Continuous choice functions and convexity

by

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1. Introduction. As is well known the existence of a choice function on an arbitrary family of sets is related to the theory of partially ordered sets, well ordered sets, and to Tychonoff's theorem [5]. Moreover choice functions subject to some restraint have been investigated. For example, if a choice function assigns to distinct sets distinct values, it is termed faithful. Conditions equivalent to the existence of a faithful choice function have been obtained. (See [3], [4], and their bibliographies.)

If the family consists of subsets of a topological space and in addition is itself topologized, then one can speak of a continuous choice function. The existence of continuous choice functions is of use in generalizing fixed point theorems to multivalued functions. The nonexistence of a continuous choice function on the family of all lines in the projective plane was demonstrated by Fenchel [2]. It might be noted that a topological space X has the fixed point property if and only if the family of sets of the form $X-P$, where $P \in X$, does not have a continuous choise function.

The object of the present note is to show the close relation between convexity and the existence of continuous choice functions. It was suggested by various results in the theory of convex bodies to be found in ([1]; p. 10-13) [6], [7], and [8].

2. Convexity of open sets. In the following E^n, $n>1$, denotes n-dimensional Euclidean space. If L^r is an r-dimensional hyperplane in E^n and $A \subset E^n$ and if $L^r \cap A \neq \emptyset$ then $L^r \cap A$ is an r-dimensional cross-section of A. Any L^{n-1} divides E^n into two (closed) subsets K_1 and K_2. A subset $B \subset A$ is a cap of A if $B = A \cap K_1$ for some K_1 or if $B = E^n$. If L^{n-1} is such that $A \subset K_1$ and $A \cap \operatorname{int} K_2 = \emptyset$ then L^{n-1} is a support plane of A. A set A is convex if $P,\ Q \in A$ implies $\overline{PQ} \subset A$.

THEOREM 2.1. *A necessary and sufficient condition that the open set A be convex is that there exists a continuous choice function f on the family of its $(n-1)$-dimensional cross-sections. Moreover any such f is onto A.*

Proof. Assume that A is convex. Define $d\colon E^n \to E^1$ by $d(P) = e^{-|P|^2}$. Define f on an $(n-1)$-dimensional cross-section B of A to be the center of gravity of B relative to the density d. (Because of the behavior of e^{-x^2} when x is large, this is a valid definition even if A is unbounded.)

Conversely assume that A is open and there exists the continuous choice function f. Let L^{n-1} be a hyperplane with the properties that $L^{n-1} \cap A = \emptyset$, $K_1 \cap A \neq \emptyset$, $K_2 \cap A \neq \emptyset$. A contradiction will now be obtained. Let $P_i \in K_i \cap A$, $i=1,2$. Let L_i^{n-1}, $i=1,2$, be the hyperplane parallel to L^{n-1} and containing P_i, $i=1,2$, respectively. It is possible to define a path in the space of $(n-1)$-dimensional hyperplanes meeting A, which begins at L_1^{n-1}, ends at L_2^{n-1}, and does not pass through L^{n-1}. (For example, first rotate L_1^{n-1} about P_1 until it meets P_2, then rotate about P_2 until it becomes L_2^{n-1}). The image of this path under f is a path in E^{n-1} which, beginning in $K_1 \cap A$ and ending in $K_2 \cap A$, must meet L^{n-1}. But $L^{n-1} \cap A = \emptyset$. This contradiction shows that any L^{n-1} which meets the interior of E, the convex hull of A, meets A.

Let $P \in \operatorname{int} E$. It will be shown that there exists L^{n-1} such that $f(L^{n-1}) = P$. Let $S^{n-1} \subset E^n$ be an $(n-1)$-dimensional Euclidean sphere with center P. Let $\lambda\colon S^{n-1} \to S^{n-1}$ be the antipodal map. Let $r\colon E^n - P \to S^{n-1}$ be radial projection. Assume that for no L^{n-1} is $f(L^{n-1}) = P$.

Define $g\colon S^{n-1} \to S^{n-1}$ by setting $g(Q) = rf(L^{n-1})$ where L^{n-1} is the hyperplane through P perpendicular to QP. Since the angle $QPg(Q) = \frac{1}{2}\pi$, g is homotopic to the identity and hence has degree one. Since $g\lambda = g$, the degree of g is even (see [8]). This contradiction implies that there is L^{n-1} so $f(L^{n-1}) = P$.

Thus $\operatorname{int} E \subset A$. But $A \subset E$, hence $A \subset \operatorname{int} E$. Thus $A = \operatorname{int} E$ and is therefore convex.

THEOREM 2.2. *A necessary and sufficient condition that the open set A be convex is that there exists a continuous choice function f on the family of r-dimensional cross-sections of A, $1 \leqslant r \leqslant n-1$. Moreover any such f is onto A.*

Proof. The case $n-r=1$ is considered in Theorem 2.1. An induction on $n-r$ easily establishes Theorem 2.2.

THEOREM 2.3. *A necessary and sufficient condition that the open set A be convex is that there exists a continuous choice function f on the family of caps of A. Moreover any such f is onto A.*

Proof. If A is convex assign to each cap, K, its center of gravity $f(K)$, relative to the density d defined above. This f is not only continuous but faithful and onto A (see [4] or [8] or the proof of Theorem 3.2 which rectifies a slight error in [8]). The proof of the converse is similar to that of the preceding proof. Incidentally one has proved

THEOREM 2.4. *A necessary and sufficient condition that the open set A be convex is that there exists a continuous faithful choice function f on the family of caps of A. Moreover any such f is onto A.*

COROLLARY 2.5. *If there is a continuous choice function on the family of caps of the open set A, then there is a continuous faithful choice function on this family.*

3. Convexity of arbitrary sets. The assumption that A is open was made for the sake of simplicity. The following theorems indicate the complications that can arise at the boundary of an arbitrary subset $C \subset E^n$.

THEOREM 3.1. *A necessary and sufficient condition that plane of C be convex and have the property that each plane of support has a unique point of contact is that there exists a continuous choice function f on the family of its $(n-1)$-dimensional cross-sections. Moreover any such f is onto C.*

Proof. Similar to that of Theorem 2.1.

The analog of Theorem 2.3 is

THEOREM 3.2. *A necessary and sufficient condition that C be convex and have the property that each plane of support has a unique point of contact is that there exists a continuous choice function on the family of caps of C. Moreover any such f is onto C.*

Proof. A proof can be based on the following: If C is convex, and has the property that each plane of support has a unique point of contact, then any continuous choice function on the family of caps of C is onto C. This fact will now be proved (for simplicity assume that C is compact).

Assume that $P \in C$ but that $P \notin \operatorname{im} f$. Let S^{n-1} be an $(n-1)$-dimensional Euclidean sphere with center P and let $I = \{t \mid 0 \leqslant t \leqslant 1\}$. For $Q \in S^{n-1}$ let $L^{n-1}(Q)$ denote the plane of support to A perpendicular to $\overline{PQ}$ and meeting the ray with end P and containing Q. Let $K(Q, t)$ denote the cap of C containing Q which is cut off by the hyperplane which is a tth of the distance from $L^{n-1}(Q)$ to $L^{n-1}(\lambda Q)$. Define $F\colon S^{n-1} \times I \to S^{n-1}$ by $F(Q, t) = rfK(Q\, t)$.

First note that $F|S^{n-1} \times 1$ is constant (with value $f(C)$). Next note that $F|S^{n-1} \times 0$ is homotopic to the identity since the angle $QPF(Q, 0) \leqslant \frac{1}{2}\pi$. Hence the identity map of S^{n-1} would be homotopic to a constant map. This contradiction proves that f is onto C.

Similar analogs of Theorems 2.2, 2.4 and Corollary 2.5 hold for arbitrary sets.

4. Questions. The preceding theorems suggest several problems.

1. What is a necessary and sufficient condition that a continuous choice function on the $(n-1)$-dimensional cross-sections of an open set A be representable as the center of gravity function induced by a suitable density on A?

2. The similar question for continuous choice functions on the caps of A.

3. Let A be a bounded convex open set. Is each choice function on the caps of A induced by a density on A also inducible by a density on the surface of A? Or conversely?

References

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