On some equation in transfinite ordinals

by

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In this paper we consider the equation

$$\xi^{\varphi} = \eta^{\psi} + q, \tag{1}$$

where $\varphi, \psi > 1$ are not limit numbers and q is finite.

THEOREM 1. *If $q = 0$, then all ordinals ξ, η which are not limit numbers and satisfy* (1) *are given by the formulae*

$$\xi = \tau^m, \quad \eta = \tau^n,$$

where the exponents m, n are finite and satisfy

$$m\varphi = n\psi .$$

THEOREM 2. *If $q > 0$, then no transfinite ordinals ξ, η satisfy* (1).

Theorem 2, for $q = 1$, $\varphi = 2$, $\psi = 3$, was proved by W. Sierpiński [2].

1. Let us denote by (a, b) the largest common divisor of a and b.

AUXILIARY LEMMA. *If k, l are natural numbers, $s = (k, l)$, $k = k's$, $l = l's$ and for an ordinal ν and a sequence of ordinals $\mu_0, \dots, \mu_t$ which has at least $kl'-1$ elements* (*i. e.*, $t \geqslant kl'-2$)

$$\nu k' + \mu_{j+k} = \nu l' + \mu_{j+l} = \mu_j \tag{2}$$

holds, then

$$\mu_j = \nu + \mu_{j+s} . \tag{3}$$

Proof. Evidently we can assume that $s < k, l$. There exist such numbers p, q that $0 < p \leqslant l'$ and

$$pk' - ql' = 1, \quad pk - ql = s . \tag{4}$$

From (2) and (4) follows

$$\mu_j = \nu pk' + \mu_{j+pk} = \nu + \nu ql' + \mu_{j+s+ql} = \nu + \mu_{j+s} \tag{5}$$

for $j + pk \leqslant t$. We shall prove first that (3) holds for $j = 0, \dots, k-1$. It is $p < l'$. Indeed, if $p = l'$ then from (4) follows $l' = 1$, which contradicts $s < l$. From $p < l'$ follows $j + pk \leqslant j + (l'-1)k \leqslant kl'-2 \leqslant i$ for $j = 0, \ldots, k-2$. Thus (5) holds for $j < k-1$ and consequently (3) also. From (3) (where $j < k-1$) it follows by easy induction that $\mu_{s-1} = \nu(k'-1) + \mu_{(k'-1)s+s-1} = \nu(k'-1) + \mu_{k-1}$. On the other hand $\mu_{s-1} = \nu k' + \mu_{k-1+s}$ by (2). Comparing the last two equalities we obtain (3) for $j = k-1$.

We proved that $\mu_i = \nu + \mu_{i+s}$ holds for $i < k$. Thus from (2) it follows that $\nu k' r + \mu_{i+kr} = \nu + \nu k' r + \mu_{i+s+kr}$ for each r and $i < k$. Therefore $\mu_{i+kr} = \nu + \mu_{i+kr+s}$. Since each integer j is of the form $i + kr$, our lemma follows.

2. Definitions. We introduce the notations

$$\omega^{\gamma} c = [\gamma, c], \qquad \sum_{r \leqslant i \leqslant s} [\gamma_i, c_i] = [\gamma_i, c_i]_{r \leqslant i \leqslant s}.$$

As we know, each ordinal δ can be uniquely represented in the *normal form*

$$\delta = \omega^{\gamma_0} c_0 + \omega^{\gamma_1} c_1 + \ldots + \omega^{\gamma_n} c_n = [\gamma_i, c_i]_{i \leqslant n},$$

where $\gamma_0 > \gamma_1 > \ldots > \gamma_n \geqslant 0$ and the numbers $n, c_0, \ldots, c_n$ are finite.

Let μ be any ordinal. If μ is transfinite, then we denote by $\bar{\mu}$ the limit number for which $-\bar{\mu} + \mu$ is finite. If μ is finite, then we set $\bar{\mu} = 0$. We suppose that

$$\xi = [a_i, a_i]_{i \leqslant k}, \qquad \eta = [\beta_i, b_i]_{i \leqslant l}, \qquad \varphi = \bar{\varphi} + c, \qquad \psi = \bar{\psi} + d.$$

3. *We assume that* (1) *holds and* ξ, η *are not limit numbers*. The normal forms of ξ^{φ} and η^{ψ} are then (see [1])

$$\begin{aligned} \xi^{\varphi} &= [a_0\varphi, a_0] + [a_0(\varphi-1) + a_i, a_i]_{1 \leqslant i < k} + [a_0(\varphi-1), a_0 a_k] + \\ &\quad + [a_0(\varphi-2) + a_i, a_i]_{1 \leqslant i < k} + [a_0(\varphi-2), a_0 a_k] + \ldots + [a_0\bar{\varphi} + a_i, a_i]_{1 \leqslant i \leqslant k} \\ &= [a_i', a_i']_{i \leqslant ck}, \\ \eta^{\psi} &= [\beta_0\psi, b_0] + [\beta_0(\psi-1) + \beta_i, b_i]_{1 \leqslant i < l} + [\beta_0(\psi-1), b_0 b_l] + \ldots + \\ &\quad + [\beta_0\bar{\psi} + \beta_i, b_i]_{1 \leqslant i \leqslant l} \\ &= [\beta_i', b_i']_{i \leqslant dl}. \end{aligned} \tag{6}$$

Equation (1) implies

$$ck = dl = u, \quad a_i' = b_i' \quad \text{for} \quad i < u, \quad a_i' = \beta_i' \quad \text{for} \quad i \leqslant u, \quad a_k = b_l + q.$$

We define s, k', l' as in the Auxiliary Lemma. Then

3.1. *There exists such an ordinal* $\varkappa$ *that* $a_0 = \varkappa k'$ *and* $\beta_0 = \varkappa l'$.

Proof. It is $a_0 c = -a_u' + a_0' = -\beta_u' + \beta_0' = \beta_0 d$. Comparing the normal forms of $a_0 c$ and $\beta_0 d$ we easily find a number $\varkappa$ for which $a_0 = \varkappa d'$ and $\beta_0 = \varkappa c'$, where $c = c'(c, d)$, $d = d'(c, d)$. It remains to observe that $ck = dl$ implies $d' = k'$, $c' = l'$.

3.2. $a_i = \varkappa + a_{i+s}$.

Proof. Let us observe that in (6)

$$a_{rk+i}' = a_0\bar{\varphi} + a_0(c-r-1) + a_i \quad \text{for} \quad i < k, \quad r < c. \tag{7}$$

We define $a_i'' = -a_u' + a_i'$, $\beta_i'' = -\beta_u' + \beta_i'$. Evidently $a_i'' = \beta_i''$. From (7) follows $a_i'' = a_0 + a_{i+k}''$ and consequently, by 3.1, $a_i'' = \varkappa k' + a_{i+k}''$. Similarly we arrive at $\beta_i'' = \varkappa l' + \beta_{i+l}''$, which implies $a_i'' = \varkappa l' + a_{i+l}''$. Since $u \geqslant c'k = l'k$, we can apply the Auxiliary Lemma. We obtain $a_i'' = \varkappa + a_{i+s}''$. From these equalities and from (7) for $i \leqslant k-s$ follows

$$a_0(c-1) + a_i = \varkappa + a_0(c-1) + a_{i+s}.$$

It remains to substitute $\varkappa k'$ for a_0.

3.3. *If* $\tau = [\varkappa, a_0] + [a_i, a_i]_{k-s < i \leqslant k}$, *then* $\xi = \tau^{k'}$.

Proof. From 3.2 follows $a_1 = \varkappa(k'-1) + a_{k-s+1}$. Since, by 3.1, $a_1 < \varkappa k'$, we obtain $\varkappa > a_{k-s+1}$. This proves that τ is defined by its normal form. Let us compute $\tau^{k'}$;

$$\begin{aligned} \tau^{k'} &= [\varkappa k', a_0] + [\varkappa(k'-1) + a_i, a_i]_{k-s < i < k} + \\ &\quad + [\varkappa(k'-1), a_0 a_k] + [\varkappa(k'-2) + a_i, a_i]_{k-s < i < k} + \ldots + [a_i, a_i]_{k-s < i \leqslant k}. \end{aligned}$$

One easily sees that $\xi = \tau^{k'}$ is equivalent to

I. $a = \varkappa + a_{i+s}$,

II. $a_i = a_{i+s}$ for $i \geqslant 1$, $i + s < k$,

III. $a_s = a_0 a_k$ if $k' > 1$.

The first of these conditions was proved in 3.2. Let us prove that II holds. We consider the sequence $a_1', \ldots, a_{u-1}'$. From (6) follows $a_i' = a_{i+k}'$. Since similarly $b_i' = b_{i+l}'$ we obtain $a_i' = a_{i+l}'$. From the Auxiliary Lemma follows $a_i' = a_{i+s}'$. This implies II.

We shall prove that III holds. If $c > 1$, then $a_k' = a_0 a_k$. Since II implies $a_k' = a_s'$, III follows. If $c = 1$, then φ is transfinite by $\varphi > 1$. Therefore ξ^{φ} is a limit number and $q = 0$. This implies $a_k = b_l$ and consequently $a_0 a_k = b_0 b_l$. Now from $k = dl$ follows $d = k'$ and thus $d > 1$. This implies $b_l' = b_0 b_l = a_0 a_k$. It remains to observe that $b_l' = a_l' = a_s$.

3.4. $k'\varphi = l'\psi$.

Proof. We have $a_0\bar{\varphi} = a_u' = \beta_u' = \beta_0\bar{\psi}$. Evidently $c\bar{\varphi} = \bar{\varphi}$ and $d\bar{\psi} = \bar{\psi}$. Thus $a_0 c\bar{\varphi} = \beta_0 d\bar{\psi}$. Since $a_0 c = \beta_0 d$ by 3.1, we obtain $\bar{\varphi} = \bar{\psi}$. The rest of the proof is given by the equalities

$$k'\varphi = d'\varphi = d'(\bar{\varphi} + c) = \bar{\varphi} + d'c = \bar{\psi} + c'd = c'\psi = l'\psi.$$

4. In this section we shall prove Theorem 1 and 2.

4.1. Proof of Theorem 1. If (1) holds, then $\eta^{\psi} = \tau^{k'\varphi} = \tau^{l'\psi}$ by

$\xi = \tau^{k'}$ and $k'\varphi = l'\psi$. Since ψ is not a limit ordinal, it follows that $\eta = \tau^{l'}$. Defining $m = k'$, $n = l'$ we have $\xi = \tau^m$, $\eta = \tau^n$.

It is evident that, conversely, the equalities $\xi = \tau^m$, $\eta = \tau^n$, $m\varphi = n\psi$ imply (1).

4.2. Proof of Theorem 2. We suppose that (1) holds for $q > 0$. Then ξ is not a limit number and thus ξ^φ is given by (6). Since ξ^φ is not a limit ordinal, we obtain $\varphi = c$.

Let us suppose that η is a limit number. Then (see [1])

$$\eta^\psi = [\beta_0(\psi-1)+\beta_i, b_i]_{i\leqslant l}. \tag{8}$$

Comparing the first and the last but one terms in (6) and (8) we obtain $a_0 c = \beta_0\psi$ and $a_{k-1} = \beta_0(\psi-1)+\beta_{l-1}$. Since $a_{k-1} \leqslant a_0$, it follows from these equalities that $(\psi-1)c < \psi$. This is impossible by $c, \psi > 1$.

We suppose that η is not a limit number. Then $\psi = d$ by 3.4. From $c, d > 1$ follows $a'_k = a_0 a_k$ and $b'_l = b_0 b_l$. Since, by 3.3, $a'_j = a'_{j+s}$ (cf. proof of II), we have $a'_s = a'_k = a'_l$. Thus, by $a'_l = b'_l$, we have $a_0 a_k = b_0 b_l$. This contradicts $a_0 = b_0$, $a_k = b_l + q$.

References

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On a closed mapping between ANR's

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1. Let X and Y be topological spaces. A continuous mapping f of X into Y is called a *closed mapping* if and only if, whenever A is a closed subset of X, $f(A)$ is a closed subset of Y. In the present note, we shall show that if X and Y are ANR's for metric spaces and there exists a closed continuous mapping f of X onto Y satisfying a certain condition, then there exist some intimate relations between combinatorial invariants of X and Y. In more detail, if Y is a finite dimensional ANR for metric spaces and for each point y of Y $f^{-1}(y)$ is an ANR for metric spaces having a certain acyclic property, we shall prove that there exists a continuous mapping g of Y into X such that $fg \simeq 1$. Moreover, if X is a finite dimensional ANR for metric spaces, we shall prove that X has the same homotopy type as Y.

In 2, several notations and lemmas which we shall need later on are given. We shall prove our main theorems in 3. In 4, we shall prove some theorems strengthening the main theorems.

2. A topological space X is called an AR (resp. ANR) *for metric spaces* if and only if, whenever X is a closed subset of a metric space Y, there exists a retraction (¹) of Y (resp. some neighborhood of X in Y) onto X. (Cf. [1] or [13], Definition 2.2, p. 790.) A topological space X is called an NES *for metric spaces* if and only if, whenever Y is a metric space and B is a closed subset of Y, any continuous mapping of B into X can be extended to a continuous mapping of some neighborhood of B in Y into X. (Cf. [13], Definition 2.1, p. 790.) A metric space X is called an LC^n *space* if and only if for each point x of X and for each neighborhood U of x there exists a neighborhood V of x such that any continuous mapping g from an i-sphere S^i to V is extended to a continuous mapping $\tilde{g}$ from an $(i+1)$-element E^{i+1} with the boundary S^i into U, $i = 0, 1, \ldots, n$. (Cf. [12], p. 79.) A metric space X is called a C^n *space* if and only if any continuous mapping g from an i-sphere S^i to X is

(¹) By a *retraction* h of Y into X we mean a continuous mapping from Y onto X such that $h(x) = x$ for each point x of X.