \times J$. Since $I \times J$ is compact, there exists a neighborhood U of C in Z such that $U \times I \times J \subset W$. Since Z is normal, there exists a continuous function μ of Z into $\langle 0, 1 \rangle$ such that $\mu(c) = 0$ for $c \in C$ and $\mu(z) = 1$ for $z \in Z - U$. Consider the subset $(0 \cup 1) \times J \cup I \times 0$ of the set $I \times J$. There exists a deformation homotopy h_s: $I \times J \to I \times J$ for $s \in \langle 0, 1 \rangle$ such that

$$
\begin{aligned}
&h_0 = \text{the identity mapping},\\
&h_s(i, t') = (i, t') \quad \text{for} \quad i = 0 \text{ or } 1 \quad \text{and} \quad t' \in J,\\
&h_s(t, 0) = (t, 0) \quad \text{for} \quad t \in I,\\
&h_1(t, t') \in (0 \cup 1) \times J \cup I \times 0 \quad \text{for} \quad (t, t') \in I \times J.
\end{aligned}
$$

Define a homotopy π: $Z \times I \times J \to X$ by putting $\pi(z, t, t') = H_2(z, h_{\mu(z)}(t, t'))$ for $(z, t, t') \in Z \times I \times J$. Then π is continuous and is an extension of H_1. Consider the homotopy T: $Z \times I \to Y$ defined by $T(z, t) = \pi(z, t, 1)$ for $(z, t) \in Z \times I$. Then we have $T(z, 0) = f_0(z)$, $T(z, 1) = f_1(z)$ and $T(C \times I) \subset A$. This completes the proof of Lemma 3.

Proof of Lemma 4. (Cf. [10], p. 96-7). By Wojdysławski's theorem ([20], p. 186), we can consider that X is a closed subset of a convex subset B of a normed vector space. Since X and A are ANR's for metric spaces, there exist an open neighborhood W_0 of X in B, an open neighborhood W_1 of A in B contained in W_0, retractions h_0: $W_0 \to X$ and h_1: $W_1 \to A$. Put $V = W_1 \cap X$. Let $S_1(x)$ be a spherical neighborhood of a point x of X in B such that

(i) if $a \in A$, $S_1(a) \cap X \subset V$ and $S_1(a) \subset W_1$,

(ii) if $x \in X-A$, $S_1(x) \cap X \subset X-A$ and $S_1(x) \subset W_0$,

(iii) $\{S_1(x) \cap X|\ x \in X\}$ forms an open refinement of $\mathfrak{U}$.

For each point x of X, take a spherical neighborhood $S_2(x)$ of x in B such that $S_2(x) \subset h_0^{-1}(S_1(x) \cap X) \cap S_1(x)$. Let $S_3(x)$ be a spherical neighborhood of x in B such that

(i) if $a \in A$, $S_3(a) \subset h_1^{-1}(S_2(a) \cap A) \cap S_2(a)$,

(ii) if $x \in X-A$, $S_3(x) \subset h_0^{-1}(S_2(x) \cap X) \cap S_2(x)$.

Let $\mathfrak{V}$ be a locally finite open covering of (X, A) which is a star refinement of $\{S_3(x) \cap X|\ x \in X\}$ and let (K', L') be the pair of nerve of $\mathfrak{V}$ with Whitehead's weak topology and let (K, L) be the barycentric subdivision of (K', L'). Then a simplex s whose vertexes belong to L belongs to L. Denote by N the closed subcomplex of K spanned by vertexes which do not belong to L. If k is a point of $K-N \cup L$, we can determine uniquely two points a, b of N, L such that $k \in$ open segment $\overline{ab} \subset K-N \cup L$. By the same way as [9], Theorem 5, p. 39, we can construct a mapping Ψ_1: $N \cup L \to \bigcup \{S_3(x)|\ x \in X\}$ such that for each simplex s of K there exists a spherical open set $S_3(x)$ containing $\Psi_1(s \cap (K \cup L)$. Define Ψ_2: $L \to A$ by $\Psi_2 = h_1 \Psi_1 | L$. Then, if $s = (v_0, \ldots, v_i, v_{i+1}, \ldots, v_n)$ is a simplex of K such that $(v_0, \ldots, v_i) \in N$ and $(v_{i+1}, \ldots, v_n) \in L$ there exists a spherical open set $S_2(x)$ containing $\Psi_1(v_0, \ldots, v_i) \cup \Psi_1(v_{i+1}, \ldots, v_n)$. Therefore there exists a continuous mapping Ψ_3: $K \to \bigcup \{S_2(x)|\ x \in X\}$ such that $\Psi_3 | N = \Psi_1 | N$, $\Psi_3 | L = \Psi_2$ and for each simplex s of K we can find a spherical open set $S_2(x)$ containing $\Psi_3(s)$. Define Ψ: $(K, L) \to (X, A)$ by $\Psi = h_0 \Psi_3$. Let Φ be a canonical mapping of (X, A) into (K', L'). Take a point x of X. Let $V_0, \ldots, V_n$ be all elements of $\mathfrak{V}$ containing x. Then $\Phi(x) \in \overline{(v_0, \ldots, v_n)}$ ([3], p. 202), where v_i is the vertex corresponding to the element V_i of $\mathfrak{V}$. Since $\mathfrak{V}$ is a star refinement of $\{S_2(x) \cap X|\ x \in X\}$, there exists a spherical open set $S_2(x')$ such that $\bigcup_{i=0}^{n} V_i \subset S_2(x') \cap X$. Hence, $x \cup \Psi\Phi(x) \subset S_1(x') \cap X$. Since $\{S_1(x) \cap X|\ x \in X\}$ is a refinement of $\mathfrak{U}$, there exists an element U_a of $\mathfrak{U}$ containing $x \cup \Psi\Phi(x)$. This completes the proof of Lemma 4.

Finally we have the following corollaries.

Corollary 3. *Under the same assumptions as in Theorem 3, the homotopy groups, homology groups, cohomotopy groups and cohomology groups of (X, A) contain the corresponding groups of (Y, B) as a direct factor for each dimension. The homomorphism induced by f is an isomorphism for each dimension $i = 0, 1, 2, \ldots, S$.*

Corollary 4. *Under the same assumptions of Theorem 4, the homotopy groups, homology groups, cohomotopy groups and cohomology groups of (X, A) are isomorphic to the corresponding groups of (Y, B) for each dimension.*

Remark 1. By simple examples, it can be shown that we cannot omit the condition "f is a closed mapping" and we cannot replace the condition "X is an ANR" by the condition "X is a compact metric space" in the above theorems.

Remark 2. In Theorems 1-4, we can replace the condition "Y is an n-dimensional ANR for metric spaces" by the condition "Y is an n-dimension LC^n metric space". (Cf. [11], Theorem 1.)

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