# Some applications of interior mappings

by

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**§ 1. Introduction.** This paper contains some applications of the following theorems on interior mappings:

THEOREM I. *A mapping $\varphi$ of a topological space* [1] *$X$ into a topological space $Y$ is interior* [2] *if and only if*

$$\varphi^{-1}(\bar{A}) = \overline{\varphi^{-1}(A)} \quad \textit{for every set } A \subset Y.$$

THEOREM II. *If a topological space has an enumerable open basis and $\bar{\bar{X}} \leqslant 2^{\aleph_0}$, then $X$ is an interior image* [3] *of a set $Z$ of irrational numbers.*

Theorem I was proved by A. D. Wallace [6] (see also [4]). Theorem II is a slight modification of a Theorem of A. Šwarc [5] and will be proved below (see § 3).

The first application concerns the problem of enumerable topological polynomials.

By a *topological polynomial* we shall understand, roughly speaking an expression $\alpha$ such that:

$\alpha$ contains some variables running through subsets of any topological space;

these variables are connected by means of finite set-theoretical operations (union, intersection, complementation) and of the closure operation only.

For instance, the expressions [4]

(1) $$\overline{A+B}\cdot\big(-(\bar{A}+\bar{B})\big),$$

(1) By a *topological space* we understand in this paper a set $X$ with a closure operation satisfying the four well-known axioms of Kuratowski (see § 4, (1)). It is not assumed that every one-point set is closed.

(2) A mapping $\varphi$ of $X$ into $Y$ is said to be *interior* provided $\varphi$ is continuous and $\varphi(G)$ is open in $Y$ for every set $G$ open in $X$.

(3) That is, there exists an interior mapping of $Z$ onto $X$.

(4) The set-theoretical union, intersection and complement will be denoted by $A+B$, $AB$ and $-A$ respectively. The empty set is denoted by 0.



(2) $$A\cdot\overline{BC} - \overline{AC}\cdot C$$

are topological polynomials with variables $A, B, C$.

If we fix a topological space $X$, then every topological polynomial $\alpha$ (with variables $A, B, \ldots$) can be interpreted as a mapping whose variables $A, B, \ldots$ run through all subsets of $X$, and whose values are also subsets of $X$. This mapping will be denoted by $\alpha_X$. We write $\alpha_X \equiv 0$ if, for arbitrary values of the variables $A, B, \ldots$ (running through subsets of $X$), $\alpha_X$ assumes always the value 0. We write $\alpha \equiv 0$, if $\alpha_X \equiv 0$ for every topological space $X$. For instance, if $\alpha$ is the polynomial (1), we have $\alpha \equiv 0$; if $\alpha$ is the polynomial (2), then $\alpha \not\equiv 0$.

Mc Kinsey and Tarski [1] have proved that if $\alpha_{X_0} \equiv 0$ for a separable metrizable dense in itself space $X_0$, then $\alpha \equiv 0$. In particular, if $\alpha_X \equiv 0$ for a dense in itself subset $X$ of the Euclidean space, then $\alpha \equiv 0$.

In this paper we shall prove an analogous theorem for enumerable topological polynomials. By *enumerable polynomials* we shall understand, roughly speaking, some expression (*i. e.*, finite sequences of signs) composed of symbols of the finite or enumerable set-theoretical operations, of the symbol of the closure operation and of symbols denoting $n$-ary sequences of subsets of any topological space.

The following problems arise: Does there exists a topological space $X_0$ such that, for every enumerable topological polynomial $\alpha$, if $\alpha_{X_0} \equiv 0$, then $\alpha \equiv 0$? Can the space $X_0$ with the above property be separable and metrizable?

The positive answer to the first question was given in [2] (in a slightly different formulation). The positive answer to the second question will be given in § 6.

The second application concerns some problems in Mathematical Logic. It was shown in [2] that every formula $\alpha$ from the Lewis functional calculus (or: from the Heyting functional calculus) can be interpreted as an enumerable topological polynomial (denoted in this section by the same letter $\alpha$). It was proved in [2] that there is a topological space $X_0$ such that a formula $\alpha$ is provable if and only if $\alpha_{X_0} \equiv X_0$. The problem whether there exists a separable metric space $X_0$ with the above property remained open. The positive answer to this problem will be given in § 7.

As the third application of Theorems I and II we shall prove in § 8 a representation theorem for closure algebras. This theorem is the solution of a problem formulated in [4].

The final section contains a theorem analogous to Theorem I. This theorem gives a new characterization of interior mappings.

**§ 2. Some topological lemmas.** A sequence $\{A_n\}$ of subsets of a set $X$ is said to *separate* the set $X$ if, for arbitrary distinct $x, y \in X$, there exists an integer $n$ such that exactly one of the elements $x, y$ belongs to $A_n$. Observe that a sequence separating $X$ exists if and only if $\overline{\overline{X}} \leqslant 2^{\aleph_0}$. In fact, the set $C$ of all the sequences $\{a_n\}$ where $a_n = 0$ or $1$ is of the power $2^{\aleph_0}$ and contains a separating sequence $A_n$ (*e. g.*, let $A_n$ be the set of all $\{a_n\}$ such that $a_n = 0$). Each subset of a set containing a separating sequence, and each one-to-one image of it contains also a separating sequence. On the other hand, if $\{A_n\}$ separates $X$, then the mapping

$$\varphi(x) = \{a_n\} \in C \quad \text{where} \quad a_n = \begin{cases} 0 & \text{if} \quad x \notin A_n, \\ 1 & \text{if} \quad x \in A_n \end{cases}$$

is a one-to-one transformations of $X$ into $C$. Hence $\overline{\overline{X}} \leqslant 2^{\aleph_0}$.

A sequence $\{G_n\}$ of open subsets of a topological space $X$ is said to be an (enumerable) open *basis* of $X$ if every open subset of $X$ is the union of some sets $G_n$. A sequence $\{G_n\}$ of open subsets of $X$ is said to be an open *subbasis* of $X$ if the set of all finite intersections $G_{n_1} G_{n_2} \ldots G_{n_k}$ is a basis of $X$.

Let $Y_0$ denote the topological space composed of four elements $g_1, g_2, f_1, f_2$ with the closure operation determined by the conditions

$$\overline{(g_1)} = \overline{(g_2)} = Y_0, \qquad \overline{(f_1)} = \overline{(f_2)} = (f_1, f_2)$$

where $(a_1, \ldots, a_n)$ denotes the set composed of elements $a_1, \ldots, a_n$.

The space $Y_0$ contains exactly three closed sets,

$$0, \quad F_0 = (f_1, f_2), \quad Y_0$$

and exactly three open sets,

$$0, \quad G_0 = (g_1, g_2), \quad Y_0.$$

The open set $G_0$ is dense in $Y_0$.

Let $Y$ be the enumerable Cartesian product $Y_0 \times Y_0 \times Y_0 \times \ldots$ with the topology determined in the usual way by the topology in $Y_0$. The sets

$$H_n = \underbrace{Y_0 \times \ldots \times Y_0}_{(n-1)\text{-times}} \times G_0 \times Y_0 \times Y_0 \times \ldots,$$

together with $0$ and $Y$, form an open subbasis of $Y$.

LEMMA 1. *Each topological space $X$ of power $\leqslant 2^{\aleph_0}$ and having an enumerable open basis is homeomorphic to a subset of $Y$.*

Let $\{A_n\}$ be a sequence separating $X$ nad let $\{G_n\}$ be an open basis of $X$. The mapping

$$\varphi(x) = \{y_n(x)\} \in Y,$$



where

$$y_n(x) = \begin{cases} g_1 & \text{if} \quad x \in G_n A_n, \\ g_2 & \text{if} \quad x \in G_n - A_n, \\ f_1 & \text{if} \quad x \in A_n - G_n, \\ f_2 & \text{if} \quad x \notin A_n + G_n, \end{cases}$$

transforms $X$ onto a subset $\varphi(X)$ of $Y$. The mapping $\varphi$ is one-to-one since $\{A_n\}$ separates $X$. It is continuous since $\varphi^{-1}(H_n) = G_n$. The mapping $\varphi$ transforms open sets into open subsets of $\varphi(X)$ since $\varphi(G_n) = H_n \varphi(X)$. This proves that $\varphi$ is a homeomorphism.

LEMMA 2. *There exists an interior mapping $\varphi$ of the set $N$ of all irrational numbers onto the space $Y$.*

Let $G$ be an open dense subset of $N$ such that the closed set $F = N - G$ is dense in itself. The set $F$ can be decomposed into two disjoint subsets $F_1, F_2$ dense in $F$. Analogously $G$ can be decomposed into two disjoint subsets $G_1, G_2$ dense in $G$ (and, consequently, in $N$ too). The mapping

$$\varphi_0(x) = \begin{cases} g_i & \text{if} \quad x \in G_i, \\ f_i & \text{if} \quad x \in F_i \end{cases}$$

$(i = 1, 2)$ is an open interior mapping of $N$ onto $Y_0$. Consequently the mapping

$$\varphi(\{x_n\}) = \{\varphi_0(x_n)\} \quad \text{for} \quad x_n \in N \times N \times \ldots$$

is an interior transformation of $N \times N \times \ldots$ onto $Y = Y_0 \times Y_0 \times \ldots$ Since $N \times N \times \ldots$ is homeomorphic to $N$, Lemma 2 is proved.

**§ 3. The proof of Theorem II.** By Lemma 1 we can assume that $X \subset Y$ where $Y$ is the space defined in § 2. Let $Z = \varphi^{-1}(X)$ where $\varphi$ is the mapping defined in Lemma 2. The mapping $\psi = \varphi | Z$ is an interior mapping of $Z$ onto $X$ [5].

**§ 4. Closure algebras.** By a *closure algebra* we shall understand a Boolean algebra [6] $\boldsymbol{A}$ with a closure operation which, with every element $A \in \boldsymbol{A}$, associates an element $\overline{A} \in \boldsymbol{A}$ in such a way that the Kuratowski axioms:

$$\overline{A+B} = \overline{A} + \overline{B}, \quad A \subset \overline{A}, \quad 0 = \overline{0}, \quad \overline{(\overline{A})} = \overline{A}$$

are satisfied.

[5] This proof of Theorem II is a slight modification of the proof of a theorem of A. Šwarc [5].

[6] The Boolean sum (join), product (meet) and complement will be denoted by $A+B$, $A \cdot B$ and $-A$ respectively ($A, B \in \boldsymbol{A}$). The enumerable sum (join) and product (meet) will be denoted (if they exist) by $\sum_{n=1}^{\infty} A_n$ and $\prod_{n=1}^{\infty} A_n$ or by $A_1 + A_2 + \ldots$ and $A_1 A_2 \ldots$ respectively ($A_n \in \boldsymbol{A}$, $n = 1, 2, \ldots$).

For instance, the class $\boldsymbol{C}(X)$ of all subsets of a topological space $X$ is a closure algebra. More generally, every class $\boldsymbol{A}$ of subsets of a topological space $X$ such that

$$\text{if} \quad A, B \in \boldsymbol{A}, \quad \text{then} \quad A+B \in \boldsymbol{A}, \quad -A \in \boldsymbol{A} \quad \text{and} \quad \bar{A} \in \boldsymbol{A}$$

is a closure algebra (viz. a closure subalgebra of $\boldsymbol{C}(X)$). Each closure algebra $\boldsymbol{A}$ of this last type is said to be a *closure algebra of subsets of X*.

A mapping $h$ of a closure algebra $\boldsymbol{A}$ into a closure algebra $\boldsymbol{B}$ is said to be a *closure homomorphism* if

$$h(A+B) = h(A)+h(B), \quad h(AB) = h(A)h(B),$$
$$h(-A) = -h(A), \quad h(\bar{A}) = \overline{h(A)} \quad \text{for any} \quad A, B \in \boldsymbol{A},$$

*i. e.*, if it is a Boolean homomorphism preserving the closure operation. A closure homomorphism $h$ of $\boldsymbol{A}$ into $\boldsymbol{B}$ is said to be a *closure isomorphism* (or: *homeomorphism*) provided it is one-to-one. If there exists a closure isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$, then the closure algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *homeomorphic*.

LEMMA 3. *For every enumerable closure algebra* $\boldsymbol{A}$ *there exists a set* $Z$ *of irrational numbers such that* $\boldsymbol{A}$ *is homeomorphic to a closure algebra* $\boldsymbol{C}$ *of subsets of* $Z$. *Moreover, if*

$$(1) \qquad A_n = \sum_{n=1}^{\infty} A_{n,i}, \qquad B_n = \sum_{n=1}^{\infty} B_{n,i}$$

*(where* $A_n, B_n, A_{n,i}, B_{n,i} \in \boldsymbol{A}$*) are given enumerable sequences of infinite joins and meets in* $\boldsymbol{A}$, *then* $Z$, $\boldsymbol{C}$ *and the homeomorphism* $h$ *of* $\boldsymbol{A}$ *onto* $\boldsymbol{C}$ *can be so chosen that* $h$ *preserves all the infinite joins and meets* (1), *i. e.*,

$$(2) \qquad h(A_n) = \sum_{i=1}^{\infty} h(A_{n,i}), \qquad h(B_n) = \sum_{i=1}^{\infty} h(B_{n,i})$$

*where* $\sum$ *and* $\prod$ *on the right side of equalities* (2) *denote the set-theoretical unions and intersections respectively.*

It follows from [2], Theorem 10.1 that there exists a topological space $X$ and a closure isomorphism $h_0$ of $\boldsymbol{A}$ onto a closure algebra $\boldsymbol{B}$ of subsets of $X$ which has the property (2). Moreover, it follows from the proof of [2], 10.1 that any pair of distinct points of $X$ is separated by a set $h_0(A) \subset X$ where $A \in \boldsymbol{A}$. Since $\bar{\bar{\boldsymbol{A}}} \leqslant \aleph_0$, we infer that $\bar{\bar{X}} \leqslant 2^{\aleph_0}$. By Theorem II, there is a set $Z$ of irrational numbers and an interior mapping $\varphi$ of $Z$ onto $X$. By Theorem I, the mapping

$$h(A) = \varphi^{-1}(h_0(A)) \quad \text{for} \quad A \in \boldsymbol{A}$$

is a closure isomorphism of $\boldsymbol{A}$ onto a closure algebra $\boldsymbol{C}$ of subsets of $Z$. The isomorphism $h$ has the property (2) since so has $h_0$.



**§ 5. Enumerable topological polynomials.** The exact definition of enumerable topological polynomials is as follows.

Consider all the expressions (inscriptions)

$$(1) \qquad \mathcal{A}^m_{\iota_j, \iota_k, \ldots, \iota_n}$$

where $\mathcal{A}$ is the letter $\mathcal{A}$, $\iota$ is the letter $\iota$, and $m, j, k, \ldots, n$ are positive integers. The number of the lower indexes is quite arbitrary, in particular it can be equal to zero, *i. e.*, the expressions of the form $\mathcal{A}^m$ are also allowed.

Let $\mathcal{P}$ be the smallest set of expressions such that

(a) $\mathcal{P}$ contains all the expressions (1);

(b) if $\alpha, \beta$ are in $\mathcal{P}$, then $(\alpha+\beta)$, $(\alpha\cdot\beta)$, $(-\alpha)$ and $\bar{\alpha}$ are in $\mathcal{P}$;

(c) if $\alpha$ is in $P$, then the expressions

$$\Big(\sum_{\iota_k=1}^{\infty} \alpha\Big) \quad \text{and} \quad \Big(\prod_{\iota_k=1}^{\infty} \alpha\Big),$$

where $\iota$ is the letter $\iota$ and $k$ is a positive integer, are in $\mathcal{P}$.

Take an expression $\beta$ belonging to $\mathcal{P}$. Some occurences of the lower indexes $\iota_j, \iota_k, \ldots, \iota_n$ (where $\iota$ — letter, $j, k, \ldots, n$ — positive integers) at the letters $\mathcal{A}$ are bound by a sign $\sum$ or $\prod$; some of them are free, *i. e.*, not bound. Replace all the free occurrences of $\iota_j, \iota_k, \ldots, \iota_n$ by some positive integers $a, b, \ldots, d$ respectively. The expression $\alpha$ obtained in this way from $\beta$ is said to be an *enumerable topological polynomial*.

For instance, the expressions

$$(2) \qquad \Big(-\overline{\Big(-\Big(\sum_{\iota_1=1}^{\infty}\big(A^1_{\iota_1}\cdot\overline{(-\bar{A}^1_{\iota_1})}\big)\Big)\Big)}\Big)$$

and

$$\Big(\sum_{\iota_2=1}^{\infty}\overline{\Big(\prod_{\iota_2=1}^{\infty}\big((-A^1_{2\iota_1})\cdot A^2_{\iota_1\iota_2 3}\big)\Big)}\Big)$$

are enumerable topological polynomials, but the expression $\big(\sum_{n=1}^{\infty} A^n\big)$ is not a topological polynomial.

Let $X$ be a topological space. Consider all the indexed sets of subsets of $X$ of the form

$$(3) \qquad \mathfrak{A} = \{A^m_{a,b,\ldots,d}\},$$

*i. e.*, mappings which with every finite sequence $m, a, b, \ldots, d$ of positive integers associates a set $A^m_{a,b,\ldots,d} \subset X$.

Clearly every enumerable topological polynomial $\alpha$ can be interpreted as a mapping which with every indexed set (5) of subsets of $X$ associates a subset of $X$. For this purpose it suffices

(a) to interpret in $\alpha$ the signs $+, \cdot, -, ^{-}, \sum, \prod$ as the sign of the usual set-theoretical or topological operations;

(b) to interpret in $\alpha$ the symbols $\iota_k$ as variables running through the set of all positive integers;

(c) to replace $\mathcal{A}$ by $A$ everywhere in $\alpha$.

The mapping determined by the enumerable polynomial $\alpha$ in the space $X$ in this way will be denoted by $\alpha_X$, and its value for the indexed set (3) will be denoted by $\alpha_X(\mathfrak{A})$.

We write $\alpha_X \equiv 0$ if $\alpha_X(\mathfrak{A}) = 0$ for every indexed set $\mathfrak{A}$ of subsets of $X$. We write $\alpha \equiv 0$ if $\alpha_X \equiv 0$ for every topological space $X$.

**§ 6. A theorem on enumerable topological polynomials.** The following theorem gives the answer to the question discussed in § 1.

THEOREM III. *There exists a set $Z$ of irrational numbers such that, for every enumerable topological polynomial $\alpha$, $\alpha_Z \equiv 0$ implies $\alpha \equiv 0$.*

Observe that a space $X$ such that

$$\text{for every } \alpha \quad \alpha_X \equiv 0 \text{ implies } \alpha \equiv 0$$

cannot have too regular properties, in particular it can be neither metrizable and topologically complete nor a locally compact regular space. In fact, the polynomial (2) defined in § 5 vanishes identically in every complete metric space (by the Baire theorem) and in every locally compact regular space, but it does not vanish identically in every topological space (*e. g.*, in a space which is of the first category on itself).

We are going to prove Theorem III. In this section the letters $\alpha, \beta$ will exclusively denote enumerable topological polynomials. We shall write

(1) $$\alpha \equiv \beta$$

if $\alpha_X(\mathfrak{A}) = \beta_X(\mathfrak{A})$ for every topological space $X$ and for every indexed set $\mathfrak{A}$ of subsets of $X$ (see § 5 (3)). Clearly the relation $\equiv$ is an equivalence relation. The class of all $\beta$ such that (1) holds will be denoted by $|\alpha|$. The set $\boldsymbol{A}$ of all $|\alpha|$, where $\alpha$ is any enumerable topological polynomial, is a closure algebra, the Boolean and the closure operations being defined by the formulae

$$|\alpha|+|\beta| = |(\alpha+\beta)|, \quad |\alpha|\cdot|\beta| = |(\alpha\cdot\beta)|, \quad -|\alpha| = |(-\alpha)|, \quad \overline{|\alpha|} = |\overline{\alpha}|.$$

$|\alpha|$ is the zero element of the Boolean algebra $\boldsymbol{A}$ if and only if $\alpha \equiv 0$.



It is easy to verify that

$$\left|\left(\sum_{\iota_k=1}^{\infty}\gamma\right)\right| = |\gamma_1|+|\gamma_2|+|\gamma_3|+\dots, \qquad \left|\left(\prod_{\iota_k=1}^{\infty}\gamma\right)\right| = |\gamma_1|+|\gamma_2|+|\gamma_3|+\dots,$$

where $\gamma_i$ is the enumerable topological polynomial obtained from $\gamma$ by replacing each free occurrence of $\iota_k$ by the positive integer $i$.

Since $\boldsymbol{A}$ is enumerable, there exist a closure algebra $\boldsymbol{C}$ of subsets of a set $Z$ of irrational numbers and an isomorphism $h$ of $\boldsymbol{A}$ onto $\boldsymbol{C}$ such that

$$h\left(\left|\left(\sum_{\iota_k=1}^{\infty}\gamma\right)\right|\right) = \sum_{i=1}^{\infty} h(|\gamma_i|), \qquad h\left(\left|\left(\prod_{\iota_k=1}^{\infty}\gamma\right)\right|\right) = \prod_{i=1}^{\infty} h(|\gamma_i|)$$

(see Lemma 3). The space $Z$ has the property mentioned in Theorem III. To prove it, first we calculate the value of $\alpha_Z(\mathfrak{A})$ of $\alpha_Z$ for the indexed set $\mathfrak{A} = \{A^m_{a,b,\dots,d}\}$ where

$$A^m_{a,b,\dots,d} = h\,(|\mathcal{A}^m_{a,b,\dots,d}|) \subset Z\,.$$

It follows easily from the commutativity of $h$ and $|\;|$ with the signs $+, \cdot, -, ^{-}, \sum, \prod$ that

$$\alpha_Z(\mathfrak{A}) = h(|\alpha|)\,.$$

The exact proof is by induction on the length of $\alpha$.

Now if $\alpha_Z \equiv 0$, then $h(|\alpha|) = \alpha_Z(\mathfrak{A}) = 0$. Since $h$ is a Boolean isomorphism, we obtain $|\alpha| = 0$, *i. e.*, $\alpha \equiv 0$, q. e. d.

**§ 7. Applications to Mathematical Logic.** This section is a supplement to the paper [2]. Notation and terminology are the same as in [2]. We recall here some fundamental notions only.

The letter $I$ will always denote the set of all positive integers.

The Lewis functional calculus (based on the Lewis system S4 of proportional calculus) will be denoted by $\mathcal{S}^*_\lambda$ (7). The logical constants in $\mathcal{S}^*_\lambda$ are $+, \cdot, \rightarrow, -, \mathbf{C}$ and quantifiers $\sum$ and $\prod$.

Each formula $\alpha$ from $\mathcal{S}^*_\lambda$ can be interpreted as a topological "functional" denoted in [2] by $(I, \boldsymbol{C}(X))\Phi_\alpha$ by treating:

(a) the individual variables as variables running over $I$,

(b) the $k$-argument relation signs as variables running over the set of all mappings of $I \times I \times \dots \times I$ ($k$-times) into $\boldsymbol{C}(X)$,

(c) the signs $+$ and $\cdot$ as the signs of the set-theoretical union and intersection respectively, the signs $\rightarrow$ and $-$ as the signs of the opera-

(7) The calculus $S^*_\lambda$ is exactly described in [2] § 10.

tions $A\to B=(X-A)+B$ and $-A=X-A$ respectively, the sign $\mathbf{C}$ as the sign of the closure operation in $X$,

(d) the signs $\sum\limits_x$ and $\prod\limits_x$ as the signs of the infinite set-theoretical union and intersection respectively, where the variable $x$ runs over $I$.

The exact definition of $(I, \boldsymbol{C}(X))\Phi_a$ is given in [2] § 5. If this functional assumes only the value $X$ (for each possible value of the variables mentioned in (a) and (b)) we write

$$(1)\qquad (I, \boldsymbol{C}(X))\Phi_a = X.$$

It was proved in [2], 10.6 that a formula $a$ is provable in $\mathcal{S}_\lambda^*$ if and only if (1) holds for every topological space $X$. Moreover there exists a special topological space, denoted in [2] by $\mathfrak{X}_\lambda^0$, such that a formula $a$ is a theorem of $\mathcal{S}_\lambda^*$ if and only if $(I, \boldsymbol{C}(\mathfrak{X}_\lambda^0))\Phi_a = \mathfrak{X}_\lambda^0$.

Now we shall prove the following more precise theorem, which is a logical analogue of the topological Theorem III.

Theorem IV. *There exists a set $Z$ of irrational numbers such that, for every formula $a$ in $\mathcal{S}_\lambda^*$, $a$ is provable in $\mathcal{S}_\lambda^*$ if and only if $(I, \boldsymbol{C}(Z))\Phi_a = Z$.*

We recall that the topological space $\mathfrak{X}_\lambda^0$ mentioned above is constructed as follows: We form the Lindenbaum algebra $\boldsymbol{L}_\lambda$ of $\mathcal{S}_\lambda^*$ (*i. e.*, the closure algebra obtained from the set of all formulas in $\mathcal{S}_\lambda^*$ by the identification of equivalent formulas). We represent isomorphically $\boldsymbol{L}_\lambda$ as a field $\boldsymbol{Ł}_\lambda$ of subsets of a space $\mathfrak{X}_\lambda^0$ (this Boolean isomorphism should preserve all the infinite sums and products corresponding to the logical quantifiers). We can assume that the sets in $\boldsymbol{Ł}_\lambda$ separate the set $\mathfrak{X}_\lambda^0$ (for if not, we can identify points which are not separate by any set in $\boldsymbol{Ł}_\lambda$). Thus

$$\overline{\overline{\mathfrak{X}_\lambda^0}} \leqslant 2^{\aleph_0}.$$

Now we assume that the sets in $\boldsymbol{Ł}_\lambda$ corresponding to the formulas of the form

$$-\mathbf{C}-\beta$$

(*i. e.*, of the formulas $\mathbf{I}\beta$ where $\mathbf{I}=-\mathbf{C}-$ is the logical constant corresponding to the topological interior operation) form the open basis of $\mathfrak{X}_\lambda^0$. Since $\overline{\overline{\boldsymbol{Ł}_\lambda}}=\aleph_0$, the topological space $\mathfrak{X}_\lambda^0$ thus defined has an enumerable open basis.

By Theorem II, there exist a set $Z$ of irrational numbers and an interior mapping $\varphi$ of $Z$ onto $\mathfrak{X}_\lambda^0$. Thus, by Theorem I, the transformation

$$h(B)=\varphi^{-1}(B)\qquad (\text{for } B\subset \mathfrak{X}_\lambda^0)$$



is a closure isomorphism of $\boldsymbol{C}(\mathfrak{X}_\lambda^0)$ into $\boldsymbol{C}(Z)$ preserving all the infinite unions and intersections. Hence, if $(I, \boldsymbol{C}(Z))\Phi_a = Z$, then, by [2], 5.5, the superposition of $h$ and $(I, \boldsymbol{C}(\mathfrak{X}_\lambda^0))\Phi_a$ is identically equal to $Z$, which implies that $(I, \boldsymbol{C}(\mathfrak{X}_\lambda^0))\Phi_a = \mathfrak{X}_\lambda^0$, *i. e.*, that $a$ is provable in $\mathcal{S}_\lambda^*$. Conversely, if is provable in $\mathcal{S}^*$, then (1) holds for every topological space $X$, in particular it holds for $Z$.

The problem analogous to that of Theorem IV can be also examined for the Heyting (*i. e.*, intuitionistic) functional calculus $\mathcal{S}_\chi^*$, described exactly in [2] § 11. The logical constants in $\mathcal{S}_\chi^*$ are $+$, $\cdot$, $\to$, $\neg$ and the quantifiers $\sum$ and $\prod$.

For every topological space $X$, let $\boldsymbol{H}(X)$ denote the class of all open subsets of $X$. We shall consider $\boldsymbol{H}(X)$ as an abstract algebra ([8]) with respect to the usual set-theoretical operations

$$(2)\qquad A+B \quad\text{and}\quad A\cdot B$$

and to the operations

$$(3)\qquad A\to B=\operatorname{int}\big((X-A)+B\big),$$

$$(4)\qquad \neg A=\operatorname{int}(X-A)$$

where $\operatorname{int}(A)$ denotes the interior of the set $A\subset X$. Clearly $\boldsymbol{H}(X)$ is a complete lattice (with respect to $+$ and $\cdot$). The infinite lattice sum (join) in $\boldsymbol{H}(X)$ is the usual set-theoretical union, the infinite lattice product (meet) in $\boldsymbol{H}(X)$ is the interior of the set-theoretical intersection.

Every formula $a$ in $\mathcal{S}_\chi^*$ can be interpreted as a topological "functional" denoted in [2] by $(I, \boldsymbol{H}(X))\Phi_a$ by treating:

(a) the individual variables as variables running through $I$,

(b) the $k$-argument relation signs as variables running through the set of all mappings of $I\times I\times\ldots\times I$ ($k$-times) into $\boldsymbol{H}(X)$,

(c) the signs $+$ and $\cdot$ as the signs of the set-theoretical union and intersection respectively, the signs $\to$ and $\neg$ as the signs of operations (3) and (4) respectively,

(d) the quantifiers $\sum\limits_x$ and $\prod\limits_x$ as the signs of the infinite lattice sum and product in $\boldsymbol{H}(X)$ respectively, where the variable $x$ runs over $I$.

The exact definition of $(I, \boldsymbol{H}(X))\Phi_a$ is given in [2] § 5. If this functional assumes only the value $X$ (for all possible values of the variables mentioned in (a) and (b)), we write

$$(5)\qquad (I, \boldsymbol{H}(X))\Phi_a = X.$$

([8]) Called "Heyting algebra" in [2].

It was proved in [2], 11.6 that a formula $\alpha$ is provable in $\mathcal{S}_\chi^*$ if and only if (5) holds for every topological space $X$. Moreover, there exists a special topological space denoted in [2] by $\mathfrak{X}_\chi^0$ such that a formula $\alpha$ in $\mathcal{S}_\chi^*$ is provable in $\mathcal{S}_\chi^*$ if and only if $(I, \boldsymbol{H}(\mathfrak{X}_\chi^0))\Phi_\alpha = \mathfrak{X}_\chi^0$.

Now we shall formulate the more exact theorem:

THEOREM V. *There exists a set $Z$ of irrational numbers such that, for every formula $\alpha$ in $\mathcal{S}_\chi^*$, $\alpha$ is provable in $\mathcal{S}_\chi^*$ if and only if $(I, \boldsymbol{H}(Z))\Phi_\alpha = Z$.*

The proof of Theorem V is the same as that of Theorem IV. Theorem V can also be easily deduced from Theorem IV and the set $Z$ mentioned in Theorem V may be identified with the set $Z$ in Theorem IV. This follows easily from the fact that with every formula in $\mathcal{S}_\chi^*$ we can associate, in a very simple manner, a formula $\psi(\alpha)$ in $\mathcal{S}_\lambda^*$ in such a way that $\alpha$ is provable in $\mathcal{S}_\chi^*$ if and only if $\psi(\alpha)$ is provable in $\mathcal{S}_\lambda^*$ (see [2] § 15). Consequently instead of $(I, \boldsymbol{H}(X))\Phi_\alpha$ it suffices to consider $(I, \boldsymbol{C}(X))\Phi_{\psi(\alpha)}$.

Statements analogous to Theorems IV and V hold also for the non-classical functional calculi $\mathcal{S}_\pi^*$, $\mathcal{S}_\mu^*$ and $\mathcal{S}_\nu^*$ examined in [2], §§ 12-14.

Observe that Theorem IV (and consequently Theorem V) can be deduced directly from Theorem III since there is a simple natural correspondence between enumerable topological polynomials and logical formulas. Consequently the set $Z$ satisfying Theorem III satisfies also Theorems IV and V.

**§ 8. A representation theorem for closure algebras.** This section is a supplement to the papers [3] and [4]. The terminology and notation are the same as in [3] and [4]. According to the hypotheses assumed in [3] and [4], we suppose in this section that all the closure algebras under consideration are $\sigma$-complete Boolean algebras.

We recall (see [3], p. 176) that two closure algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *weakly homeomorphic* if the closure subalgebras of all Borel elements of $\boldsymbol{A}$ and $\boldsymbol{B}$ are homeomorphic. If $\boldsymbol{A}$ is a closure algebra with an enumerable open basis, and $\boldsymbol{I}$ is a $\sigma$-ideal of $\boldsymbol{A}$, then $\boldsymbol{A}/\boldsymbol{I}$ denotes the closure algebra defined in [3], § 9, *i. e.*, $\boldsymbol{A}/\boldsymbol{I}$ is the Boolean factor algebra and the open elements of $\boldsymbol{A}/\boldsymbol{I}$ are those determined by the open elements of $\boldsymbol{A}$.

Combining Theorems I, II and [4] (xii) we obtain the following

THEOREM VI. *For every closure algebra $\boldsymbol{A}$ with an enumerable open basis there exist a closure subalgebra $\boldsymbol{S}$ of the closure algebra of all subsets of the set of all irrational numbers, and a $\sigma$-ideal $\boldsymbol{I}$ of $\boldsymbol{S}$ such that $\boldsymbol{A}$ is weakly homeomorphic to $\boldsymbol{S}/\boldsymbol{I}$.*



**§ 9. A theorem on interior mappings.** As in § 7, let $\boldsymbol{H}(X)$ denote the class of all open subsets of a topological space $X$. We shall consider $\boldsymbol{H}(X)$ as an abstract algebra with the three binary operations (see § 7, (2) and (3))

$A+B$ = the set-theoretical union of $A$ and $B$,

$A\cdot B$ = the set-theoretical intersection of $A$ and $B$,

$A \to B$ = the interior of the set $(X-A)+B$.

As seen in § 7, such operations appear, in a natural way, in logical investigations connected with the Heyting functional calculus.

Let $X$ and $Y$ be topological spaces. A mapping $h$ of $\boldsymbol{H}(Y)$ into $\boldsymbol{H}(X)$ is said to be an *H-homomorphism* provided

$$h(A+B) = h(A)+h(B), \tag{1}$$

$$h(A\cdot B) = h(A)\cdot h(B), \tag{2}$$

$$h(A\to B) = h(A)\to h(B) \tag{3}$$

for every sets $A, B \in \boldsymbol{H}(Y)$.

If $\varphi$ is a mapping of $X$ into $Y$, and

$$h(A) = \varphi^{-1}(A) \quad \text{for all} \quad A \in \boldsymbol{H}(Y), \tag{4}$$

then $h$ maps $\boldsymbol{H}(Y)$ into $\boldsymbol{H}(X)$ if and only if $\varphi$ is continuous. Obviously the mapping $h$ defined by (4) satisfies (1) and (2). However, condition (3) is not always satisfied. This fact is nearly explained by the following theorem:

THEOREM VII. *Let $h$ be defined by (4). If $\varphi$ is an interior mapping of $X$ into $Y$, then $h$ is an H-homomorphism. Conversely, if $Y$ is a Hausdorff space satisfying the first axiom of countability* [9] *and $h$ is an H-homomorphism, then $\varphi$ is an interior mapping.*

If $\varphi$ is an interior mapping, then, by Theorem I,

$$\varphi^{-1}(\mathrm{int}(C)) = \mathrm{int}(\varphi^{-1}(C)) \quad \text{for every set} \quad C \subset Y.$$

Thus

$$h(A\to B) = \varphi^{-1}(\mathrm{int}((Y-A)+B)) = \mathrm{int}(\varphi^{-1}(Y-A)+\varphi^{-1}(B)) = h(A)\to h(B).$$

Conversely, if $h$ is an H-homomorphism, then

$$\varphi^{-1}(\mathrm{int}(C)) = \mathrm{int}(\varphi^{-1}(C))$$

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[9] That is, if every point $y \in Y$ has an enumerable complete system of neighbourhoods.

for every set $C \subset Y$ of the form $C = F + G$ where $F$ is closed and $G$ is open (to prove it, assume that $A = X - F$ and $B = G$ in (3)). By passage to complements, we infer that

$$\varphi^{-1}(\bar{D}) = \overline{\varphi^{-1}(D)} \tag{5}$$

for every set $D \subset Y$ of the form $D = F \cdot G$ where $F$ is closed and $G$ is open. In particular, (4) holds in $D$ is the set of all terms of a convergent sequence. This implies, under the hypothesis that $Y$ satisfies the first axiom of countability, that

$$\varphi^{-1}(\bar{C}) \subset \overline{\varphi^{-1}(C)} \quad \text{for every set} \quad C \subset Y. \tag{6}$$

In fact, if $\varphi(x) \in \bar{C}$, then $\varphi(x) \in \bar{D}$ where $D \subset C$ is the set of all terms of a sequence convergent to $\varphi(x)$. Hence

$$x \in \varphi^{-1}(\bar{D}) = \overline{\varphi^{-1}(D)} \subset \overline{\varphi^{-1}(C)}.$$

On the other hand, $\varphi$ is continuous since $h$ transforms $\boldsymbol{H}(Y)$ into $\boldsymbol{H}(X)$. Therefore

$$\overline{\varphi^{-1}(C)} \subset \varphi^{-1}(\bar{C}) \quad \text{for every set} \quad C \subset Y. \tag{7}$$

It follows from (6), (7) and Theorem I that $\varphi$ is an interior mapping.

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# On some equation in transfinite ordinals

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In this paper we consider the equation

$$\xi^{\varphi} = \eta^{\psi} + q, \tag{1}$$

where $\varphi, \psi > 1$ are not limit numbers and $q$ is finite.

THEOREM 1. *If $q = 0$, then all ordinals $\xi, \eta$ which are not limit numbers and satisfy* (1) *are given by the formulae*

$$\xi = \tau^m, \quad \eta = \tau^n,$$

*where the exponents $m, n$ are finite and satisfy*

$$m\varphi = n\psi.$$

THEOREM 2. *If $q > 0$, then no transfinite ordinals $\xi, \eta$ satisfy* (1).

Theorem 2, for $q = 1$, $\varphi = 2$, $\psi = 3$, was proved by W. Sierpiński [2].

**1.** Let us denote by $(a, b)$ the largest common divisor of $a$ and $b$.

AUXILIARY LEMMA. *If $k, l$ are natural numbers, $s = (k, l)$, $k = k's$, $l = l's$ and for an ordinal $\nu$ and a sequence of ordinals $\mu_0, \dots, \mu_t$ which has at least $kl' - 1$ elements* (*i. e.*, $t \geqslant kl' - 2$)

$$\nu k' + \mu_{j+k} = \nu l' + \mu_{j+l} = \mu_j \tag{2}$$

*holds, then*

$$\mu_j = \nu + \mu_{j+s}. \tag{3}$$

Proof. Evidently we can assume that $s < k, l$. There exist such numbers $p, q$ that $0 < p \leqslant l'$ and

$$pk' - ql' = 1, \quad pk - ql = s. \tag{4}$$

From (2) and (4) follows

$$\mu_j = \nu pk' + \mu_{j+pk} = \nu + \nu ql' + \mu_{j+s+ql} = \nu + \mu_{j+s} \tag{5}$$

for $j + pk \leqslant t$. We shall prove first that (3) holds for $j = 0, \dots, k-1$. It is $p < l'$. Indeed, if $p = l'$ then from (4) follows $l' = 1$, which contra-