exists, where $a$ is a generator of the group $H'(S^n) \cong \mathbb{Z}$. It follows that $g^*h^*(a) \neq 0$, whence the homomorphism

$$g^*h^* = (hg)^* : H'(S^n) \to H'(S^n)$$

is nontrivial. This proves that $h_0$ is an essential map, contradicting the assumption that each composition $h_0$ with $g: S^3 \to X$, $h: X \to S^n$ is inessential.

Remark 1. In particular, the map $\Phi$ given above yields an example of an inessential map of $S^3$ into $S^3$ which is not factorizable through $R^n$. Indeed, setting $X = R^n$ and assuming $\Phi = f_0; \psi : S^3 \to R^n$, then $R^n \to S^n$, we would get as above $H'(R^n) \neq 0$, which is absurd.

Remark 2. If we choose $r$ sufficiently near to the identity, we get $|\Phi(x) - \Phi_0(x)| < \epsilon$, but it is easy to check that the map $\Phi_0$ (see (1)) can be factored through $R^n$. This shows that the set of maps which are factorizable through $R^n$ is not open in the set of all inessential maps of $S^3$ into $S^n$.

References


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After this paper had been submitted to the Fundamenta, professor K. Borsuk kindly informed me that U. Hopf had already communicated to him the solution of problem 1 of [4].

Note on dimension theory for metric spaces

by

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Recently, a dimension theory for general metric spaces has been established by M. Katětov and by K. Morita (see [4] and [5]) independently. They have extended the sum, decomposition and product theorems to non-separable metric spaces and have shown the equivalence of the Lebesgue dimension and the inductive dimension (4). On the other hand, the following theorem of F. Alexandroff and P. Urysohn is well known:

In order that a $T_0$-topological space $X$ be metrizable it is necessary and sufficient that there exists a sequence $\emptyset \neq \emptyset_1 \neq \emptyset_2 \neq \emptyset_3 \neq \cdots$ of open coverings such that $\{S(p, \emptyset_m) \mid m = 1, 2, \ldots\}$ is a ndc (neighbourhood) basis for each point $p$ of $X$.

The purpose of the present note is to refine this theorem to a theorem concerning $\alpha$-dimensionality of metric spaces and to develop the dimension theory for general metric spaces. In § 1 we shall prove that Alexandroff-Urysohn's theorem turns into a theorem asserting a necessary and sufficient condition for $\alpha$-dimensionality if we add the condition order $\mathfrak{m} < \alpha + 1$ ($m = 1, 2, \ldots$) to the original condition. Furthermore, concerning that theorem it will be shown that we may replace order $\mathfrak{m} < \alpha + 1$ by Alexandroff-Kolmogoroff's length of $\mathfrak{m} < \alpha + 1$ (see [1]). In § 2 we shall apply the result of § 1 to the study of the connection, dimension and metric function. § 3 contains applications of the result of § 1 to the embedding of $\alpha$-dimensional metric spaces into products of $1$-dimensional spaces. The final section is devoted to
the embedding of n-dimensional metric spaces into a product of Euclidean 
(2m+1)-space with a zero-dimensional space and to its modifications.

Throughout this paper all spaces are metric or metrizable, and all 
coverings are open, unless the contrary is explicitly stated.

§ 1. The main theorem.

Definition. For two collections $\mathcal{U}, \mathcal{U}'$ of open sets we denote by $\mathcal{U} \prec \mathcal{U}'$ the fact that $\mathcal{V} \subseteq \mathcal{U}'$ for every $\mathcal{V} \in \mathcal{U}$ and for some $\mathcal{V}' \in \mathcal{U}'$.

Definition. We mean by a disjointed collection a collection $\mathcal{U}$ of open sets such that $\mathcal{U}, \mathcal{U}' \in \mathcal{U}$ and $\mathcal{U} \cap \mathcal{U}' = \emptyset$.

Theorem 1. In order that $\dim R < n$ for a metric space $R$ it is necessary and sufficient that there exist $n+1$ sequences $\mathcal{U}_1 > \mathcal{U}_2 > \ldots$ 
$(i = 1, 2, \ldots, n+1)$ of disjointed collections such that

$$(i = 1, 2, \ldots, n+1)$$

is an open basis of $R$.

Proof. If $\dim R = 0$, then by [5] there exists a sequence $\mathcal{U}_m$ 
$m = 1, 2, \ldots$ of finite coverings (1) consisting of open and closed 
sets such that $\mathcal{S}(p, \mathcal{U}_m) (m = 1, 2, \ldots)$ is a nbhd basis at each point $p$ of $R$.

Let $\mathcal{U}_m = \{ \mathcal{V}_n | a < \tau \}$, then we define a sequence of coverings by

$$\mathcal{U}_m = \{ \mathcal{V}_n \cup \mathcal{V}_j | a < \tau \}$$

and

$$\mathcal{U}_m = \{ \mathcal{V}_n \cup \mathcal{V}_j | a < \tau \}$$

It is clear that $\mathcal{U}_1 > \mathcal{U}_2 > \ldots$ is a sequence of disjointed collections, and

$$(i = 1, 2, \ldots, n+1)$$

is an open basis of $R$.

Conversely, if there exists a sequence $\mathcal{U}_1 > \mathcal{U}_2 > \ldots$ of disjointed 
sequences such that $(\mathcal{U}_m = 1, 2, \ldots, n+1)$ is an open basis of $R$, then for any

arbitrary point $p$ of $R$, $p \in \mathcal{U} \in \mathcal{U}_m$ implies

$$\mathcal{U} \cap \mathcal{U}' = \emptyset$$

for $\mathcal{U}'$ with $\mathcal{U} \neq \mathcal{U}' \in \mathcal{U}_m$, and $p \in U$ for every $U \in \mathcal{U}_m$ implies

$$(i = 1, 2, \ldots)$$

by the fact that $(\mathcal{U}_m = 1, 2, \ldots, n+1)$ is an open basis of $R$, and $\mathcal{U}_m > \mathcal{U}_m+1$. Hence each $\mathcal{U}_m$ is locally finite and consists of open and closed sets, and hence $\dim R = 0$ follows from [5].

Now we proceed to n-dimensional cases. Let $\dim R < n$; then we can decompose $R$ into $n+1$ n-dimensional spaces $R_i (i = 1, \ldots, n+1)$

by the general decomposition theorem due to Katětov and to Morita, i.e.,

$$R = \bigcup_{i=1}^{n+1} R_i, \quad \dim R_i = 0.$$
Proof. Necessity. If \( R \) is a metric space with \( \dim R < \infty \), then by the general decomposition theorem \( R = \sqcup_{i=1}^{n+1} R_i \) for some \( 0 \)-dimensional spaces \( R_i \) \((i = 1, \ldots, n+1)\).

(A) Let \( \mathcal{U} = (U_a | a \in A) \) be an arbitrary locally finite covering of \( R \); then there exists a disjointed covering \( \mathcal{V}_1 = (V_{a_1} | a_1 \in A_1) \) of \( R \) such that \( V_{a_1} \subseteq U_{a_1} \). By putting

\[
V'_1 = \bigcup \{ S_{a_1}(x) | x \in V_{a_1} \}
\]

such that

\[
R \setminus S_{a_1}(x) \subseteq V_{a_1}, \quad S_{a_1}(x) \subseteq U_{a_1}.
\]

we get a disjointed collection \( \mathcal{V}'_1 = (V'_1 | a_1 \in A_1) \) of \( R \) such that \( \mathcal{V}'_1 \upless \mathcal{U} \). Hence \( \mathcal{V}' = \bigcup_{i=1}^{n+1} \mathcal{V}'_i \) is a locally finite covering of \( R \) of order \( < n+1 \) and is a refinement of \( \mathcal{U} \) (1). Hence there exists an open covering \( \mathcal{V}'' = (V''_p | p \in B) \) of \( R \) such that \( V''_p \subseteq V'_i \) for \( \mathcal{V}'_i = (V'_i | i \in B) \). It is easily seen from order \( \mathcal{V}' \leq < n+1 \) and from the property of \( \mathcal{V}' \) that every point \( p \in R \) has some nbd intersecting at most \( n+1 \) of the sets belonging to \( \mathcal{V}''_i \); we call such a covering to be of local order \( < n+1 \).

Now let \( \mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots \supseteq \mathcal{U}_n \) be a sequence of coverings of \( R \) such that \( \mathcal{S}(p, U_{\alpha}) \) \((\alpha = 1, 2, \ldots, n)\) is a nbd basis for each point \( p \), then from the paracompactness (1) of \( R \) we may assume that all \( U_{\alpha} \) are locally finite. Hence from (A) we get a refinement \( \mathcal{V}_i \) of \( \mathcal{U}_1 \) such that the local order of \( \mathcal{U}_i \leq \mathcal{U}_{i+1} \). Furthermore, we can select locally finite coverings \( \mathcal{V}_i, \mathcal{V}_i' \) such that \( \mathcal{V}_i' \subseteq \mathcal{V}_i \) and such that each set of \( \mathcal{V}_i \) intersects at most \( n+1 \) of the sets of \( \mathcal{V}_i' \) such that the local order of \( \mathcal{V}_i \leq \mathcal{V}_{i+1} \). By using local finite coverings \( \mathcal{V}_i, \mathcal{V}_i' \), we obtain a refinement \( \mathcal{V}_i' \) of \( \mathcal{U}_i \) and \( \mathcal{V}_i \subseteq \mathcal{V}_i' \) with the local order of \( \mathcal{V}_i \leq \mathcal{V}_{i+1} \). Then it follows clearly that \( \mathcal{V}_i \subseteq \mathcal{U}_i, \mathcal{V}_i' \subseteq \mathcal{V}_i \) and each set of \( \mathcal{V}_i \) intersects at most \( n+1 \) of sets of \( \mathcal{U}_i \). By repeating such processes we obtain a sequence \( \mathcal{V}_i \supseteq \mathcal{V}_i' > \mathcal{V}_i > \mathcal{V}_i > \cdots \) of open coverings such that \( \mathcal{V}_i \supseteq \mathcal{U}_i \) and such that each set of \( \mathcal{V}_i \) intersects at most \( n+1 \) of the sets belonging to \( \mathcal{U}_i \).

Since \( \mathcal{S}(p, U_{\alpha}) \) \((\alpha = 1, 2, \ldots, n)\) is a nbd basis for \( p \), \( \mathcal{S}(p, U_{\alpha}) \) \((\alpha = 1, 2, \ldots, n)\) is also a nbd basis of \( p \), and hence the necessity is proved.

Sufficiency. The metrizability of such a space is obvious from Urysohn-Alexandroff's theorem. We divide the proof of \( n \)-dimensionality into three parts.

1. If \( \mathcal{V}_1 > \mathcal{V}_2 > \cdots \) is a sequence satisfying the condition of this proposition, then it is easily seen that for each point \( p \) of \( R \), \( \mathcal{S}(p, U_{\alpha_1}) \) \((\alpha_1 = 1, 2, \ldots)\) intersects at most \( n+1 \) sets of \( \mathcal{U}_m \). Putting

\[
U_m = \bigcup_{i=1}^{n+1} (U_{\alpha_i} \cap \mathcal{V}_{i+1}) \quad (m = 1, 2, \ldots),
\]

we get a sequence \( U_1 > U_2 > U_3 > \cdots \) of open coverings such that \( \mathcal{S}(p, U_m) \) \((m = 1, 2, \ldots)\) is a nbd basis of \( p \) and \( R \) and such that each \( \mathcal{S}(p, U_m) \) \((m = 1, 2, \ldots)\) intersects at most \( n+1 \) sets of \( \mathcal{U}_m \).

Let \( U_m = (U_{\alpha_i} | \alpha_i < \infty) \); then we can prove first that there exist open sets \( U_i \) such that

\[
\bigcup_{i=1}^{n+1} U_i \subseteq U_n, \quad U_i \cap \mathcal{V}_{i+1} = \emptyset \quad \text{for} \quad \alpha_i \neq \beta
\]

and such that

\[
U_n \supseteq M \subseteq U_{n+1} \quad \text{implies} \quad M \subseteq U_i \quad \text{for some} \quad U_i.
\]

To prove this we define \( U_i \) \((a < \infty)\) by induction such that

1. \( \bigcup U_i \subseteq U_{n+1} \)
2. \( U_i \cap U_i = \emptyset \quad \text{for} \quad \beta < a \)
3. \( U_i \supseteq M \subseteq U_{n+1} \quad \text{implies} \quad M \subseteq U_i \quad \text{for some} \quad U_i \)
4. \( U_i \cap W_{\sigma_i} = \emptyset \quad (i = 1, \ldots, n+1) \)

where we put \( S(p, U_{\alpha_1}) \) \((\alpha_1 = 1, 2, \ldots)\) intersects some \( k \) sets of \( U_r \) \((r > a)\) \((r = 1, \ldots, n+1)\) and \( W_{\alpha} = S(p, U_{\alpha_1}) \cup \cdots \cup S(p, U_{\alpha_k}) \)

For \( a = 0 \) we define

\[
U_0 = U_{n+1}, \quad U_i = U_i \quad (i = 2, \ldots, n+1).
\]

Since \( \mathcal{S}(p, U_{n+1}) \) intersects at most \( n+1 \) of \( U_{n+1} \) \((a < \infty)\), \( U_i \cap W_{\sigma_i} = \emptyset \) is obvious from the definition of \( W_{\sigma_i} \) \((i = 1, \ldots, n+1)\) and also the other three conditions are obviously satisfied.

Let us assume that \( U_i \) are defined for \( \beta < a \); then putting

\[
U'_{\alpha} = \bigcup_{\beta < \alpha} U_i \quad \text{and} \quad U'_{\alpha} = \bigcup_{\beta < \alpha} U_i \cap V_{\beta} \quad (i = 1, \ldots, n+1)
\]

we get \( U_{\alpha} \) satisfying 1) - 4). Since the validity of 1), 2), 4) for \( U'_{\alpha} \) is clear from the above definition, we prove 3) only. If \( M \subseteq U_{n+1} \) is an arbitrary set contained in \( U_n \), then

\[
M \cap W_{\sigma_i} = U_i \cap W_{\sigma_i} = U_n \cap W_{\sigma_i + 1}.
\]

1) \( \mathcal{S}(p, U) \supseteq \mathcal{S}(p, \mathcal{V}), \mathcal{S}(p, U) \supseteq \mathcal{S}(p, \mathcal{V}), \mathcal{S}(p, U) \supseteq \mathcal{S}(p, \mathcal{V}). \)
On the other hand, \( U_n \cap S_{n+1}^{\omega+1} = \emptyset \) is easily seen from the fact that every \( S_n^{\omega+1}(\mathcal{P}, U_{n+1}) \) intersects at most \( n+1 \) sets of \( U_n \). For \( \eta \in U_n \cap S_n^{\omega+1} \) then \( S_n^{\omega+1}(x, U_n+1) \) intersects \( U_n \) and some \( n+1 \) sets of \( U_n \), which is impossible. In consequence we get \( \mathcal{M} \cap W_t^{\omega+1} = \emptyset \). It follows that either \( \mathcal{M} \cap W_t^{\omega} = \emptyset \) for some \( i \) or \( \mathcal{M} \cap W_t^{\omega} = \emptyset \) for some \( i \) such that \( 2 \leq n-i+3 < n+3 \).

If the former is the case, then \( \mathcal{M} \cap W_t^{\omega} = \emptyset \). Since \( U_n \cap V_{n+1}^{\omega+1} \) is obvious for every \( \beta < \alpha \), and since \( U_n^{\omega+1} \cap W_t^{\omega} = \emptyset \), then from the assumption of induction, it follows that \( \mathcal{M} \cap V_{n+1}^{\omega+1} = \emptyset \) for every \( \beta < \alpha \).

Therefore \( U_{n+1} \cap V_{n+1}^{\omega+1} = \emptyset \), which implies \( \mathcal{M} \cap V_{n+1}^{\omega+1} = \emptyset \). Combining this with \( \mathcal{M} \cap V_{n}^{\omega+1} = \emptyset \), we conclude that \( \mathcal{M} \cup U_n^{\omega+1} = \emptyset \).

If the latter is the case, i.e.,

\[ y \in \mathcal{M} \cap W_t^{\omega} \cap W_{n+1}^{\omega+1}, \quad \mathcal{M} \cap W_{n+1}^{\omega+1} = \emptyset, \quad 2 \leq n-i+3 \leq n+1, \]

then \( y \in S_n^{\omega+1}(x, U_{n+1}) \) for some \( k > 0 \), i.e., \( S_n^{\omega+1}(y, U_{n+1}) \) intersects some \( n-i+2+k \) sets of \( U_n \). Accordingly \( S_n^{\omega+1}(x, U_{n+1}) \) intersects \( n-i+2+k+1 \) sets of \( U_n \) for every \( \eta \in \mathcal{M} \). Hence

\[ \mathcal{M} \cap W_{n+1}^{\omega+1} \subseteq W_n^{\omega+1}, \]

and hence \( \mathcal{M} \subseteq W_n^{\omega+1} \). Since \( U_n^{\omega+1} \cap W_n^{\omega+1} = \emptyset \), it follows that \( \mathcal{M} \cap W_n^{\omega+1} = \emptyset \). Combining this conclusion with the assumption \( \mathcal{M} \cap V_{n+1}^{\omega+1} = \emptyset \), we obtain

\[ \mathcal{M} \cap (V_{n+1}^{\omega+1} \setminus W_{n+1}^{\omega+1}) = \emptyset \]

from the openness of \( \mathcal{M} \). Therefore \( \mathcal{M} \subseteq U_n^{\omega+1} \). Thus the condition 3) is valid for \( a \), and hence we can define \( U_n^a \) (\( i = 1, \ldots, n+1 \)) satisfying 1)-3) for every \( a < \tau \).

2. Since \( U_{n+1} \subseteq U_n \subseteq U_{n+1}^{a} \) for \( i = 1, \ldots, n+1 \), if we put

\[ U_n^0 = \bigcup U_n^{a} \]

then \( U_n^0 \) is an open covering refining \( U_n \), and \( U_n \subseteq U_n^0 \) and \( U_n \neq U_n^0 \).

By condition 2) of \( U_n^0 \), from now on let us denote \( U_{n+1} \) and \( U_{n+1}^0 \) by \( U_n \) and \( U_n^0 \) (\( m = 1, 2, \ldots \)) respectively for brevity; then \( U_n \) and \( U_n^0 \) satisfy

\[ S(U_n, U_{n+1}) \cap S(U_n, U_{n+1}^0) = \emptyset \]

by the fact that \( U_n \neq U_n^0 \).
we obtain

(D) \( V \subseteq U' \) for every \( V \in \bigcup_{i=1}^{m} \mathcal{S}(U_i) \), \( U_i \subseteq \mathcal{U}_{\mathcal{S}(U_i)} \) and for some \( U' \subseteq \mathcal{U}_{\mathcal{S}(U_i)} \) with \( U' \cap U_i \neq \emptyset \).

It follows from (D) that \( \mathcal{S}(U_i) \subseteq \mathcal{S}(U_j) \) for every \( V \in \bigcup_{i=1}^{m} \mathcal{S}(U_i) \). Therefore we can conclude that

(E) \( \mathcal{S}(U_i) \subseteq \mathcal{S}(U_j) \) for every \( U_i, U_j \subseteq \mathcal{U}_{\mathcal{S}(U_i)} \).

In consequence, if \( U_i, U_k \subseteq \mathcal{U}_{\mathcal{S}(U_i)} \) and \( U_i \neq U_k \), then by (E) we can conclude that \( \mathcal{S}(U_i) \cap \mathcal{S}(U_k) = \emptyset \). As is easily seen, it follows from (E) that

(F) \( \{ \mathcal{S}(U_i) : U_i \subseteq \mathcal{U}_{\mathcal{S}(U_i)} \} \) is a family of sets, which will be used later.

Next we proceed to the case of (ii). If \( \mathcal{S}(U_i) \cap \mathcal{S}(U_j) \neq \emptyset \) for \( U_i, U_j \subseteq \mathcal{U}_{\mathcal{S}(U_i)} \) and \( U_i \neq U_j \), then there exist some \( V_p \in \mathcal{S}(U_i) \), \( V_q \in \mathcal{S}(U_j) \) with \( V_p \cap V_q \neq \emptyset \) and consequently two sequences

\[
V_1 = V_p \leftarrow V_2 \leftarrow V_3 \leftarrow \ldots \leftarrow V_m \quad (m \geq 1, \ldots, p),
V_2 = V_q \leftarrow V_3 \leftarrow V_4 \leftarrow \ldots \leftarrow V_m \quad (m \geq 1, \ldots, q).
\]

We take \( k > 0 \) such that

\[
2k - 1 + n(j) < 2k - 1 + 1 < 2k - 1 + n(j + 1);
\]
we notice that

(G) \( 2k - 1 + j < 2k - 1 + n(j + 1) \),

because \( j \) and \( n(j + 1) \) are even. Since \( 2k - 1 + n(j) < 2k - 1 + 1 \) implies \( \mathcal{S}(U_i) \neq \mathcal{S}(V_j) \subseteq \mathcal{S}(U_j) \) by (i), we assume

(H) \( 2k - 1 + n(j) < 2k - 1 + 1 \).

If \( j = p \), i.e., \( 2k - 1 + n(p) < 2k - 1 + j \), then since by (D) there exists \( W' \subseteq \mathcal{U}_{\mathcal{S}(U_k)} \) such that \( W' \subseteq W_i \) and \( W' \cap U_i \neq \emptyset \), we get \( \mathcal{S}(U_i) \subseteq \mathcal{S}(U_j) \) by (E). If \( j = p \), then by (E) we exist \( W' \subseteq \mathcal{U}_{\mathcal{S}(U_k)} \) and \( W' \subseteq \mathcal{U}_{\mathcal{S}(U_j)} \) such that

(I) \( W' \cap U_i \neq \emptyset \), \( \mathcal{S}(U_i) \cap \mathcal{S}(U_j) \neq \emptyset \).

(by (B), \( \mathcal{S}(U_i) \subseteq \mathcal{S}(U_j) \), (G) is an open covering of \( \mathcal{S}(U_i) \), moreover it is a refinement of \( \mathcal{S}(U_i) \) by (F). In consequence, \( \mathcal{S}(p, \mathcal{S}(U_i)) \subseteq \mathcal{S}(U_i) \) is a nbd for every point \( p \) of \( \mathcal{S}(U_i) \); hence it follows from \( \mathcal{S}(p, \mathcal{S}(U_i)) \subseteq \mathcal{S}(U_i) \) that \( (2) \), \( n+1 \), \( m+1 \), \( j \), \( 1 \), \( 2 \), \( \ldots \) is an open basis of \( \mathcal{S}(U_i) \). Combining this conclusion with (I), we get \( n+1 \) sequences \( 2 \leq 2 \leq \ldots \) (i.e., \( n+1 \) and \( m+1 \)) of disjointed collections such that \( (2) \) is an open basis of \( \mathcal{S}(U_i) \). Thus we conclude that \( \dim \mathcal{S}(U_i) = n \) by Theorem 1.

From this theorem we easily obtain the following main theorem.

**Theorem 3.** In order that a \( T \) topology space \( R \) be a metrizable space with \( \dim R = n \) it is necessary and sufficient that there exists a sequence \( \mathcal{S}(U_i) \subseteq \mathcal{S}(U_j) \subseteq \mathcal{S}(U_k) \subseteq \mathcal{S}(U_{m-1}) \subseteq \mathcal{S}(U_m) \) of open coverings such that \( \mathcal{S}(p, \mathcal{S}(U_i)) \subseteq \mathcal{S}(U_i) \) and \( \mathcal{S}(p, \mathcal{S}(U_i)) \subseteq \mathcal{S}(U_i) \) is a nbd basis for each point \( p \) of \( R \) and such that \( \mathcal{S}(U_{m-1}) \subseteq \mathcal{S}(U_m) \).

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Proof. Since the necessity is contained in Theorem 2, we prove only the sufficiency. Let $B_{\epsilon} = (V_{\epsilon})_{\alpha \in A}$ then we define $U_{\epsilon}$ by

(A) $U_{\epsilon} = \bigcup \{V_{\epsilon} | V \in B_{\epsilon}, \forall \alpha \in A \}$.

Since $B_{\epsilon} < B_{\delta}$, $U_{\epsilon} = \{U_{\epsilon} | \alpha \in A \}$ is an open covering of $X$ such that $U_{\epsilon} > B_{\delta}$.

Furthermore we notice that

(B) each set of $B_{\epsilon}$ intersects at most $\alpha + 1$ sets of $U_{\epsilon}$ by (A) and the condition $\dim R < \alpha + 1$.

Next, we assume $B_{\epsilon} = (V_{\beta})_{\beta \in B}$ and define $U_{\beta}$ by

$$U_{\beta} = \bigcup \{V_{\beta} | V \in B_{\epsilon}, \forall \beta \in B \}.$$ 

Then $U_{\beta} = \{U_{\beta} | \beta \in B \}$ is an open covering of $X$ and it follows from $B_{\epsilon} < B_{\beta}$ that

$$U_{\beta} > B_{\beta} > B_{\epsilon} > U_{\epsilon} > U_{\delta} > B_{\delta}.$$ 

We notice that

(C) each set of $B_{\epsilon}$ intersects at most $\alpha + 1$ sets of $U_{\epsilon}$.

Thus we can repeat this process and get a sequence

$$U_{\epsilon} > B_{\beta} > B_{\epsilon} > U_{\epsilon} > U_{\beta} > U_{\epsilon} > B_{\beta} > ...$$

of open coverings such that

(D) each set of $B_{\beta}$ intersects at most $\alpha + 1$ sets of $U_{\beta}$. Hence by (D) $U_{\epsilon} > B_{\beta} > B_{\epsilon} > U_{\beta} > ...$ is a sequence such that each set of $U_{\beta + 1}$ intersects at most $\alpha + 1$ sets of $U_{\beta}$ and such that $\{S(p, U_{\beta + 1}) | \epsilon = 1, 2, ... \}$ is a nbd basis of $p$. Therefore we conclude that $\dim X < \alpha$ from Theorem 2.

Let us apply our theorem to the notion "length of a multiplicative covering" due to Alexandroff and Kolmogoroff (see [1]).

**Definition.** We call a covering $U$ a multiplicative covering if every non-empty intersection $\bigcap_{i=1}^{k} U_{i}$ of elements $U_{i} (i = 1, ..., k)$ of $U$ is an element of $U$.

**Definition.** The maximal number $n$ such that there exists a sequence $U_{1}, U_{2}, ..., U_{n}$ of elements of a multiplicative covering $U$ is called the length of $U$.

**Definition.** We mean by the rank of an element $U$ of a multiplicative covering $U$ the maximal number $r$ such that there exists a sequence $U = U_{1} U_{2} U_{3} ... U_{r}$ of elements of $U$.

**Theorem 4.** In order that a $T_{1}$-space $X$ be a metrizable space with $\dim R < \alpha$ it is necessary and sufficient that there exists a sequence $U_{1}, U_{2}, ..., U_{\alpha}$ of multiplicative coverings with length $\leq \alpha + 1$ such that $\{S(p, U_{\alpha}) | m = 1, 2, ... \}$ is a nbd basis of $p$.

Note on dimension theory for metric spaces

Let us assume the existence of a sequence satisfying the condition of the proposition. If we denote by $U_{\alpha} (\alpha \in A)$ all the elements of $U_{\alpha}$ with rank $r$, then

$$U = (U_{\alpha} | \alpha \in A, r = 1, ..., \alpha + 1).$$

We define $V_{m+1}^{(m)} (i = 1, ..., \alpha + 1)$ by

$$V_{m+1}^{(m)} = U_{\alpha}, \quad V_{m}^{(m+1)} = \{x | S(x, U_{\alpha}) \subseteq V_{m+1}^{(m+1)} \} \quad (i = 2, 3, ..., \alpha + 1).$$

It follows directly from the above definition and $U_{\alpha + 1} < U_{\alpha}$ that

(A) $V_{m+1}^{(m)} \subseteq V_{m}^{(m+1)} \subseteq V_{m}^{(m+1)} = U_{\alpha}$,

(B) $U_{\alpha} < (V_{m+1}^{(m)})_{\alpha \in A_{\alpha}} \quad r = 1, ..., \alpha + 1$ ($r = 1, ..., \alpha + 1$),

(C) $S(V_{m+1}^{(m)}, U_{\alpha}) \subseteq V_{m+1}^{(m+1)}$ ($r = 2, ..., \alpha + 1$).

Next we define $M_{\alpha}$ ($r = 1, ..., \alpha + 1$) by

$$M_{\alpha} = V_{m}^{(m+1)} - \bigcup_{1 \leq 2} \{S(V_{m+1}^{(m+1)}, U_{\alpha+1}) \subseteq S(V_{m+1}^{(m+1)}, U_{\alpha+1}) \}.$$ 

Let us show that

(D) $M_{\alpha} = V_{m}^{(m+1)} - \bigcup_{1 \leq 2} \{S(V_{m+1}^{(m+1)}, U_{\alpha+1}) \subseteq S(V_{m+1}^{(m+1)}, U_{\alpha+1}) \}.$

Let $U_{i}$ be an arbitrary set of $U_{\alpha+1}$; then by using (B) for $i = \alpha + 1$ we get $V_{m+1}^{(m+1)}$ with $U \subseteq V_{m}^{(m+1)}$. $U \subseteq V_{m}^{(m+1)}$ follows from (A), and hence

$$U_{\alpha+1} \subseteq (V_{m+1}^{(m)} | \alpha \in A, r = 1, 2, ..., \alpha + 1).$$

Therefore we can find for every $U \in U_{\alpha+1}$ the minimum number $r$ such that $U \subseteq V_{m}^{(m+1)}$. To prove (E) we show that

$$U \cap S(V_{m+1}^{(m)}, U_{\alpha+1}) = \emptyset$$

for this $r$ and every $h$ with $1 < h < r$ and for every $\alpha \in A$. If we assume the contrary: $U \cap S(V_{m+1}^{(m)}, U_{\alpha+1}) \neq \emptyset$, $1 < h < r - 1$, then we have, from $U_{\alpha+1} \subseteq U_{\alpha}$, (A) and (C),

$$U \subseteq S(V_{m+1}^{(m)}, U_{\alpha}) \subseteq V_{m+1}^{(m)} \subseteq V_{m+1}^{(m+1)}.$$ 

(1) $A^{*}$ denotes the interior of $A$. 

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which contradicts the character of \( r \) because \( k < r \). Hence we must have \( U \cap S(V_{m_0}^0, U_{m_1}) = \emptyset \) (1 \( k < r \) - 1, \( a \in A_k \)). This combined with \( U \subseteq V_{m_0}^0 \)
for a definite \( a \in A_k \), implies \( U \subseteq M_{m_0} \) by (D), proving (E).

Now, to show that

(F) order \( M_k \ll n + 1 \),

we prove

(G) \( M_{r_0} \cap M_{r_0} = \emptyset \) for \( a \neq \beta \).

In the case \( a, \beta \in A_k \), \( a \neq \beta \) implies clearly

\[
M_{r_0} \cap M_{r_0} = U_{m_1} \cap U_{m_1} = \emptyset,
\]

because the ranks of \( U_{m_1} \) and of \( U_{m_1} \) are 1.

To show the same assertion for \( r > 1 \), we prove that

(H) \( U_{m_0} \cap U_{m_0} = U_{m_0} \), for \( a, \beta \in A_r \), \( \gamma \in A_r \) implies \( V(r_{m_0}^0) \cap V(r_{m_0}^0) = V(r_{m_0}^0) \).

First, \( V(r_{m_0}^0) \subseteq V(r_{m_0}^0) \cap V(r_{m_0}^0) \) is obvious from the definition of \( V(r_{m_0}^0) \).

Conversely, suppose that \( \varepsilon \in V(r_{m_0}^0) \cap V(r_{m_0}^0) \), then there exist nbd's \( P(\varepsilon), Q(\varepsilon) \) of \( \varepsilon \) such that

\[
S(P(\varepsilon), U_{m_0}) \subseteq U_{m_0}, \quad S(Q(\varepsilon), U_{m_0}) \subseteq U_{m_0}.
\]

Hence

\[
S(P(\varepsilon) \cap Q(\varepsilon), U_{m_0}) \subseteq U_{m_0} \cap U_{m_0} = U_{m_0}.
\]

This means that \( \varepsilon \in V(r_{m_0}^0) \), proving \( V(r_{m_0}^0) = V(r_{m_0}^0) \cap V(r_{m_0}^0) \). Repeating this process, we conclude that \( V(r_{m_0}^0) = V(r_{m_0}^0) \).

We now return to the proof of (G). By using (H) and (D), we have

\[
M_{r_0} \cap M_{r_0} \subseteq V(r_{m_0}^0) \cap V(r_{m_0}^0) = \cdots \subseteq (S(V(r_{m_0}^{m_0}), U_{m_1}; a) \cap a \in A_k) \cap \cdots \subseteq (S(V(r_{m_0}^{m_0}), U_{m_1}; a) \cap a \in A_k) \cap \cdots = \emptyset
\]

for \( r' \) determined by \( U_{m_0} \cap U_{m_0} = U_{m_0} \), because \( r' < r \) and consequently

\[
S(V(r_{m_0}^{m_0}), U_{m_1}; a) \subseteq \cdots \subseteq (S(V(r_{m_0}^{m_0}), U_{m_1}; a) \cap a \in A_k) \cap \cdots = \emptyset.
\]

Thus (G) is proved for \( r = 1, \ldots, n + 1 \). Since \( M_k = (M_{r_0} : a \in A_k, \ r = 1, \ldots, n + 1) \), the assertion (F): order \( M_k \ll n + 1 \) follows directly from (G).

Since \( M_k \ll n \), it is obvious, for (D) combined with (F) we obtain a covering \( M_k \) satisfying

\[
U_{m_1} < M_k < U_1, \quad \text{order } M_k < n + 1.
\]

Repeating the same process, we get a sequence \( M_k \) \((m = 1, 2, \ldots)\) of coverings of order \( < n + 1 \) such that

\[
U_{m_1} < M_k < U_1, \quad \text{order } M_k < n + 1.
\]

Therefore we have, from Theorem 3, \( \dim R < n \).

§ 2. Dimension and metric function. We know that if \( \varphi(x, y) < \varepsilon^{(y)} \) (\( i = 1, 2, 3 \)) in Euclidean 1-space \( E_1 \), then \( \varphi(y, y) < \varepsilon \) for some two points \( y, y \) of the three points \( y, y, y \), and the same is also valid for seven points \( y, y \) (\( i = 1, \ldots, 7 \)) and a point \( x \) of \( E_1 \), and that the number of \( y \) having such character increases with the dimension \( n \) of \( E_1 \). To begin with, we shall characterize generally the dimension of a metric space with a similar property of metric function.

**Theorem 5.** In order that \( \dim R < n \) for a metrizable space \( R \), it is necessary and sufficient to be able to define a metric \( \varphi(x, y) \) agreeing with the topology of \( R \) such that for every \( \varepsilon > 0 \) and for every point \( x \) of \( R \),

\[
\varphi(S(x; \varepsilon), y) < \varepsilon \quad (i = 1, \ldots, n + 2)
\]

imply

\[
\varphi(y, y) < \varepsilon \quad \text{for some } i, j \text{ with } i \neq j.
\]

**Proof.** Necessity. 1. Let \( R \) be a metrizable space with \( \dim R < n \); then by Theorem 2 there exists a sequence \( U_n > U^{(n)} > U^{(n-1)} > \cdots \)
(\( U^{(n)} = (U^{(n)})^* \)) of open coverings of \( R \) such that \( (S(y, U_n) : m = 1, 2, \ldots) \) is a nbhd basis for each point \( p \) of \( R \) and such that each \( S(p, U_{m_1}; A_k) \) intersects at most \( n + 1 \) sets of \( U_{m_1} \). Now we define \( S_{m_0 - m_0}(U) \) for \( 1 < m_0 < m_0 \) for some \( i, j \) with \( i \neq j \).

Furthermore we define open coverings of \( R \) by

\[
S_{m_0}(U) = \cup \{U' | S(U', U_n) \cap U \neq \emptyset, U' \in U_n\} = S(U, U_n),
\]

\[
S_{m_0 - m_0}(U) = \cup \{U' | S(U', U_n) \cap S_{m_0 - m_0}(U) \neq \emptyset, U' \in U_n\} = S(U, U_n),
\]

and

\[
S_{m_0 - m_0}(U) = U \quad \text{for } p = 1.
\]

We show first that

(A) \( \frac{1}{2^{m_0}} + \cdots + \frac{1}{2^n} > \frac{1}{2^n} \cdots \cdots + \frac{1}{2^n} \) implies \( S_{m_0 - m_0} \geq S_{m_0 - k} \).

We show that for every \( U' \in U_{m_0} \),

\[
S_{m_0 - k}(U') \subseteq S(U', U_{m_0})
\]

for every \( U' \in U_{m_0} \).

\( (\cdot) \) We denote by \( \varphi(x, y) \) the distance of \( x \) and \( y \); see (\( \cdot \)).
which will often be used in the remainder of this proof. For it follows from \( k_i < k_2 < \ldots < k_t \), that
\[
U_{k_1} > U_{k_2} > U_{k_3} > \ldots > U_{k_t},
\]
and hence
\[
S_{k_1-\frac{1}{2}}(U') \subseteq S(U, U_{k_1}) \subseteq S(U, U_{k_2}) \subseteq \ldots \subseteq S(U, U_{k_t}) \subseteq S(U, U_{k_1}).
\]
Therefore, if
\[
\frac{1}{2^{m_1}} + \ldots + \frac{1}{2^{m_p}} > \frac{1}{2^{m_1}} + \ldots + \frac{1}{2^{m_p}}
\]
and
\[
m_{i-1} = l_i, m_i = l_i, \ldots, m_{t-1} = l_t, m_t < l_i \quad \text{for a definite } i \quad \text{with } 2 < i < p, q,
\]
then from (B) we have
\[
(C) \quad S_{k_1-\frac{1}{2}}(U') \subseteq S(U', U_{k_1}) \subseteq S(U', U_{k_2}) \quad \text{for every } U' \in U_{k_1} \quad \text{and for some } U'' \in U_{k_1}.
\]
If further this \( U' \) satisfies \( S_{k_1-\frac{1}{2}}(U') \subseteq U' \) \( \text{for } U' \in U_{k_1} \), then from (C) we have
\[
(D) \quad U'' \cap S_{k_1-\frac{1}{2}}(U') = U'' \cap S_{k_1-\frac{1}{2}}(U') \neq \emptyset.
\]
Therefore by (C) and (D)
\[
(E) \quad S_{k_1-\frac{1}{2}}(U') \subseteq S(U'' \cap S_{k_1-\frac{1}{2}}(U')) \subseteq S(U, U_{k_1})(U') \quad \text{holds for every } U''
\]
with \( S_{k_1-\frac{1}{2}}(U') \subseteq U' \in U_{k_1}. \) Hence it follows from (E) that
\[
S_{k_1-\frac{1}{2}}(U') = S_{U'' \cap S_{k_1-\frac{1}{2}}(U')}(U'') \subseteq S(U, U_{k_1})(U') \subseteq S(U, U_{k_1})(U'),
\]
proving \( S_{U'' \cap S_{k_1-\frac{1}{2}}(U')} \subseteq S(U, U_{k_1})(U') \)

In the case of \( m_i < l_i \)
\[
S_{k_1-\frac{1}{2}}(U') \subseteq S(U', U_{k_1}) \subseteq U'' \subseteq S(U, U_{k_1})(U')
\]
for every \( U' \in U_{k_1} \) and for some \( U'' \in U_{k_1} \) follows directly from (C). This completes the proof of proposition (A).

2. Now we define a non-negatively valued function \( \varphi(x, y) \) on \( R \times R \) by
\[
(F) \quad \varphi(x, y) = \inf \{1/2^{m_1} + \ldots + 1/2^{m_p} \mid y \in S(x, U_{m_{p+1}}) \},
\]
\[
\varphi(x, y) = 1 \quad \text{if } y \notin S(x, U_{m_{p+1}}) \quad \text{for every } m_i \quad (i = 1, \ldots, p).
\]
Let us show that \( \varphi(x, y) \) satisfies the axiom of metric function.

\[\text{(c)} \] We use the notation \( A \cap U' \) in a somewhat different sense from that of the proof of Theorem 3, i.e., \( A \cap U' \) for \( U' \in U_0 \) means \( S(U', U_0) \cap A \neq \emptyset \) in this proof.

Note on dimension theory for metric spaces

Since \( (S, \mathcal{U}_m, \mathcal{U}_n) \mid m = 1, 2, \ldots \) is a nbd basis of \( p, \varphi(x, y) \) obviously agree with the topology of \( R, i.e., (S(x) \mid x > 0) \) is a nbd basis of each point \( p \) of \( R \).

To prove the triangle axiom \( \varphi(x, y) + \varphi(y, z) \geq \varphi(x, z) \) we assume that \( \varphi(x, y) = a > b = \varphi(y, z) \). For an arbitrary \( r > 0 \) we can select \( m_1, \ldots, m_p, k_1, \ldots, k_t \) such that
\[
1 < m_k < \ldots < m_p, \quad 1 < k_1 < \ldots < k_t,
\]
\[
a + \varepsilon > 1/2^m + \ldots + 1/2^p > a,
\]
\[
b + \varepsilon > 1/2^k + \ldots + 1/2^k > b
\]
and such that
\[
1/2^m + \ldots + 1/2^p > 1/2^k + \ldots + 1/2^k,
\]
Since \( y \in S(x, U_{m_{p+1}}) \), \( x \in S(y, U_{k_{p+1}}) \) are obvious from \( (F) \), we assume
\[
(G) \quad x, y \in S(m_{p+1}, m_1), \quad U \in U_{m_1}; \quad y, z \in S(k_{p+1}, k_1), \quad V \in U_{k_1}.
\]
Moreover we notice that we can assume
\[
(H) \quad p > 2, m_p > k_1.
\]
Without loss of generality.

(i) Let us consider the case of \( m_1 = 1 \). Since \( S(m_{p+1}, m_1)(U) \subset S(U, U_{m_1}) \) and \( S(k_{p+1}, k_1)(V) \subset S(U, U_{k_1}) \subset S(V, U_{k_1}) \) hold by \( (B) \), it follows from \( (G) \) and \( U_{m_1} \subset U_{k_1} \) that
\[
x, z \in S(U, U_{m_1}) \subset S(U, U_{k_1}) \subset W
\]
for some \( W \in U_{m_1-1} \). Hence \( x \in S(x, U_{m_{p+1}}) \) which implies
\[
\varphi(x, y) < 1/2^{m_1} + \ldots + 1/2^p < a + b + 2\varepsilon
\]
because \( m_p = 1 \).

(ii) To consider the case of \( m_1 < 1 \), we notice that there exist two sequences
\[
(i_1) \quad U = U_1 \leftarrow U_2 \leftarrow \ldots \leftarrow U_p, \quad V = V_1 \leftarrow V_2 \leftarrow \ldots \leftarrow V_q, \quad \text{with } U_i \in U_{m_i},
\]
\[
(i = 1, \ldots, p), \quad V_j \in U_{k_j} \quad (j = 1, \ldots, q), \quad y \in S(U \cap V \cap U_{m_i} \cap V_j).
\]
By \( (H) \) we can take \( i > 1 \) such that \( m_i < k_i < m_{i+1} \).

(a) In the case of \( i < m_{i+1} \) we can select \( S_1, S_2 \in U_{m_{i+1}} \) such that
\[
y \in S_1 \cap S_2, \quad S_1 \cap U_1 \neq \emptyset, \quad S_1 \cap V_2 \neq \emptyset.
\]
For it follows from \( m_{i+1} \), \( i > 1 \) that \( U_{i+1} \in U_{m_{i+1}} \subset U_{k_{i+1}} \) and \( V_2 \in U_{k_i} \subset U_{k_{i+1}} \). Hence
\[
y \in S(U_{i+1}, U_{m_{i+1}}) \subseteq S(U_{i+1}, U_{k_{i+1}}) \subseteq S(U_{i+1}, U_{k_{i+1}}) \subseteq S(U, U_{k_{i+1}})
\]
for some $S_1 \in \mathcal{U}_{n+1}$ and
\[ y \in S_{n+1}(V_0) \subseteq S(V_1, \mathcal{U}_1) \subseteq S_2 \]
for some $S_2 \in \mathcal{U}_{n+1}$ follows from (B). Then $S_1 \cap U_i \neq \emptyset$ and $S_2 \cap V_i \neq \emptyset$ are obvious because $U_i \subseteq U_i^{++1}$, $V_0 \subseteq V_i$. On the other hand, since $y \in S_1 \cap U_i$ and $J_i < \mathcal{U}_1$, $S_2 \cap U_i \subseteq W$ holds for some $W \in \mathcal{U}_1$. Hence $S(V_1, \mathcal{U}_1) \cap U_i \neq \emptyset$ for $i = 1, \ldots, n$, $U_i \subseteq U_i$ and $m_i < l_i$, we can consider $S_{n+1}(U_i)$ because $U_i \subseteq \mathcal{U}_1$ and $m_i = l_i$, and hence from the above discussion we get
\[ \exists \ z \in S_{n+1}(U_i) \subseteq S_{n+1}(U_1), \]
By (I), there exists a sequence
\[ (K) \quad U_i \supseteq U_i^{++1} \supseteq \cdots \supseteq U_0 \supseteq \emptyset \]
with $U_i \in \mathcal{U}_1$ (j = 2, \ldots, p). It follows from (B) and from $l_i < m_i - 1$, then we take $k$ such that
\[ (L) \quad 0 \leq k < i \quad m_i - 1 = m_i - 1 = m_{i-2} - 1 = \cdots = m_{i-k-1}, \]
where $k = 0$ means $m_i - 1 > m_i$, and $k = i$ means $m_i - 1 = m_i$ (j = 1, 2, \ldots, i).

In the case of $k = i$ it follows from (I) and (B) that
\[ S(U_i^{++1}, U_{i-k}) \cap U_i \neq \emptyset \quad \text{and} \quad y \in S(U_i^{++1}, U_{i-k}) \subseteq S(V_1, \mathcal{U}_1) \neq \emptyset, \]
which implies
\[ (M) \quad W \subseteq S(V_1, \mathcal{U}_1) \]
for some $W \in \mathcal{U}_{n+1}$ because $l_i < m_i + 1$ and $U_i - \mathcal{U}_1$, we can consider $S_{n+1}(U_i)$ because $m_{i-k} = m_i$, and since $x \in S(V_1, \mathcal{U}_1) \subseteq W$ by (I) and (B), we can conclude from (M) and (I) that
\[ \exists \ z \in S_{n+1}(U_i) \]
with respect to $x$, we select a sequence satisfying (K). Then
\[ x \in S(U_i^{++1}, U_{i-k}) \subseteq S_{n+1}(U_i) \]
by (B), and hence there exists $W' \in \mathcal{U}_{n+1}$ satisfying $x \in W' \subseteq S_{n+1}(U_i)$. Hence $x \in S_{n+1}(U_i)$ which implies $x \in S(U_i^{++1}, U_{i-k})$, and hence
\[ \exists \ x \in S(U_i^{++1}, U_{i-k}) \subseteq S_{n+1}(U_i) \]
by $l_i = m_i + 1$ and (L)

\[ \text{Note on dimension theory for metric spaces} \]

In the case of $k = i$ we select, by $(I_1)$, $(I_2)$ and (B), $P, \mathcal{U}_i \in \mathcal{U}_{n+1}$ with $x \in P$, $x \in \mathcal{U}_i$, $P \cap \mathcal{U}_i \neq \emptyset$. Then $x \in S(x, \mathcal{U}_{n+1})$ which implies
\[ g(x, y) < 1/2^{m_i} + \cdots + 1/2^{m_i+1} + 1/2^k < a + b + 2e. \]

Thus we get, in every case, $g(x, y) < a + b + 2e$ for an arbitrary $e > 0$,

proving
\[ g(x, y) < a + b = g(x, y) + g(y, y). \]

3. Now it remains to prove that $g(x, y) < \varepsilon$ for some distinct two points $y$, $x$. Since $g(x, y) < \varepsilon$, we can choose $m_1 + 1$ points $y_i$ and a positive number $s$ such that
\[ g(x, y_i) < \varepsilon, \]
\[ g(x, y_i) < \varepsilon. \]

Let $m_1, \ldots, m_p$ be positive integers satisfying
\[ 2^m < 1/2^m + \cdots + 1/2^m < \varepsilon; \]
then there exist $S_1 \subseteq S_{n+1}$ satisfying $x_i, y_i \in S_1$ because of $g(x_i, y_i) < \varepsilon$. On the other hand, since $\delta < 1/2^{m_1} + \cdots + 1/2^{m_2}$, we have
\[ \mathcal{U}_i \subseteq S(x, \mathcal{U}_{n+1}) \]
and hence $g(x, y_i) < \varepsilon$. Let $S_i = S_{n+1}(U_i) \\ U_i \subseteq \mathcal{U}_1 \cup \mathcal{U}_1$, then by (B) there exists $S_i \subseteq \mathcal{U}_1$ satisfying $S_i \cap U_i \neq \emptyset$. Hence it follows from (O) and $S_{n+1} \subseteq S_{n+1}$, that
\[ S_i \cap S(x, \mathcal{U}_{n+1}) \neq \emptyset \quad (i = 1, \ldots, n + 2), \]
which implies
\[ \exists \mathcal{U}_i \subseteq S(x, \mathcal{U}_{n+1}) \]
\[ (i = 1, \ldots, n + 2), \]
then by the first assumption $S(x, \mathcal{U}_{n+1})$ intersects at most $n + 2$ sets of $\mathcal{U}_i$, we must have $U_i = U_i$ for some $i$, with $i \neq i$. Then $y_i \in S(y_i, \mathcal{U}_{n+1})$, hence we conclude that
\[ g(y_i, y_i) < 1/2^{m_i} + \cdots + 1/2^{m_i} < \varepsilon. \]

Sufficiency. We denote by $\mathcal{U}(x, y)$ a metric satisfying the condition of this theorem. Then we denote by $\mathcal{M}_1$ a maximal subset of $R$ such that $x, y \in \mathcal{M}_1$ and $x \neq y$ imply $\mathcal{U}(x, y) > 1/2$. By the maximal property of $\mathcal{M}_1$, $\mathcal{M}_1 = (\mathcal{U}(x) \cap (\mathcal{M}_1))$ is evidently an open covering of $R$. Let $\mathcal{M}_2(x)$ intersect each of $\mathcal{S}_n(x)$ for $x \in \mathcal{M}_1$ (i = 1, \ldots, n + 2); then it follows from the property of $\mathcal{U}(x, y)$ that $g(x, y) < 1/2$ for some distinct points.

Since $g(x, y) < a + b + 2e$ is obvious in the case of $m_i = 1$, we assume $m_i > 1.$
Conversely, let \( \varrho(x, y) \) be a metric satisfying \( \varrho(x, z) \leq \max\{\varrho(x, y), \varrho(y, z)\} \), and let us assume that \( \varrho(S_{n}(x), y) < \varepsilon / i \) (\( i = 1, 2 \)). Then there exist \( x_{1}, x_{2} \in S_{n}(x) \) such that \( \varrho(x_{1}, y) < \varepsilon / i \). Since \( \varrho(x_{1}, x_{2}) < \varepsilon / i \), we get \( \varrho(y, y_{0}) < \max\{\varrho(y, y_{0}), \varrho(x_{1}, y_{0})\} \leq \max\{\varrho(y, y_{0}), \varrho(x_{1}, y_{0})\} < \varepsilon / i < \varepsilon / i < \varepsilon \).

**Definition.** A real-valued function \( \varrho \) of two points of a topological space \( R \) is a non-Archimedean parametric if

(i) \( \varrho(x, y) > 0 \),

(ii) \( \varrho(x, y) = \varrho(y, x) \),

(iii) \( \varrho(\varrho(x, y), \varepsilon) \) is open for each \( \varepsilon > 0 \),

(iv) \( \varrho(x, y) < \max\{\varrho(x, z), \varrho(y, z)\} \).

Now let us prove the following decomposition theorem for the metric function.

**Theorem 6.** In order that \( \dim R \leq n \) for a metrisable space \( R \) it is necessary and sufficient to be able to define a metric \( \varrho(x, y) \) agreeing with the topology of \( R \) such that for every \( \varepsilon > 0 \) and for some \( \varrho(\varepsilon(x), y) < \varepsilon \) (for \( i = 1, ..., n + 2 \)) imply \( \varrho(y_{i}, y_{j}) < \varepsilon \) for some \( i, j \) with \( i \neq j \).

In the compact case we get the following corollary.

**Corollary 2.** In order that \( \dim R < n \) for a compact metrisable space \( R \) it is necessary and sufficient to be able to define a metric \( \varrho(x, y) \) agreeing with the topology of \( R \) such that for every \( \varepsilon > 0 \), \( \varrho(x, y) < \varepsilon \) (for \( i = 1, ..., n + 2 \)) imply \( \varrho(y_{i}, y_{j}) < \varepsilon \) for some \( i, j \) with \( i \neq j \).

We can deduce the following theorem proved by J. de Groot (see [3]) from our theorem for the special case of \( n = 0 \).

**Corollary 3.** A metrisable space \( R \) is 0-dimensional if and only if one can define a metric which satisfies

\[ \varrho(x, z) < \max\{\varrho(x, y), \varrho(y, z)\}. \]

**Proof.** Let \( \dim R = 0 \); then by our theorem we can define a metric \( \varrho(x, y) \) such that \( \varrho(S_{n}(x), y) < \varepsilon / i \) (for \( i = 1, 2 \)) \( \varrho(y, y_{0}) < \varepsilon / i \). Hence, if we assume \( \varrho(x, z) < \varepsilon > \max\{\varrho(x, y), \varrho(y, z)\} \) for some \( x, y, z \in R \), then \( \varrho(S_{n}(x), z) < \varepsilon / i \) \( \varrho(y, y_{0}) < \varepsilon / i \) \( \varrho(x, z) = \varepsilon \), which contradicts the character of \( \varrho(x, y) \). Therefore we must have \( \varrho(x, z) < \max\{\varrho(x, y), \varrho(y, z)\} \).

\( \varepsilon \) This theorem is also a generalization of the above theorem of J. de Groot to the \( n \)-dimensional case.
if \( \mathcal{W}_m \) is defined, then we define \( \mathcal{W}_{m+1} \) by

\[
\mathcal{W}_{m+1} = \mathcal{W}_m \cup \mathcal{W}_{m+1}^c \quad (i = 1, \ldots, n+1, m = 0, 1, \ldots).
\]

Then it follows that \( \mathcal{W}_{m+1}^c \subset \mathcal{W}_m^c \subset \mathcal{S}_m \). Now we define a real-valued function \( g \) of two points by

\[
g(x, y) = \inf \{1/2^{m-1} | y \in S(x, \mathcal{W}_m^c) \}.
\]

Then it is easy to see that \( g \) is a non-Archimedean parametric. For \( g(x, y) < \max \{g(x, x), g(y, y)\} \), \( g(x, y) \) is evident from the disjointedness of \( \mathcal{W}_m \). Since

\[
S_i(y) = \{y \in S(x, \mathcal{W}_m^c) \mid g(x, y) < e \} = \bigcup_{i=1}^{n+1} (S(x, \mathcal{W}_m^c))_e > 1/2^{m-1},
\]

\( S_i(y) \) is open for every \( i > 0 \). Moreover, (i) and (ii) are clearly satisfied, and hence \( g \) is a non-Archimedean parametric of \( R \).

Since \( y \in S(x, \mathcal{W}_m^c) \) implies \( \phi(x, y) < 1/2^{m-1} \) by \( \mathcal{W}_m \subset \mathcal{S}_m \), we have

\[
(2) \quad g(x, y) > \phi(x, y) \quad \text{for every } x, y \in R.
\]

Now we can easily see that

\[
(3) \quad \phi(x, y) = \inf \{g(x, x), g(x, y) + \phi(x, y), g(y, y) \}_{x \in R}
\]

\[
(4) \quad g(x, y) = \min \{g(x, y) | x = 1, \ldots, n+1 \}
\]

is a metric of \( R \). It is enough to show only the agreement of \( g \) with the topology of \( R \). For a given \( e > 0 \) and \( x \in R \) we take \( m \) such that \( e > 1/2^{m-1} \) and \( R \subset S(x, \mathcal{W}_m^c) \). For \( y \in S(x, \mathcal{W}_m^c) \) we generally imply \( \phi(x, y) \leq e \) and consequently \( g(x, y) < e \) by (2). On the other hand \( g(x, y) > \phi(x, y) \) is obvious from (2) and (3), and hence \( g(x, y) < e \) implies \( \phi(x, y) < e \), which proves that \( \{y \mid g(x, y) < e \} \) is an open basis of \( x \), \( e \), \( g \) agrees with the topology of \( R \). Thus we deduce the necessity of the condition.

Sufficiency. Let \( g(x, y) \) be a metric of \( R \) satisfying the condition; then we easily see that \( R = \bigcup_{i=1}^{n+1} R_i \) if we put \( R_i = \{x \mid g(x, x) = 0\} \). To see this we assume the existence of \( x \in R \) such that \( x \not\in \bigcup_{i=1}^{n+1} R_i \). Then we must have \( g(x, x) = e_i > 0 \) \((i = 1, \ldots, n+1, m)\), hence by the property of \( g \),

\[
g(x, y) = \max \{g(x, x), g(x, y), g(y, y)\} > g(x, x) = e_i
\]

for every \( y \in R \). Therefore \( g(x, y) > \min e_i > 0 \), and hence \( g(x, y) > \min e_i > 0 \) for every \( x \in R \), which is a contradiction.

Putting

\[
S_i^{-1}(x) = \{y \mid g(x, y) < 1/m \}
\]

we see that \( S_i^{-1}(x) \cap S_i^{-1}(y) \neq \emptyset \) implies \( S_i^{-1}(x) = S_i^{-1}(y) \). For, if we choose \( z \in S_i^{-1}(x) \cap S_i^{-1}(y) \), then \( g(x, x) < 1/m \), \( g(y, y) < 1/m \), and hence

\[
g(x, y) < 1/m \text{ by (4)}. \quad \text{Therefore } g(y, y) < 1/m \text{ implies } g(x, y) < 1/m,
\]

which proves \( S_i^{-1}(x) \subseteq S_i^{-1}(y) \). In the same way we get \( S_i^{-1}(y) \supseteq S_i^{-1}(x) \) and consequently \( S_i^{-1}(x) = S_i^{-1}(y) \). Thus

\[
U_i = (S_i^{-1}(x) \cap R_i | x \in R_i) \quad \text{for } m = 1, 2, \ldots
\]

are open disjointed covering of \( R_i \). Moreover, since \( y \in S(x, U_i) \) implies \( g(x, y) < g(y, y) < 1/m \) \((U_i | x = 1, 2, \ldots) \) is an open basis of \( R_i \). Thus we conclude that \( \dim R_i = 0 \) by Theorem 1. This combined with \( R = \bigcup_{i=1}^{n+1} R_i \) implies \( \dim R = n \) by the generalized decomposition theorem.

§ 3. Imbedding of a metric space in a product of 1-dimensional spaces. We shall start with the following theorem, which is a generalization of the sufficiency part of Theorem 3.

**Theorem 7.** Let \( n = n_1 + n_2 + \ldots + n_k \) for non-negative integers \((i = 1, \ldots, k)\). If there exist sequences

\[
U_{1,i} > U_{2,i} > U_{3,i} > \ldots \quad (i = 1, \ldots, k)
\]

of open coverings of a \( T \)-space such that order \( U_{3,i} < U_{1,i} + 1 \) \((m = 1, 2, \ldots) \) and such that \( \{S(p, U_{1,i}) \mid m = 1, 2, \ldots \} \) for \( U_{1,i} = \bigcup_{i=1}^{n_k} U_{3,i} \) is a nbd basis of \( p \), then \( R \) is a metrizable space with \( \dim R_i = n \) and can be imbedded in a product of \( k \) metrizable spaces \( R_i = (i = 1, \ldots, k) \) with \( \dim R_i = n_i \).

Proof. As is easily seen from the proof of Theorem 3, we can select sequences

\[
U_{1,i} > U_{2,i} > U_{3,i} > \ldots \quad (i = 1, \ldots, k)
\]

of open coverings such that \( S(p, U_{1,i+1}) \) intersects at most \( n_i + 1 \) sets of \( U_{1,i} \) and such that \( \{S(p, U_{1,i}) \mid m = 1, 2, \ldots \} \) for \( U_{1,i} = \bigcup_{i=1}^{n_k} U_{3,i} \) is a nbd basis of \( p \). Let \( U_{1,i} = \{U_i | a \in A \} \) for fixed \( m, i \); then we put

\[
(5) \quad V_a = S(U_{1,i} \cup \alpha \cup a); \quad W_{2,i} = S(V_i, U_{1,i+1});
\]

\[
W_{2,i+1} = S(W_{2,i}, U_{1,i+1}); \quad W_{2,i} = S(V_i, U_{1,i}),\quad W_{2,i+1} = S(W_{2,i}, U_{1,i+1});
\]

\[
W_{3,i+1} = S(W_{3,i}, U_{1,i+1}); \quad W_{3,i+1} = S(W_{3,i}, U_{1,i+1}); \quad W_{3,i+1} = S(W_{3,i}, U_{1,i+1});
\]

Defining \( f_{m+1}(x) = \inf \{|x | x \in W_r| \}, \) we get continuous functions \( f_a (a \in A) \) satisfying
(B) \( \text{f}_{\text{smd}}(V_i^2) = 0 \), \( \text{f}_{\text{smd}}(U_a) = 1/2^n = 1 \) (11).

Clearly, for every \( \varepsilon > 0 \) there exists \( l_\varepsilon = l_\varepsilon(\varepsilon) \) such that
\[
\text{f}_{\text{smd}}(x) - \text{f}_{\text{smd}}(y) < \varepsilon \quad (a \in A_m, \quad m = 1, 2, \ldots).
\]

We consider a topological product
\[
P_i = \prod \{ I_a | a \in A_{m,i}, \quad m = 1, 2, \ldots \}
\]
of
\[
I_a = \{(x) | 0 < x < 1/2^{m-i} \} \quad (a \in A_m),
\]
and consider \( \text{f}_{\text{smd}} \) a mapping of \( R \) into \( I_a \). Then we define a continuous mapping \( F_i \) of \( R \) into \( P_i \) by
\[
F_i(x) = \{ f_{\text{smd}}(x) | a \in A_{m,i}, \quad m = 1, 2, \ldots \} \quad (x \in R).
\]

Now we proceed to prove that \( F_i(R) \subseteq P_i \) is a metrizable space with \( \text{dim} F_i(R) = n_i \).

Since
\[
N_a = F_i(R) \cap \{ (p_\alpha) | ||p_\alpha - p_\beta|| > 0 \} \quad (a \in A_{m,i})
\]
are open sets, and since
\[
\text{f}_{\text{smd}}(U_a) = 1/2^n, \quad \bigcup \{ U_a | a \in A_{m,i} \} = R
\]
by (B), \( \mathfrak{R}_{m,i} = \{ N_a | a \in A_{m,i} \} \) is an open covering of \( F_i(R) \). First we choose an integer \( i \) such that \( y \in S(x) \in N_{m,i} \) implies
\[
\text{f}_{\text{smd}}(x) - \text{f}_{\text{smd}}(y) < \varepsilon \quad (a \in A_{m,i}, \quad m = 1, 2, \ldots).
\]

We prove that \( F_i(R) \subseteq P_i \) is a metrizable space with \( \text{dim} F_i(R) = n_i \).

Next let us show that
\[
\text{order} \mathfrak{R}_{m,i} < n_i + 1, \quad \text{dim} \mathfrak{R}_{m,i} < n_i + 1.
\]

If \( \text{f}_{\text{smd}}(x_0) = 0 \) and \( a \in A_{m,i} \), then we can choose \( p = (p_\alpha) \in F_i(R) \) and \( x \in R \) such that
\[
\text{f}_{\text{smd}}(x) - \text{f}_{\text{smd}}(y) < \varepsilon \quad (a \in A_{m,i}, \quad m = 1, 2, \ldots).
\]

We consider a topological product
\[
P_i = \prod \{ I_a | a \in A_{m,i}, \quad m = 1, 2, \ldots \}
\]
of
\[
I_a = \{(x) | 0 < x < 1/2^{m-i} \} \quad (a \in A_m),
\]
and consider \( \text{f}_{\text{smd}} \) a mapping of \( R \) into \( I_a \). Then we define a continuous mapping \( F_i \) of \( R \) into \( P_i \) by
\[
F_i(x) = \{ f_{\text{smd}}(x) | a \in A_{m,i}, \quad m = 1, 2, \ldots \} \quad (x \in R).
\]

Now we proceed to prove that \( F_i(R) \subseteq P_i \) is a metrizable space with \( \text{dim} F_i(R) = n_i \).

Since
\[
N_a = F_i(R) \cap \{ (p_\alpha) | ||p_\alpha - p_\beta|| > 0 \} \quad (a \in A_{m,i})
\]
are open sets, and since
\[
\text{f}_{\text{smd}}(U_a) = 1/2^n, \quad \bigcup \{ U_a | a \in A_{m,i} \} = R
\]
by (B), \( \mathfrak{R}_{m,i} = \{ N_a | a \in A_{m,i} \} \) is an open covering of \( F_i(R) \). First we choose an integer \( i \) such that \( y \in S(x) \in N_{m,i} \) implies
\[
\text{f}_{\text{smd}}(x) - \text{f}_{\text{smd}}(y) < \varepsilon \quad (a \in A_{m,i}, \quad m = 1, 2, \ldots).
\]

We prove that \( F_i(R) \subseteq P_i \) is a metrizable space with \( \text{dim} F_i(R) = n_i \).

Next let us show that
\[
\text{order} \mathfrak{R}_{m,i} < n_i + 1, \quad \text{dim} \mathfrak{R}_{m,i} < n_i + 1.
\]
Theorem S. Every metric space $R$ with $\dim R < n$ can be topologically imbedded in a topological product of $n + 1$ at most 1-dimensional metric spaces.

Proof. If $\dim R < n$, then it is easily shown that

(A) we can define a covering $B$ and open collections $U_i (i = 1, \ldots, n + 1)$ to every covering $H$ of $R$ such that $B < \bigcup_{i=1}^{n+1} U_i < H$ and such that each $S^h(p, B)$ intersects at most one of sets belonging to $U_i$ for a fixed $i$. For each $R$ with $\dim R < n$, there exists a disjointed collection $S_i$ of $R$ satisfying $B_i < H$. For every point $x$ of $R$, we denote by $\varepsilon(x)$ a positive number such that

$$S(x) \cap R \subseteq V \subseteq U, \quad S(x) \subseteq U < H$$

for $U$ defined by $V$ so that $V \subseteq U$. Then $B_i = \{ \bigcup \{ (S(x) | x \in U_i) | V \in B_i \}$

is an open collection of $R$ with $\bigcup_{i=1}^{n+1} B_i < H$. Selecting a covering $B$ with $B < \bigcup_{i=1}^{n+1} B_i$, we can define an open collection $U_i$ by

$$U_i = \{ \bigcup \{ (S(W, B) \subseteq V_i | V \in B_i \}$.

It is easy to see from the disjointedness of $B_i$ that $\bigcup_{i=1}^{n+1} U_i$ covers $B$ and that each set of $B$ intersects at most one of sets of $U_i$. Choosing a covering $B$ with $B < B_i$, we have open collections and a covering satisfying the required condition ($A$).

We denote by $\mathcal{G} > \mathcal{G}_2 > \mathcal{G}_3 > \mathcal{G}_2 > \ldots$ a sequence of coverings such that $(\mathcal{S}(p, \mathcal{G}))_{n = 1, 2, 3, \ldots}$ is a nbd basis for each point of $R$, and take a covering $B$ and collections $U_i, (i = 1, \ldots, n + 1)$ satisfying (A) for $\mathcal{G}_i$, i.e.,

$$B < \bigcup_{i=1}^{n+1} U_i, \quad \mathcal{G}_i$$

and $S(p, B)$ intersects at most one set of $U_i$. Let $U_i = \{ U_i | a \in A \}$ and define $\mathcal{R}_l$ by

$$\mathcal{R}_l = \{ S(U_i, A), R - \bigcup_{a \in A} U_i | a \in A \}$$

for a fixed $i$;

then $\mathcal{R}_l$ is a covering of order $< 2$. Moreover, it follows from $\mathcal{S}_l < \mathcal{G}_i$ and $\bigcup_{i=1}^{n+1} B_i \subseteq \mathcal{G}_i$.

Now we notice that

(B) every covering $\mathcal{N}$ of order $< 2$ has a locally finite star-refinement $\mathcal{N}'$ with order $< 2$.

To show this we put $\mathcal{N}' = \{ P \delta \in D \}$ and denote by $\mathcal{N}'$ a star-refinement of $\mathcal{N}$. Then

(C) $\mathcal{M} = \{ M_a = \{ \bigcup \{ (S(P, f \subseteq \mathcal{P}), \mathcal{F} \subseteq \mathcal{P}, P \mathcal{F} \delta) | \delta \in D \}$

is a locally finite refinement of $\mathcal{N}$ of order $< 2$. We define an open set $L_0$ for every $\delta \in D$ such that

$$\mathcal{D}_a = \bigcup_{a \in A} M_a \subseteq U_a \subseteq L_a \subseteq M_a$$

and put

$$Q_0 = L_0 - \bigcup_{\delta \in D} L_0, \quad \Omega = \{ Q_0, M_a \cup M_b | a \neq b, a \neq b \}.$$

It is easy to see that $\Omega$ is an open covering satisfying $\mathcal{G}_2$, but order $\mathcal{N} < 2$.

To show that $\Omega$ covers $R$ we take an arbitrary point $p$ of $R$. If $p \in M_0$, and $p \neq M_b (b' \neq b)$, then $p \in L_0$ by (D). Since $\{ L_0 | \delta \in D \}$ is locally finite, it follows from (D) that there exists a nbd $U(p)$ of $p$ such that $U(p) \cap L_0 = \emptyset$ for every $\delta' \neq \delta$. Hence $p \notin \bigcup_{\delta' \neq \delta} L_0$ and consequently $p \notin Q_0$ by (E).

Therefore from (C), (D), (E) we get that $\Omega$ covers $R$ and that $S(p, \mathcal{O}) = Q_0 \subset M_0 \subset P_0$. If $p \notin M_0 \cup M_b$, then $p \notin L_0$ for every $\gamma$ with $p \neq a, b$. It follows from (E) that either $p \notin Q_0$ or $p \notin Q_0$. Therefore either

$$S(p, \mathcal{O}) = Q_0 \cup (M_0 \cup M_b) \subseteq M_0 \subseteq P_0$$

or

$$S(p, \mathcal{O}) = Q_0 \cup (M_0 \cup M_b) \subseteq M_b \subseteq P_b,$$

which shows that $\mathcal{N}' < \mathcal{N}$ and order $\mathcal{N} < 2$. Repeating such a process we have a locally finite $\delta$-refinement $\mathcal{N}'$ of $\mathcal{O}$ with order $< 2$. $\mathcal{N}'$ satisfies the required condition of (B).

To show the existence of sequences $\mathcal{R}_a > \mathcal{R}_b > \mathcal{R}_d > \mathcal{R}_d > \ldots$

$(i = 1, \ldots, n + 1)$ of coverings of order $< 2$ such that $\mathcal{R}_a < \mathcal{S}_i$,

($\star$) $\mathcal{Q} = \{ (S(p, \mathcal{O}) | p \in E \}$.  

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we assume the existence of such $R_i$ for $i < n$. Then there exists by (B) a locally finite covering $\mathfrak{R}$, of order $< 2$, such that $R_i < R_{m_i}$. Let us select a covering $\mathfrak{R}$ with $\mathfrak{R}^* \subseteq \bigwedge_{i=1}^{n+1} \mathfrak{R}_i$. Next we select by (A) a covering $\mathfrak{Q}$ and open collections $\mathfrak{R}_i (i = 1, ..., n+1)$ such that

$$(P) \quad \mathfrak{Q} \subseteq \bigwedge_{i=1}^{n+1} \mathfrak{R}_i \subseteq \mathfrak{R} \cap \bigwedge_{i=1}^{n+1} \mathfrak{R}_i$$

and such that each $S(p, \mathfrak{Q})$ intersects at most one of sets belonging to $\mathfrak{E}_i$ for a fixed $i$. We put $\mathfrak{E}_i := \{ p \mid p \in B \} \cup \{ X \mid X < \tau \} \cup \{ \gamma \mid \gamma \in \mathfrak{R}_i \}$ and denote by $\gamma(\beta)$ the first ordinal $\gamma$ satisfying

$$(G) \quad S(p, \mathfrak{Q}) \subseteq \mathfrak{R}_i$$

for $\beta \in B$. Then we define a covering $\mathfrak{R}_{m+1}$ by

$$(H) \quad \mathfrak{R}_{m+1} := \{ (p, \beta, S(p, \mathfrak{Q}) \mid \gamma < \tau, \beta \neq \beta \}$$

where we put

$$(I) \quad \mathfrak{R}_i := \mathfrak{R}_i \cap \bigwedge_{j=1}^{n+1} \mathfrak{R}_j$$

It easily follows that

$$(J) \quad \mathfrak{R}_{m+1} \subseteq \mathfrak{R}_i$$

Since $\mathfrak{R}_{m+1} \subseteq \mathfrak{R}_1$ is obvious, let us prove the latter assertion. We denote by $x$ an arbitrary point of $\mathfrak{R}$. First we consider the case of $x \in S(p, \mathfrak{Q}) \cap S(p, \mathfrak{Q})$ for some $\beta \neq \beta$. Then we have $S(p, \mathfrak{Q}) \cap S(p, \mathfrak{Q})$ for every $\beta \neq \beta$. For the former case, then $x \in S(p, \mathfrak{Q}) \cap S(p, \mathfrak{Q})$ for every $\beta \neq \beta$ because $S(a, \mathfrak{Q}) \cap S(a, \mathfrak{Q})$ intersects at most one of sets $\mathfrak{R}_i$. Hence it follows from (G) that $x \in \mathfrak{R}$ and $x \in S(p, \mathfrak{Q})$ for every $\beta \neq \beta$. If the latter is the case $x \in S(p, \mathfrak{Q})$ for every $\beta \neq \beta$ and $x \in S(p, \mathfrak{Q})$ for every $\beta \neq \beta$ because $S(a, \mathfrak{Q}) \cap S(a, \mathfrak{Q})$ intersects at most one of sets $\mathfrak{R}_i$. Hence it follows from (G) that $x \in \mathfrak{R}$ and $x \in S(p, \mathfrak{Q})$ for every $\beta \neq \beta$. If the latter is the case, then it follows from (I) and order $\mathfrak{R} < 2$ that $x \in \mathfrak{R}_i$. Next we consider the case of $x \in S(p, \mathfrak{Q}) \cap S(p, \mathfrak{Q})$. Then $x \in S(p, \mathfrak{Q}) \cap S(p, \mathfrak{Q})$ for every $\beta \neq \beta$ as in the above discussion. Since $x \in S(p, \mathfrak{Q}) \cap S(p, \mathfrak{Q})$ is obvious from (G), we can find $x \in \mathfrak{R}_i$ in at most two sets of $\mathfrak{R}_{m+1}$. Thus in every case $x$ is contained in at most two sets of $\mathfrak{R}_{m+1}$. By (I) in $\mathfrak{R}_{m+1}$, $x \in \mathfrak{R}_i$ is a covering of order $< 2$.

Let $\mathfrak{Q} \subseteq \mathfrak{Q}_i \subseteq \mathfrak{R}_i (i = 1, ..., n+1)$ and let $x$ be an arbitrary point of $\mathfrak{R}$; then $x \in S(p, \mathfrak{Q})$ for some $i$ because $\bigwedge_{i=1}^{n+1} \mathfrak{Q}_i$ covers $\mathfrak{R}$. Hence $x \in \mathfrak{Q}_n$ by (I), which shows $\bigwedge_{i=1}^{n+1} \mathfrak{Q}_i = \emptyset$. Therefore every set of $\bigwedge_{i=1}^{n+1} \mathfrak{R}_i$ is contained in $S(p, \mathfrak{Q})$ with $p \in \mathfrak{Q}_i$ for some $i$, which implies

$$(K) \quad \bigwedge_{i=1}^{n+1} \mathfrak{R}_{m+1} \subseteq \bigwedge_{i=1}^{n+1} \mathfrak{R}_i$$

by (P). It follows from (J) and $\mathfrak{R} \subseteq \mathfrak{R}_i$ that $\mathfrak{R}_{m+1} \subseteq \mathfrak{R}_i$. This combined with (I), (K) completes the induction, and hence we get sequences $\mathfrak{R}_{m+1} \subseteq \mathfrak{R}_2 \subseteq \mathfrak{R}_1 \subseteq \mathfrak{R}_i > ... (i = 1, ..., n+1)$ such that $\bigwedge_{i=1}^{n+1} \mathfrak{R}_i < \mathfrak{R}_i < \mathfrak{R}_{m+1} \subseteq \mathfrak{R}_i$ order $\mathfrak{R} < 2$. Hence by Theorem 7 we can imbed $\mathfrak{R}$ into a topological product of $n+1$ metrizable spaces $\mathfrak{R}_i$, with $\dim R < 1$.

§ 4. Imbedding $n$-dimensional spaces in $\mathfrak{R}(\mathfrak{Q})$.

Definition. We call a covering $\mathfrak{Q}$ star-finite (star-countable) if every $\mathfrak{Q}$ intersects finitely (countably) many sets of $\mathfrak{R}$.

An open basis consisting of an enumerable number of star-finite (star-countable) open covers is called a $\alpha$-star-finite (star-countable) open basis.

Remark. A regular space $\mathfrak{R}$ has a $\alpha$-star-finite basis if and only if $\mathfrak{R}$ has a $\alpha$-star-countable basis. Moreover K. Morita has proved the following theorem: A regular space having a $\alpha$-star-finite (star-countable) basis can be imbedded in the topological product $N(\mathfrak{Q}) \times \mathfrak{T}$ of a generalized Baire 0-dimensional space $N(\mathfrak{Q})$ (ii) and Hilbert cube $\mathfrak{T}$, and the converse is also true.

Remark. A metric space having a $\alpha$-star-finite basis need not have the star-finite property or the star-countable property (ii). For example, $\mathfrak{R}(\mathfrak{Q})$ for $\mathfrak{Q} = \mathfrak{Q}$ has obviously a $\alpha$-star-finite basis, but it has not the star-countable property if the cardinal number of $\mathfrak{Q}$ is greater than $\aleph_0$. For if we put

$$S(a_1, a_2, ..., a_n) := \{ p \mid p = (a_1, a_2, ..., a_n) \in N(\mathfrak{Q}) \}$$

then it is easily seen that the open covering

$$\{ N(\mathfrak{Q}) \times \{ \alpha \} | \alpha < 1 \}, S(a_1, a_2, ..., a_n) \times \{ \alpha \} | \alpha < 1 \} \subseteq \{ \alpha \} \times \{ \alpha \} | \alpha < 1 \}$$

of this space has no star-countable refinement and accordingly no star-finite refinement. To see this we assume that $\mathfrak{Q}$ is a star-countable refinement of this covering. Then $S(a_1, a_2, ..., a_n) \subseteq \mathfrak{Q}$ consisting of countably many sets of $\mathfrak{Q}$. We can select $S(a_1, a_2, ..., a_n) \subseteq \mathfrak{Q}$.

It follows from the connectedness of $\{ \alpha \} \times \{ \alpha \} | \alpha < 1 \}$ that $\bigcup_{a \in \mathfrak{Q}} S(a_1, a_2, ..., a_n) \subseteq \mathfrak{Q}$ with $\mathfrak{Q} < 1$.

(ii) This notion is due to [12]. For any two sequences of elements from an abstract set $\mathfrak{Q} = \{ a_1, a_2, ..., \}$, $\beta = \{ \beta_1, \beta_2, ..., \}$, we define the metric $d(\alpha, \beta)$ by

$$d(\alpha, \beta) = \min | \alpha_\beta \neq \beta_\beta | .$$

Then the set $\mathfrak{Q}(\mathfrak{Q})$ of all such sequences turns out to be a zero-dimensional space.

(ii) We say that $\mathfrak{Q}$ has the star-finite (star-countable) property if only if every open covering of $\mathfrak{Q}$ has a star-finite (star-countable) open refinement.
Hence \( \bigcup_{n=1}^{\infty} S^n(U, H) \) contains every set of \( H \) contained in \( \bigcup_{n=1}^{\infty} S^n(U, H) \times \{ a(1/2k^{1/2} < x < 1/2k^{1/2} + 1/2k^{1/2}) \) for some \( a_k, e, \zeta \) in \( \bigcup_{n=1}^{\infty} S^n(U, H) \); it contains non-enumerably many sets of \( H \), which is a contradiction.

**Theorem 9.** Suppose that \( R \) is a regular space having a \( \sigma \)-star-finite (\( \sigma \)-star-countable) basis and \( \dim R < n \). Then \( R \) is homeomorphic to a subset of \( N(D) \times I_{m+1} \), where \( I_{m+1} \) is a \( (2m+1) \)-dimensional Euclidean cube and \( N(D) \) is the generalized Baire 0-dimensional space for a set \( D \) whose cardinal number is not less than the cardinal number of an open basis of \( R \).

**Proof.** 1. There exists, as is seen from the above Morita's theorem, a sequence \( R_0 > R_1 > R_2 > \ldots \) of \( \sigma \)-star-finite open coverings \( R_n \) of \( R \) such that \( \delta(p, R_n) = 1, 2, \ldots \) is a nbd of every point of \( R \). We define a disjointed covering \( \mathfrak{S} \) of \( R \) by \( \mathfrak{S}_n = \bigcup_{n=1}^{\infty} S^n(N, R_n) \), where \( S^n(N, R_n) = \bigcup_{n=1}^{\infty} S^n(N, R_n) \). Let \( \mathfrak{S}_0 = \bigcup_{n=1}^{\infty} S^n(N, R_n) \) and \( \mathfrak{S}_n \cap \mathfrak{S}_m = \emptyset \); then for every \( a \in A_n \), \( S_n \) is a countable set of sets \( R_n \), i.e.,

\[
\mathfrak{S}_n = \bigcup_{i=1}^{\infty} S_{n_i}^n(i = 1, 2, \ldots ), \quad N_{n_i}^n \subseteq R_n \quad (i = 1, 2, \ldots )
\]

Since \( R_n \) is locally finite, there exists an open covering \( R_n \) of \( R \) such that

\[
\mathfrak{S}_n = \bigcup_{i=1}^{\infty} S_{n_i}^n(i = 1, 2, \ldots ), \quad N_{n_i}^n \subseteq R_n
\]

Next, we define a sequence of open coverings by

\[
U_m = \bigcup_{n=1}^{\infty} S_{n_i}^n(i = 1, 2, \ldots )
\]

Then \( U_0 \supset U_1 \supset U_2 \supset \cdots \) and \( \delta(p, U_n) = 1, 2, \ldots \) is a nbd of each point of \( R \). Then

(A) \( \mathfrak{S}_n \) is finite in every \( S_n(a \in A_n, k > m) \).

Let \( \mathfrak{S}_n \) be the disjointed covering \( \mathfrak{S} \) of \( \mathfrak{S} \) so that \( \mathfrak{S}_n \cap \mathfrak{S}_m = \emptyset \). Then \( \mathfrak{S}_n \) is a nbd of every point of \( \mathfrak{S}_n \) and \( \mathfrak{S}_k \) is a nbd of every point of \( \mathfrak{S}_k \). We define a continuous mapping \( c(\pi) \) of \( R \) into \( N(D) \) and \( c(\gamma) \) of \( R \) into \( I_{m+1} \).

Moreover we define the following notions, which will be needed later on.

\[
T_n = c(S_n) = \{(a_1, a_2, \ldots | a_m = a) \in c(R) \quad \text{for } n \in A_n, \}
\]

where we denote by \( S_{n_i}(a) \) the spherical nbd of radius \( 1/2m \) around \( x \) in \( I_{m+1} \). We mean by a star-decomposition a disjointed covering of \( R \) consisting of open sets contained in \( \bigcup_{n=1}^{\infty} T_n \). Let \( C \leq \mathfrak{S}_n \) be a star-decomposition; then for every \( \gamma \in C \) we denote by \( m(\gamma) \) such a number that \( \gamma \in A_{m(\gamma)} \).

We denote by \( M(R, m) \) the totality of mappings of \( M(R) \) satisfying

\[
f^{-1}(T) \subseteq f^{-1}(T) \subseteq U_m \quad (\gamma \in C)
\]

for some star-decomposition \( C \) of \( \bigcup_{n=1}^{\infty} T_n \). Finally we define \( C \)-neighborhood \( N_C(f) \) of \( f \in M(R) \) by

\[
N_C(f) = \{ \gamma \in M(R) | \sup \{ d(\sigma(\pi), \sigma(\gamma)) | x \in S_{n_i}^n(x) \} < 1/m(\gamma) \}
\]

for a star-decomposition \( C \) of \( \bigcup_{n=1}^{\infty} T_n \), where \( \pi, \sigma \) denote the projection of \( N(D) \) into \( I_{m+1} \) onto \( I_{m+1} \) and the metric of \( I_{m+1} \) respectively.

2. First we prove

\( \textbf{(B)} \) \( N_C(f) \cap (M(R, m) \neq \emptyset \) for every \( f \in M(R) \), every star-decomposition \( C \) and every positive integer \( m \).

Take \( \gamma \in C \) and put

\[
D_\gamma = \{ \delta \in A_{m(\gamma)} \}, \quad T_\gamma \subseteq T_\gamma \quad (\gamma \in C) = \text{the same}
\]

Since we can cover \( I_{m+1} \) by a finite subcover of \( \bigcup_{n=1}^{\infty} T_n \), we may denote by \( \bigcup_{n=1}^{\infty} T_n \) a nbd of every point of \( R \), where \( \bigcup_{n=1}^{\infty} T_n \) is a nbd of every point of \( R \). We denote \( \delta \in A_{m(\gamma)} \) by \( \delta \notin A_{m(\gamma)} \) so that \( \delta \notin A_{m(\gamma)} \) is a nbd of every point of \( R \). Then \( \delta \in A_{m(\gamma)} \) is a nbd of every point of \( R \). Then

\[
N_\gamma(f) = \{ \delta \in A_{m(\gamma)} | d(\sigma(\pi), \sigma(\gamma)) < 1/m(\gamma) \}
\]

\( \gamma \in C \) is an open covering of \( R \) of order \( \gamma \notin \gamma \).

Let us consider fixed \( \gamma \in C \) and \( \delta \in D_\gamma \), and assume that \( V_1, \ldots, V_k \) are all the numbers of \( R \). Then we select vertices \( x_0(V_i) \) \( (i = 1, \ldots, k) \) in \( I_{m+1} \) for which it is true that \( d(\pi(V_i), \pi(V_j)) < 1/3m(\gamma) \) \( (i = 1, \ldots, k) \), the \( \pi(V_i) \) are in a general position in \( E_{m+1} \). Then, the \( \pi(V_i) \) lie in an \( m \)-dimensional linear subspace of \( E_{m+1} \). We define a barycentric mapping \( \Phi_i \) of \( S_0 \) into \( I_{m+1} \) by

\[
\Phi_i(x) = (x_0 + s_1 V_1 + \cdots + s_k V_k) / (s_1 + \cdots + s_k)
\]
hence \( p, p' \in S, p \subseteq S \). Therefore we get \( d(\Phi(p), \Phi(p')) < 2k(h, \delta) \), which implies \( L_0(p) \cap L_0(p') = \emptyset \). If we suppose that \( L_0(p') \) is spanned by \( x(V_1), \ldots, x(V_{n+1}) \), then since \( x(V_1), \ldots, x(V_{n+1}) \) are in a general position in \( E_{n+1} \), it follows that at least one of \( x(V_1), \ldots, x(V_{n+1}) \) is also one of \( x(V_1), \ldots, x(V_{n+1}) \). Hence \( p \) and \( p' \) are contained in a common member \( V_i \) of \( \mathcal{F} \), i.e., \( p, p' \in B(p, 3\delta) \). It follows from \( S \subseteq \mathcal{F} \) that \( \gamma^{-1}(T_0 \times S_{1m+1}(x)) \subseteq U \) for some \( U \subseteq N_\gamma \). Thus we get \( \gamma^{-1}(T_0) \subseteq N_\gamma \) for every \( \gamma \in E \), proving \( \gamma \subseteq N_\gamma \). We now prove that

\[ \Phi(p) = \Phi(p) \quad (p \in S_0, \delta \in D_\gamma, \gamma \in C) \]

of \( E \) into \( E_{n+1} \). We now prove that the mapping \( \Phi(p) = \{ \Phi(p), \Phi(p') \} \in M(E) \) is contained in the common part of \( N_\gamma(f) \) and \( M(E) \).

To prove \( \Phi \in N_\gamma(f) \) we take an arbitrary point \( p \in S \) for \( \gamma \in C \). Then \( p \in S_0 \) for some \( \delta \in D_\gamma \). Assume that \( V_i \) are so numbered that \( V_1, \ldots, V_t \) is the set of all the \( V_i \) which contain \( p \). Then \( \Phi(p, V_i) = 0 \) for \( i < t \).

From \( d[\sigma(V_i)] < 2l(\gamma) < 1/m(\gamma) \) and \( d[\sigma(V_i), \sigma(V_j)] < 1/m(\gamma) \) we get

\[ d[\sigma(p), \sigma(p')] < 2l(\gamma) < 1/m(\gamma) \]

A fortiori, the centre of gravity \( \Phi(p) \) of the \( V_i \) satisfies

\[ d(\Phi(p), \sigma(p)) < 2l(\gamma) < 1/m(\gamma) \]

Therefore \( \Phi \in N_\gamma(f) \).

Next in order to show that \( \Phi \in M(E, m) \) we fix \( \gamma \in C \) and \( \delta \in D_\gamma \) and suppose that \( V_1, \ldots, V_t \) are all the members of \( V_t \) containing a given point \( p \in S_0 \). Consider the linear \((t-1)\)-space \( L_0(p) \) in \( E_{n+1} \) spanned by the vertices \( x(V_1), \ldots, x(V_t) \). Since there are only a finite number of linear subspaces \( L_0(p) \), there exists a positive number \( h(\delta) > l(\delta) \) such that any two of these linear subspaces \( L_0(p) \) and \( L_0(p') \) either meet or are at a distance \( > 2h(\delta) \) from each other.

Putting \( S_0 = \{ x \in A_{mg}, T_2 = T_2 \} \), we consider a star-decomposition

\[ E = \{ S_0 \in E_{mg}, \delta \in D_\gamma, \gamma \in C \} \]

Hence \( \Phi(p), \Phi(p') \in T_2 \times S_{1m+1}(x) \) for \( p, p' \in E \), then it follows that \( c(p), c(p') \in T_2, \) and

\[ d(\Phi(p), x) < 1/k(\delta), \quad d(\Phi(p'), x) < 1/k(\delta); \]

\[ (\gamma) \Phi(p, q) \] denotes the metric of \( R \).
3. We can select by (B) and (D) two sequences $G_1 > G_2 > G_3 > \ldots$, $D_1 > D_2 > D_3 > \ldots$ of star-decompositions and a sequence $f_1, f_2, \ldots$ of elements of $M(R)$ such that

$$
\gamma \in G_m \implies m(\gamma) > m,
$$

$$
N_{C^\infty}(f) \subseteq \text{M}(R, \lambda),
$$

$$
D_1 : (T_1) \delta \in D_1 \gamma \in G_1 \quad \left[ D_{x_H} = \{ \delta \in A_1, T_1 \subseteq T_2 \mid \gamma \in G_1 \} \right],
$$

$$
N_{C^\infty}(f) \subseteq N_{D_1}(f) \cap \text{M}(R, 2),
$$

$$
D_2 : (T_2) \delta \in D_2 \gamma \in G_2 \quad \left[ D_{x_H} = \{ \delta \in A_2, T_2 \subseteq T_3 \mid \gamma \in G_2 \} \right],
$$

$$
\vdots
$$

$$
N_{C^\infty}(f) \subseteq N_{D_1, \ldots, D_{k-1}}(f) \cap \text{M}(R, k),
$$

$$
D_h : (T_h) \delta \in D_h \gamma \in G_h \quad \left[ D_{x_H} = \{ \delta \in A_2, T_h \subseteq T_{h+1} \mid \gamma \in G_h \} \right],
$$

Then, since $f_h \in N_{D_h}(f) \ (h \geq k)$, we have

$$
d(\pi_{\gamma}(p), \pi_{\gamma}(p)) < 1/n \quad (h \geq k)
$$

for some $\gamma \in G_h$, and hence $(\pi_{\gamma}(p) | h = 1, 2, \ldots)$ uniformly converges to a continuous mapping $\phi(p)$ of $R$ into $I_{\infty+1}$.

Let us show that

$$
(c(p), \phi(p)) = \phi(p) \in \bigcap_{n=1}^{\infty} M(R, m).
$$

Since $f_h \in N_{D_1}(f) \ (h \geq k)$, if we take, for given $p \in G_h$ and $p \in S$, $\delta \in D_{x_H} : \gamma \in G_h$, then

$$
d(\pi_{\gamma}(p), \pi_{\gamma}(p)) < 1/2m(\gamma) \quad (h \geq k);
$$

hence

$$
d(\phi(p), \pi_{\gamma}(p)) < 1/2m(\gamma) \quad (p \in S).
$$

Therefore

$$
\phi(p) \in N_{C^\infty}(f) \subseteq M(R, k), \quad i.e., \quad \phi(p) \in \bigcap_{n=1}^{\infty} M(R, m).
$$

Thus we get a homeomorphic mapping $\phi$ of $R$ into $N(\Omega) \times I_{\infty+1}$.

\section*{Corollary 4}

Let $R$ be a metric space having the local Lindelöf property \((\text{L})\) with $\text{dim } R < n$. Then $R$ is homeomorphic to a subset of $N(\Omega) \times I_{\infty+1}$.

\textbf{Proof.} Since every metric space with the local Lindelöf property has a $\sigma$-star-countable basis, this proposition is a direct consequence of Theorem 9.

\section*{Theorem 10}

In order that a metric space $R$ with a $\sigma$-star-finite (countable) basis has dimensions $\leq n$ and have an open basis whose cardinal number is not greater than $n$ it is necessary and sufficient that $R$ be homeomorphic with a subset of $N(\Omega) \times M_{\infty+1}$, where $\Omega$ is a set with $|\Omega| = m$ and $M_{\infty+1}$ is the set of points in $I_{\infty+1}$ at most $n$ of whose coordinates are rational.

\textbf{Proof.} Since it is well known that $\text{dim } M_{\infty+1} = 10$, $\text{dim } N(\Omega) \times M_{\infty+1} = 10$ from the generalized product theorem (see [4] and [5]). Hence the sufficiency is obvious.

The proof of the necessity is analogous to that of Theorem 9. Let $I_1, I_2, \ldots$ be a sequence of $n$-dimensional linear subspaces in $I_{\infty+1}$; then we shall prove generally that $R$ is homeomorphic with a subset of $N(\Omega) \times I_{\infty+1} \setminus \{0\}$.

Let $I_1, I_2, \ldots$ are all the linear spaces in $I_{\infty+1}$ of the form $x_0 = r_1, \ldots, x_{n+1} = r_{n+1}$, the $r_i$'s being rational, then we get the necessity part of this proposition. To show this we generally use the same notation as the above, but we replace $M(R, m)$ in the above proof by

$$
\text{N}(R, m) = \{\phi(p) \in M(R), \quad \varphi^{-1}(x) < \mu \quad (p \in C),
$$

$$
\bigcap_{n=1}^{\infty} \text{M}(R, m) = \Theta \quad (\phi \in C) \text{ for some star-decomposition } C \}.
$$

The part 1 of the above proof (of Theorem 9) is suitable for the present proof too.

We now prove $N_{C}(f) \cap N(R, m) \neq \emptyset$ for every $f \in M(R)$, every star-decomposition $C$ and every positive integer $m$. We define $D_1 (\delta \in D_1, \gamma \in C)$ and $D_2 (\delta \in D_2, \gamma \in C)$ in the same way as in the proof of Theorem 9 \((\text{L})\) and consider fixed $\gamma \in C$ and $\delta \in D_1$. Assume that $V_1, \ldots, V_n$ are all the members of $S$.

Then we select vertices $x(V)$ \((i = 1, \ldots, n) \in I_{\infty+1}\) and $p_1, \ldots, p_n$ in $I_{\infty+1}$ for which it is true that $d(\pi(x(V)), x(V)) < 1/n \mu(\gamma)$, the $x(V)$ and $p_i$ are in a general position in $I_{\infty+1}$. Defining $\phi(p) \in M(R)$ by \((C)\) in the above proof, we see $\phi \in N_{C}(f)$ in the same way.

\section*{Footnotes}

\((\text{L})\) We mean by Lindelöf property the property that every open covering has a countable subcovering. It every point of $R$ has a subneighborhood has the Lindelöf property, then $R$ is said to have the local Lindelöf property.

\((\text{L})\) From now on we omit "in the proof of Theorem 9" for brevity.
To show that \( \varphi \in N(\mathbb{R}, m) \) we consider fixed \( \gamma \in C \) and \( \delta \in \mathcal{D}_r \) and suppose that \( V_1, \ldots, V_\gamma \) are all the members of \( \mathcal{B} \) containing a given point \( p \) of \( S_\gamma \). We denote by \( L_\gamma(p) \) the linear \((\gamma-1)\)-space in \( L_{m+1} \) spanned by the vertices \( s(V_1), \ldots, s(V_\gamma) \). Then \( \varphi_\gamma(p) = L_\gamma(p) \subseteq L_{m+1} - L_m \) and there exists \( h(\beta) > 0 \) such that \( L_\gamma(p) \cap L_\gamma(p') = 0 \) implies \( d(L_\gamma(p), L_\gamma(p')) > 2h(\beta) \). Defining a star-decomposition \( E \) in the same way, we have \( \varphi^{-1}(T_\beta) \subseteq U_m \) for all \( \beta \in E \) and proving \( \varphi \in N(\mathbb{R}, m) \).

Next, in order to show that for every \( \varphi \in N(\mathbb{R}, m) \) there exists a star-decomposition \( E \) satisfying \( N_C(\mathbb{R}, m) \subseteq N(\mathbb{R}, m) \), we shall prove \( N_C(\mathbb{R}, m) \subseteq N(\mathbb{R}, m) \) for some star-decomposition \( E \). Since \( \varphi \in N(\mathbb{R}, m) \)

\[
\varphi^{-1}(T_\beta) \subseteq U_m \quad (\beta \in E),
\]

for positive integers \( k(\beta) \) \( (\beta \in E) \) and \( \max(2m, k(\beta)) = \lambda(\beta) \), \( (\beta \in E) \), we have a star-decomposition \( E : \{ (\beta) \subseteq E \} \) \( (\beta \in E) \). For an arbitrary \( \varphi \in N_C(\mathbb{R}, m) \text{ } \varphi^{-1}(T_\beta) \subseteq U_m \) \( (\beta \in E) \) is proved in the same way. Moreover \( p \in S_\beta \) implies \( d(L_\beta(p), p) > h(\beta) \) \( (\beta \in E) \) for some \( \eta(\beta) > 0 \). Therefore \( d(L_\beta(p), p) > h(\beta) \) \( (\beta \in E) \)

\[
\frac{1}{h(\beta)} \varphi(L_\beta(p), p) > \eta(\beta) > 0 \quad (\beta \in E),
\]

which means \( \varphi(L_\beta(p), p) \subseteq \mathcal{S} \subseteq \mathcal{B} \). Since we can prove \( \mathcal{S} \subseteq \mathcal{B} \) \( (\beta \in E) \), we have \( \varphi(p) = \bigcap_{m=1}^\infty \mathcal{N}(\mathbb{R}, m) \) which topologically maps \( R \) into \( \mathcal{N}(\mathbb{Q}) \times M_{m+1} \).

**Definition.** We say that the \( p \)-dimensional density of a subset \( S \) of a metric space is zero if and only if for every \( \epsilon > 0 \) there exists a decomposition \( S = \bigcup_{i=1}^\infty A_i \) \( (\gamma \in C) \), such that \( \delta(A_i) < \epsilon \) \( (\gamma \in C) \) and such that \( \bigcap_{i=1}^\infty A_i = \emptyset \). We denote by \( \delta(A) \) the diameter of \( A \).

**Theorem 11.** Every metric space \( R \) of \((n+1)\)-density zero has dimension \( < n \).

**Proof.** Let us show that \( \text{ind} \text{dim} R \leq n \). We consider an arbitrary pair \( (F', G) \) of closed sets with \( g(F', G) > 0 \). If we can show the existence of an open set \( U \) with \( \text{dim} \text{dim} \text{dim} \text{dim} U < n \), then \( \text{ind} \text{dim} R < n \) is proved \( (\alpha) \). Select a positive integer \( m \) with \( 1/m < g(F', G) \), and let \( R = \bigcup_{x \in A} A \text{ } \{ (\gamma \in C) \}, \) \( i = 1, 2, \ldots \) be a decomposition of \( R \) such that

\[
\delta(A_i) < 1/m,
\]

and such that \( \mathcal{S}_x = \bigcup_{x \in A} A \text{ } \{ (\gamma \in C) \}, \) \( (\gamma \in C) \), is open for every \( \gamma \in C \). We put

\[
u_{i+1} = \sup_{x \in A} \{ (\gamma \in C) \},
\]

\[
u_{i+1} = \inf_{x \in A} \{ (\gamma \in C) \},
\]

Then it easily seen that \( \nu_{i+1} = \delta(A_{i+1}) \). We define a non-negative valued function \( d_i(r) \) for every \( r \in \mathcal{D}_r \) by

\[
d_i(r) = \sum_{(\gamma \in C) \subseteq A} \delta(A_{i+1})^x \text{ } (0 < r < \nu_{i+1}, \text{ or } y < r)
\]

\[
d_i(r) = \sum_{(\gamma \in C) \subseteq A} \delta(A_{i+1})^x \text{ } (y < r < \nu_{i+1})
\]

It follows from

\[
\int_0^{\nu_{i+1}} d_i(r) dr < \delta(A_{i+1})^{x+1}
\]

that

\[
\int_0^{\nu_{i+1}} d_i(r) dr = \sum_{(\gamma \in C) \subseteq A} \delta(A_{i+1})^x < \delta(A_{i+1})^{x+1} < 1/m
\]

since considering \( d_i(r) > 0 \) we may interchange integration and summation by Lebesgue's theorem. This implies \( \delta_i(r) < 1/m \) for some \( r \) with \( 0 < r < \nu_{i+1} \). We denote by \( \delta_i(r) \) the set of all the points satisfying \( g(F', r) < r \) and by \( \mathcal{S}_i(r) \) the boundary of \( \mathcal{S}(F', r) \). Then

\[
[\delta(A_{i+1}) \cap \mathcal{S}(F', r)] \cap d_i(r) < \delta_i(r)
\]

combined with \( d_i(r) \) implies

\[
\int_0^{\nu_{i+1}} d_i(r) dr < \delta(A_{i+1})^{x+1} < 1/m
\]

We notice that \( \bigcap_{x \in A} \mathcal{S}(F', r) \cap \mathcal{S}(F', r) \cap \mathcal{S}(F', r) \) is open for every \( r \in \mathcal{D}_r \) and \( \mathcal{S}_x \cap \mathcal{S}_y = \emptyset \). (\( \gamma \neq \gamma' \) \( (\gamma \in C) \)).

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(\( \gamma \neq \gamma' \) \( (\gamma \in C) \)).
is evidently an open set of $R$ satisfying $F \subset U \subset G$. Since $\bar{U} - U = \bigcup (A_i \setminus \mathcal{N}(\varphi)) \cap \mathcal{N}(\varphi)) \{ \varphi \in C, i = 1, 2, \ldots \}$, the $n$-dimensional density of $\bar{U} - U$ is zero. The above argument is also valid for $n = 0$; hence $\bar{U} - U = O$ for a space $R$ of 1-dimensional density zero, proving indim $R < 0$.

Thus we can inductively establish this theorem.

**Theorem 12.** If a metric space $R$ has dimension $< n$ and has a $\Delta$-star-finite (countable) basis, then it is homeomorphic to a subset $S$ of $N(Q) \times I_{n+1}$ such that $(n + 1)$-density of $S$ is zero.

**Proof.** We define the distance $d'(x, y)$ between two points $x = (x_1, \ldots, x_n, y = (y_1, y_2)$ of $N(Q) \times I_{n+1}$ for $x_1, y_1 \in N(Q), \, x_2, y_2 \in I_{n+1}$ by $d'(x, y) = d(x_1, y_1) + d(x_2, y_2)$. Replacing $M(R, m)$ in the proof of Theorem 9 by

$O(R, m) = \{ \varphi \in M(R) \mid \varphi^{-1}(\mathcal{U}_m) \not\subseteq \mathcal{U}_m \}$

for some star-decomposition $C$ and there exist decompositions $\varphi(R) \cap T_i = \bigcup_{\varphi \in C} A_{i, \varphi}$ such that

$$\delta(A_{i, \varphi}) < 1/m, \quad \sum_{i=1}^{\infty} [\delta(A_{i, \varphi})]^{n+1} < 1/m \quad (\varphi \in C, i = 1, 2, \ldots),$$

we can analogously prove $O(R, m) = \emptyset$. Part 1 of the proof of Theorem 9 is suitable for the present proof.

To prove $N_{\varphi}(f) \cap O(R, m) = \emptyset$ for every $f \in M(R)$, every star-decomposition $C$ and every positive integer $m$, we define $\varphi \in M(R)$ by (C) in the proof of Theorem 9. Since $\pi[\varphi(R) \cap T_i] = \varphi(S_{i, \varphi})$ for a fixed $\delta$ is contained in an $n$-dimensional polytope in $I_{n+1}$, $\pi[\varphi(R) \cap T_i]$ is also contained in an $n$-dimensional polytope because

$$\pi[\varphi(R) \cap T_i] \subset \pi[\varphi(S_{i, \varphi})] \subset \pi[\mathcal{N}(\varphi)] .$$

It is well-known that the $(n + 1)$-dimensional measure of an $n$-dimensional polytope is zero (see [5]), and hence, by the compactness of $\pi[\varphi(R) \cap T_i]$, there exist open sets $K_{i, \varphi}$ ($i = 1, 2, \ldots, p(\varphi)$) of $I_{n+1}$ such that

$$\pi[\varphi(R) \cap T_i] \subset \bigcup_{i=1}^{p(\varphi)} K_{i, \varphi}, \quad \delta(K_{i, \varphi}) < 1/m, \quad \sum_{i=1}^{p(\varphi)} [\delta(K_{i, \varphi})]^{n+1} < 1/m .$$

We can select a positive integer $h(\varphi)$ satisfying

$$\sum_{i=1}^{p(\varphi)} [\delta(K_{i, \varphi}) + 1/h(\varphi)]^{n+1} < 1/m, \quad \delta(K_{i, \varphi}) + 1/h(\varphi) < 1/m \quad (i = 1, \ldots, p(\varphi)) .$$

$L(p) \cap L(p') = \emptyset$ implies $d(L(p), L(p')) > 2h(\varphi)$.

Then we consider a star-decomposition

$$E : (S_i \mid i \in E_i, \, \delta \in E, \, \gamma \in O)$$

for $E_i = (e \mid e \in A_{i, \varphi}, \, T_i \subset T)$ .

$\varphi^{-1}(\mathcal{U}_m) \not\subseteq \mathcal{U}_m$ is proved in the same way. Moreover it follows from

$$\delta(T_i) < 1/h(\delta)$$

that

$$\delta(T_i, K_{i, \varphi}) < \delta(K_{i, \varphi}) + 1/h(\delta) < 1/m, \quad \sum_{i=1}^{p(\varphi)} [\delta(T_i, K_{i, \varphi})]^{n+1} < \sum_{i=1}^{p(\varphi)} [\delta(K_{i, \varphi}) + 1/h(\delta)]^{n+1} < 1/m .$$

Since $\varphi(R) \cap T_i \subset \bigcup_{i=1}^{p(\varphi)} (T_i \times K_{i, \varphi})$ is obvious, we have $\varphi \in O(R, m)$.

Next let us prove that $\varphi \in N_{\varphi}(f) \cap O(R, m)$ implies $N_{\varphi}(\gamma) \subset N_{\varphi}(f)$ for a suitable star-decomposition $C$. We have, by $\varphi \in O(R, m)$, a star-decomposition $B$ such that $\varphi^{-1}(T_i) \subset \mathcal{U}_m$ and such that

$$\varphi(R) \cap T_i = \bigcup_{i=1}^{p(\varphi)} A_{i, \varphi}, \quad \delta(A_{i, \varphi}) < 1/m, \quad \sum_{i=1}^{p(\varphi)} [\delta(A_{i, \varphi})]^{n+1} < 1/m$$

for some $A_{i, \varphi}$. This implies

$$\pi[\varphi(R) \cap T_i] \subset \bigcup_{i=1}^{p(\varphi)} A_{i, \varphi}, \quad \delta(A_{i, \varphi}) < 1/m, \quad \sum_{i=1}^{p(\varphi)} [\delta(A_{i, \varphi})]^{n+1} < 1/m$$

for some $A_{i, \varphi}$.

Hence, by the compactness of $\pi[\varphi(R) \cap T_i]$, there exist open sets $H_{i, \varphi}$ ($i = 1, \ldots, q(\varphi)$) of $I_{n+1}$ satisfying

$$\pi[\varphi(R) \cap T_i] \subset \bigcup_{i=1}^{q(\varphi)} H_{i, \varphi}, \quad \delta(H_{i, \varphi}) < 1/m, \quad \sum_{i=1}^{q(\varphi)} [\delta(H_{i, \varphi})]^{n+1} < 1/m$$

for some $H_{i, \varphi}$.

We choose a positive integer $h(\varphi)$ for every $\beta \in B$ satisfying

$$\sum_{i=1}^{q(\varphi)} [\delta(H_{i, \varphi}) + 1/h(\varphi)]^{n+1} < 1/m, \quad \delta(H_{i, \varphi}) + 1/h(\varphi) < 1/m$$

for $i = 1, \ldots, q(\varphi)$.
Letting

$$h(\beta) = \max \{ \beta m(\beta), h(\beta) \}, \quad E_B = \{ \beta \in E \mid T_\beta \subseteq T_0 \}$$

we have a star-decomposition $E = \{ S_{\beta} \in E | \beta \in B \}$.

To prove $N_0(p) \subseteq O(R, m)$, we consider a given $p \in N_0(p)$. Then $\psi^{-1}(T_0) \subseteq H_0$ is proved in the same way. On the other hand, for any $x \in \psi(E) \cap T$, there exists $y \in \psi(E) \cap T_0 \subseteq \psi(S_{\beta})$ with $d(\pi(x), \pi(y)) < 1/\beta$. Let $\psi(\beta) = y, \beta \in S_{\beta}$; then it follows from

$$d(\pi(x), \pi(y)) < 1/\beta$$

for any $x \in \psi(E) \cap T_0$. That is to say for any $y \in \psi(E) \cap T_0$, we can select $x \in \psi(E) \cap T_0$ satisfying $d(\pi(x), \pi(y)) < 1/\beta$. Hence letting

$$B_{\beta} = \{ x \in \psi(E) \cap T : d(\pi(x), \pi(y)) < 1/\beta \}$$

we have $\psi(E) \cap T = \bigcup_{\beta \in E} B_{\beta}$. For given $z_1, z_2 \in B_{\beta}$, we take $x_1, x_2$ with

$$d(\pi(x_1), \pi(x_2)) < 1/\beta$$

and $d(\pi(z_1), \pi(z_2)) < 1/\beta$. Thus we have

$$d(B_{\beta}) < \delta(\pi(x), \pi(y)) < 1/\beta$$

proving $\psi \subseteq O(R, m)$.

References