Note on dimension theory for metric spaces *

by

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Recently, a dimension theory for general metric spaces has been established by M. Katětov and by K. Morita (see [4] and [5]) independently. They have extended the sum, decomposition and product theorems to non-separable metric spaces and have shown the equivalence of the Lebesgue dimension and the inductive dimension (1). On the other hand, the following theorem of P. Alexandroff and P. Urysohn is well known:

In order that a T_1-topological space R be metrizable it is necessary and sufficient that there exists a sequence $\mathfrak{V}_1 > \mathfrak{V}_2^* > \mathfrak{V}_2 > \mathfrak{V}_3^* > \ldots$ of open coverings such that $\{S(p, \mathfrak{V}_m) | m = 1, 2, \ldots\}$ (2) is a nbd (neighbourhood) basis for each point p of R.

The purpose of the present note is to refine this theorem to a theorem concerning n-dimensionality of metric spaces and to develop the dimension theory for general metric spaces. In § 1 we shall prove that Alexandroff-Urysohn's theorem turns into a theorem asserting a necessary and sufficient condition for n-dimensionality if we add the condition order $\mathfrak{V}_m \leqslant n+1$ $(m = 1, 2, \ldots)$ to the original condition. Furthermore, concerning that theorem it will be shown that we may replace order $\mathfrak{V}_m \leqslant n+1$ by Alexandroff-Kolmogoroff's length of $\mathfrak{V}_m \leqslant n+1$ (see [1]). In § 2 we shall apply the result of § 1 to the study of the connections between dimension and metric function. § 3 contains applications of the result of § 1 to the embedding of n-dimensional metric spaces into products of 1-dimensional spaces. The final section is devoted to

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(1) $\operatorname{ind}\dim \emptyset = -1$ for a vacuous set $\emptyset$, and $\operatorname{ind}\dim R \leqslant n$ if and only if for any pair of a closed set F and an open set G with $F \subseteq G$ there exists an open set U such that $F \subseteq U \subseteq G$, $\dim B(U) \leqslant n-1$, where we denote by $B(U)$ the boundary of U.

(2) $S(p, \mathfrak{V}) = \bigcup \{V | p \in V \in \mathfrak{V}\}$ for a covering $\mathfrak{V}$ of R, $S(A, \mathfrak{V}) = \bigcup \{V | V \cap A \neq \emptyset, V \in \mathfrak{V}\}$ for a subset A of R, $\mathfrak{V}^* = \{S(V, \mathfrak{V}) | V \in \mathfrak{V}\}$. $\mathfrak{V}$ is called a star-refinement of $\mathfrak{U}$ if $\mathfrak{V}^* < \mathfrak{U}$. The notation of this paper is chiefly due to [8]. See also [2] with respect to the notions.

the embedding of n-dimensional metric spaces into a product of Euclidean $(2n+1)$-space with a zero-dimensional space and to its modifications.

Throughout this paper all spaces are metric or metrizable, and all coverings are open, unless the contrary is explicitly stated.

§ 1. The main theorem.

DEFINITION. For two collections $\mathfrak{U}, \mathfrak{U}'$ of open sets we denote by $\mathfrak{U} < \mathfrak{U}'$ the fact that $U \subseteq U'$ for every $U \in \mathfrak{U}$ and for some $U' \in \mathfrak{U}'$.

DEFINITION. We mean by a *disjointed collection* a collection $\mathfrak{U}$ of open sets such that $U, U' \in \mathfrak{U}$ and $U \neq U'$ imply $U \cap U' = \emptyset$.

THEOREM 1. *In order that* $\dim R \leqslant n$ *for a metric space* R *it is necessary and sufficient that there exist* $n+1$ *sequences* $\mathfrak{U}_1^i > \mathfrak{U}_2^i > \ldots$ $(i = 1, 2, \ldots, n+1)$ *of disjointed collections such that* $\{\mathfrak{U}_m^i | i = 1, \ldots, n+1;$ $m = 1, 2, \ldots\}$ *is an open basis of* R.

Proof. If $\dim R = 0$ (³), then by [5] there exists a sequence $\mathfrak{V}_m$ $(m = 1, 2, \ldots)$ of locally finite coverings (⁴) consisting of open and closed sets such that $S(p, \mathfrak{V}_m^*)$ $(m = 1, 2, \ldots)$ is a nbd basis of each point p of R. Let $\mathfrak{V}_m = \{V_\alpha | \alpha < \tau\}$, then we define a sequence of coverings by

$$\mathfrak{V}_m' = \{V_\alpha - \bigcup_{\beta<\alpha} V_\beta | \alpha < \tau\}$$

and

$$\mathfrak{U}_1 = \mathfrak{V}_1', \quad \mathfrak{U}_2 = \mathfrak{U}_1 \wedge \mathfrak{V}_2', \quad \mathfrak{U}_3 = \mathfrak{U}_2 \wedge \mathfrak{V}_3', \ldots$$

It is clear that $\mathfrak{U}_1 > \mathfrak{U}_2 > \ldots$ is a sequence of disjointed collections, and $\{\mathfrak{U}_m | m = 1, 2, \ldots\}$ is an open basis of R.

Conversely, if there exists a sequence $\mathfrak{U}_1 > \mathfrak{U}_2 > \ldots$ of disjointed collections such that $\{\mathfrak{U}_m | m = 1, 2, \ldots\}$ is an open basis of R, then for an arbitrary point p of R, $p \in U \in \mathfrak{U}_m$ implies

$$U \cap U' = \emptyset \quad \text{for} \quad U' \text{ with } U \neq U' \in \mathfrak{U}_m,$$

and $p \notin U$ for every $U \in \mathfrak{U}_m$ implies

$$\bigcap \{S(p, \mathfrak{U}_j) | j = 1, \ldots, m-1;\ S(p, \mathfrak{U}_j) \neq \emptyset\} = p$$

by the fact that $\{\mathfrak{U}_m | m = 1, 2, \ldots\}$ is an open basis of R, and $\mathfrak{U}_m > \mathfrak{U}_{m+1} > \ldots$ Hence each $\mathfrak{U}_m$ is locally finite and consists of open and closed sets, and hence $\dim R = 0$ follows from [5].

Now we proceed to n-dimensional cases. Let $\dim R \leqslant n$; then we can decompose R into $n+1$ 0-dimensional spaces R_i $(i = 1, \ldots, n+1)$ by the general decomposition theorem due to Katětov and to Morita, *i. e.*,

$$R = \bigcup_{i=1}^{n+1} R_i, \quad \dim R_i = 0.$$

Then there exists a sequence $\mathfrak{V}_1^i > \mathfrak{V}_2^i > \ldots$ of disjointed collections of R_i such that $\{\mathfrak{V}_m^i | m = 1, 2, \ldots\}$ is an open basis of R_i. As is obvious from the above discussion for 0-dimensional cases, we may assume that every $\mathfrak{V}_i^m$ covers R_i. We put $\mathfrak{V}_m^i = \{V_{\alpha m} | \alpha \in A\}$ and take the maximal positive number ε for each $x \in V_{\alpha m}$ such that $S_\varepsilon(x) \cap R_i \subseteq V_{\alpha m}$ (⁵). Furthermore we define

$$\varepsilon(m, x) = \min(1/m, \varepsilon/2), \quad U_{\alpha m} = \bigcup \{S_{\varepsilon(m,x)}(x) | x \in V_{\alpha m}\},$$
$$\mathfrak{U}_m^i = \{U_{\alpha m} | \alpha \in A\}.$$

Then it easily follows from $\mathfrak{V}_1^i > \mathfrak{V}_2^i > \ldots$ and from the disjointedness of $\mathfrak{V}_m^i$ that $\mathfrak{U}_1^i > \mathfrak{U}_2^i > \ldots$ and each $\mathfrak{U}_m^i$ is a disjointed collection. Next we take an arbitrary point x of R and a positive number δ. We can select positive integers m, l such that

$$2/m < \delta, \quad l \geqslant m, \quad x \in V_{\alpha l} \subseteq S_{1/m}(x) \quad \text{for some} \quad V_{\alpha l} \in \mathfrak{V}_l^i.$$

Since for these integers

$$x \in U_{\alpha l} \subseteq S_{2/m}(x) \subseteq S_\delta(x)$$

is obvious, it follows that $\{\mathfrak{U}_m^i | i = 1, \ldots, n+1;\ m = 1, 2, \ldots\}$ is an open basis of R.

Conversely, if R admits $n+1$ sequences $\mathfrak{U}_1^i > \mathfrak{U}_2^i > \ldots$ $(i = 1, \ldots, n+1)$ such that $\{\mathfrak{U}_m^i | i = 1, \ldots, n+1;\ m = 1, 2, \ldots\}$ is an open basis of R, then we define $n+1$ subspaces R_i of R by

$$R_i = \{x | S(x, \mathfrak{U}_m^i)\ (m = 1, 2, \ldots) \text{ is a nbd basis of } x\}.$$

Considering $\mathfrak{U}_m^i$ a disjointed collection of R_i, $\{\mathfrak{U}_m^i | m = 1, 2, \ldots\}$ is an open basis of R_i; hence $\dim R_i = 0$ follows from the 0-dimensional case. Thus we deduce $\dim R \leqslant n$ from $R = \bigcup_{i=1}^{n+1} R_i$.

THEOREM 2. *In order that a* T_1*-topological space* R *be a metrizable space with* $\dim R \leqslant n$ *it is necessary and sufficient that there exists a sequence* $\mathfrak{V}_1 > \mathfrak{V}_2^* > \mathfrak{V}_2 > \mathfrak{V}_3^* > \ldots$ *of open coverings such that* $\{S(p, \mathfrak{V}_m) | m = 1, 2, \ldots\}$ *is a nbd basis for each point* p *of* R *and such that each set of* $\mathfrak{V}_{m+1}$ *intersects at most* $n+1$ *sets of* $\mathfrak{V}_m$.

(³) From now on we assume $R \neq \emptyset$.

(⁴) We call $\mathfrak{V}$ a locally finite covering if every point of R has some nbd intersecting only finitely many elements of $\mathfrak{V}$.

(⁵) $S_\varepsilon(x) = \{y | \varrho(x, y) = \text{the distance between } x \text{ and } y < \varepsilon\}$.

Proof. Necessity. If R is a metric space with $\dim R \leqslant n$, then by the general decomposition theorem $R = \bigcup_{i=1}^{n+1} R_i$ for some 0-dimensional spaces R_i $(i = 1, \ldots, n+1)$.

(A) Let $\mathfrak{U} = \{U_a | a \in A\}$ be an arbitrary locally finite covering of R; then there exists a disjointed covering $\mathfrak{V}_i = \{V_a | a \in A\}$ of R_i such that $V_a \subseteq U_a$. By putting

$$V'_a = \bigcup \{S_{\varepsilon(x)/2}(x) | x \in V_a\} \quad \text{for} \quad \varepsilon(x) > 0$$

such that

$$R_i \cap S_{\varepsilon(x)}(x) \subseteq V_a, \quad S_{\varepsilon(x)}(x) \subseteq U_a,$$

we get a disjointed collection $\mathfrak{V}'_i = \{V'_a | a \in A\}$ of R such that $\mathfrak{V}'_i < \mathfrak{U}$. Hence $\mathfrak{V}' = \bigcup_{i=1}^{n+1} \mathfrak{V}'_i$ is a locally finite covering of R of order $\leqslant n+1$ and is a refinement of $\mathfrak{U}$ [6]. Hence there exists an open covering $\mathfrak{V}'' = \{V''_\beta | \beta \in B\}$ of R such that $\overline{V}''_\beta \subseteq V'_\beta$ for $\mathfrak{V}' = \{V'_\beta | \beta \in B\}$. It is easily seen from order $\mathfrak{V}' \leqslant n+1$ and from the property of $\mathfrak{V}''$ that every point p of R has some nbd intersecting at most $n+1$ of the sets belonging to $\mathfrak{V}''$; we call such a covering to be of *local order* $\leqslant n+1$.

Now let $\mathfrak{U}_1 > \mathfrak{U}_2 > \ldots$ be a sequence of coverings of R such that $\{S(p, \mathfrak{U}_m) | m = 1, 2, \ldots\}$ is a nbd basis for each point p, then from the paracompactness [7] of R we may assume that all $\mathfrak{U}_m$ are locally finite. Hence from (A) we get a refinement $\mathfrak{V}_1$ of $\mathfrak{U}_1$ such that the local order of $\mathfrak{U}_1 \leqslant n+1$. Furthermore, we can select locally finite coverings $\mathfrak{P}, \mathfrak{Q}$ such that $\mathfrak{P}^* < \mathfrak{V}_1$ and such that every set of $\mathfrak{Q}$ intersects at most $n+1$ sets of $\mathfrak{V}_1$. Since $\mathfrak{U}_2 \wedge \mathfrak{P} \wedge \mathfrak{Q}$ is locally finite, from (A) we obtain a refinement $\mathfrak{V}_2$ of $\mathfrak{U}_2 \wedge \mathfrak{P} \wedge \mathfrak{Q}$ with the local order of $\mathfrak{V}_2 \leqslant n+1$. Then it follows clearly that $\mathfrak{V}_2 < \mathfrak{U}_2$, $\mathfrak{V}_2^* < \mathfrak{V}_1$ and each set of $\mathfrak{V}_2$ intersects at most $n+1$ sets of $\mathfrak{V}_1$. By repeating such processes we obtain a sequence $\mathfrak{V}_1 > \mathfrak{V}_2^* > \mathfrak{V}_2 > \mathfrak{V}_3^* > \ldots$ of open coverings such that $\mathfrak{V}_m < \mathfrak{U}_m$ and such that each set of $\mathfrak{V}_{m+1}$ intersects at most $n+1$ of the sets belonging to $\mathfrak{V}_m$. Since $\{S(p, \mathfrak{U}_m) | m = 1, 2, \ldots\}$ is a nbd basis of p, $\{S(p, \mathfrak{V}_m) | m = 1, 2, \ldots\}$ is also a nbd basis of p, and hence the necessity is proved.

Sufficiency. The metrizability of such a space is obvious from Urysohn-Alexandroff's theorem. We divide the proof of n-dimensionality into three parts.

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(6) We call $\mathfrak{V}'$ a refinement of $\mathfrak{U}$ if $\mathfrak{V}' < \mathfrak{U}$, *i. e.*, for every $V \in \mathfrak{V}'$ there exists $U \in \mathfrak{U}$ with $U \supseteq V$.

(7) Every fully normal space is paracompact by [6]. R is called paracompact if every covering of R has a locally finite refinement, and it is called fully normal if every covering has a star-refinement. It is well known that every metric space is fully normal.

1. If $\mathfrak{V}_1 > \mathfrak{V}_2^* > \ldots$ is a sequence satisfying the condition of this proposition, then it is easily seen that for each point p of R $S^{n+2}(p, \mathfrak{V}_{m+1+n+2})$ [8] is contained in some set of $\mathfrak{V}_{m+1}$. Therefore each $S^{n+2}(p, \mathfrak{V}_{m+1+n+2})$ intersects at most $n+1$ sets of $\mathfrak{V}_m$. Putting

$$\mathfrak{U}_m = \mathfrak{V}_{1+(m-1)(n+3)} \quad (m = 1, 2, \ldots),$$

we get a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \ldots$ of open coverings such that $\{S(p, \mathfrak{U}_m) | m = 1, 2, \ldots\}$ is a nbd basis of $p \in R$ and such that each $S^{n+2}(p, \mathfrak{U}_{m+1})$ intersects at most $n+1$ sets of $\mathfrak{U}_m$.

Let $\mathfrak{U}_m = \{U_a | a < \tau\}$; then we can prove first that there exist open sets U_a^i such that

$$\bigcup_{i=1}^{n+1} U_a^i \subseteq U_a, \quad U_a^i \cap U_\beta^i = 0 \quad \text{for} \quad a \neq \beta$$

and such that

$$U_a \supseteq M \in \mathfrak{U}_{m+1} \quad \text{implies} \quad M \subseteq U_a^i \quad \text{for some} \quad U_a^i.$$

To prove this we define U_a^i $(a < \tau)$ by induction such that

1) $\bigcup_{i=1}^{n+1} U_a \subseteq U_a$,

2) $U_a^i \cap U_\beta^i = \emptyset$ for $\beta < a$,

3) $U_a \supseteq M \in \mathfrak{U}_{m+1}$ implies $M \subseteq U_a^i$ for some U_a^i,

4) $U_a^i \cap W_a^{n-i+2} = \emptyset$ $(i = 1, \ldots, n+1)$,

where we put $S_a^k = \{p | S^k(p, \mathfrak{U}_{m+1})$ intersects some k sets of U_γ $(\gamma > a)\}$ $(k = 1, \ldots, n+1)$ and $W_a^k = S_a^k \cup S_a^{k+1} \cup \ldots \cup S_a^{n+1}$.

For $a = 0$ we define

$$U_0^1 = U_0, \quad U_0^i = \emptyset \quad (i = 2, \ldots, n+1).$$

Since $S(p, \mathfrak{U}_{m+1})$ intersects at most $n+1$ of U_a $(a < \tau)$, $U_0^1 \cap W_0^{n+1} = \emptyset$ is obvious from the definition of $W_0^{n+1} = S_0^{n+1}$, and also the other three conditions are obviously satisfied.

Let us assume that U_β^i are defined for $\beta < a$; then putting

$$V_a^i = \bigcup_{\beta < a} U_\beta^i \quad \text{and} \quad U_a^i = U_a - \overline{V}_a^i \cup \overline{W}_a^{n-i+2} \quad (i = 1, \ldots, n+1)$$

we get U_a^i satisfying 1)-4). Since the validity of 1), 2), 4) for U_a^i is clear from the above definition, we prove 3) only. If $M \in \mathfrak{U}_{m+1}$ is an arbitrary set contained in U_a, then

$$M \cap W_a^{n+1} \subseteq U_a \cap W_a^{n+1} = U_a \cap S_a^{n+1}.$$

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(8) $S^1(p, \mathfrak{V}) = S(p, \mathfrak{V})$, $S^{n+1}(p, \mathfrak{V}) = S(S^n(p, \mathfrak{V}), \mathfrak{V})$.

On the other hand $U_a \cap S_a^{n+1} = \emptyset$ is easily seen from the fact that every $S^{n+2}(p, \mathfrak{U}_{m+1})$ intersects at most $n+1$ sets of U_γ $(\gamma \geq a)$. For let $x \in U_a \cap S^{n+1}$; then $S^{n+1}(x, \mathfrak{U}_{m+1})$ intersects U_a and some $n+1$ sets of U_γ $(\gamma > a)$, which is impossible. In consequence we get $M \cap W_a^{n+1} = \emptyset$. Hence it follows that either $M \cap W_a^i = \emptyset$ $(i = 1, \ldots, n+1)$ or $M \cap W_a^{n-i+2} \neq \emptyset$, $M \cap W_a^{n-i+3} = \emptyset$ for some i such that $2 \leq n-i+3 \leq n+1$.

If the former is the case, then $M \cap W_a^1 = \emptyset$. Since $U_a \subset S_\beta^1 \subseteq W_\beta^1$ is obvious for every $\beta < a$, and since $U_\beta^{n+1} \cap W_\beta^1 = \emptyset$ $(\beta < a)$ from the assumption of induction, it follows that $U_a \cap U_\beta^{n+1} = \emptyset$ for every $\beta < a$. Therefore $U_a \cap V_a^{n+1} = \emptyset$, which implies $M \cap V_a^{n+1} = \emptyset$. Combining this with $M \cap W_a^1 = \emptyset$ we conclude that $M \subseteq U_a^{n+1}$.

If the latter is the case, *i. e.*,

$$y \in M \cap W_a^{n-i+2} \neq \emptyset, \quad M \cap W_a^{n-i+3} = \emptyset, \quad 2 \leq n-i+3 \leq n+1,$$

then $y \in S_a^{n-i+2+k}$ for some $k \geq 0$, *i. e.* $S^{n-i+2+k}(y, \mathfrak{U}_{m+1})$ intersects some $n-i+2+k$ sets of U_γ $(\gamma > a)$. Accordingly $S^{n-i+2+k+1}(x, \mathfrak{U}_{m+1})$ intersects $n-i+2+k+1$ sets of U_γ $(\gamma \geq a)$ for every $x \in M$. Hence

$$x \in S_\beta^{n-i+3+k} \subseteq W_\beta^{n-i+3} \quad \text{for every} \quad \beta < a,$$

and hence $M \subseteq W_\beta^{n-i+3}$. Since $U_\beta^{i-1} \cap W_\beta^{n-i+3} = \emptyset$ $(\beta < a)$ from the assumption of induction, we get $M \cap U_\beta^{i-1} = \emptyset$ $(\beta < a)$ and consequently $M \cap V_a^{i-1} = \emptyset$. Combining this conclusion with the assumption $M \cap W_a^{n-i+3} = \emptyset$ we obtain

$$M \cap (\overline{V}_a^{i-1} \cup \overline{W}_a^{n-i+3}) = \emptyset$$

from the openness of M. Therefore $M \subseteq U_a^{i-1}$. Thus the condition 3) is valid for a, and hence we can define U_a^i $(i = 1, \ldots, n+1)$ satisfying 1)-3) for every $a < \tau$.

2. Since

$$\mathfrak{U}_{m+2}^* < \mathfrak{U}_{m+1} < \{U_a^i \mid i = 1, \ldots, n+1;\ a < \tau\},$$

if we put

$$\mathfrak{U}_m^i = \{U_a^i - \overline{S(R - U_a^i,\ \mathfrak{U}_{m+2})} \mid a < \tau\},$$

then $\bigcup_{i=1}^{n+1} \mathfrak{U}_m^i$ is an open covering refining $\mathfrak{U}_m$, and $U_1, U_2 \in \mathfrak{U}_m^i$ and $U_1 \neq U_2$ imply

$$S(U_1, \mathfrak{U}_{m+2}) \cap S(U_2, \mathfrak{U}_{m+2}) = \emptyset$$

by condition 2) of $\mathfrak{U}_m^i$. From now on let us denote $\mathfrak{U}_{2m-1}$ and $\mathfrak{U}_{2m-1}^i$ by $\mathfrak{U}_m$ and $\mathfrak{U}_m^i$ $(m = 1, 2, \ldots)$ respectively for brevity; then $\mathfrak{U}_m$ and $\mathfrak{U}_m^i$ satisfy

(B) $\quad \mathfrak{U}_{m+1} < \bigcup_{i=1}^{n+1} \mathfrak{U}_m^i < \mathfrak{U}_m,$

(B') $\quad S(U_1, \mathfrak{U}_{m+1}) \cap S(U_2, \mathfrak{U}_{m+1}) = \emptyset$

if $U_1, U_2 \in \mathfrak{U}_m^i$, $U_1 \neq U_2$.

For a fixed i and for every $U \in \mathfrak{U}_{2k-1}^i$ we define inductively

$$\mathfrak{S}(U) = \mathfrak{S}^1(U) = \{U' \mid U' \in \mathfrak{U}_{2k-1+2j}^i \text{ for some positive integer } j,\ S(U', \mathfrak{U}_{2k-1+2j}) \cap U \neq \emptyset\},$$

$$\mathfrak{S}^{m+1}(U) = \bigcup \{\mathfrak{S}(U') \mid U' \in \mathfrak{S}^m(U)\} \quad (m = 1, 2, \ldots).$$

(C) From now on we denote by $U \leftarrow U'$ the fact that

$$S(U', \mathfrak{U}_{2k-1+2j}) \cap U \neq \emptyset \quad \text{for} \quad U' \in \mathfrak{U}_{2k-1+2j}^i, \quad U \in \mathfrak{U}_{2k-1}^i.$$

Then

$$\mathfrak{S}^m(U) = \{U' \mid U \leftarrow U_1 \leftarrow U_2 \leftarrow \ldots \leftarrow U_m = U' \text{ for } U_j \in \mathfrak{U}_{2k-1+n(j)}^i\ (j = 1, 2, \ldots),\ 0 < n(1) < n(2) < \ldots < n(m)\}.$$

Furthermore, we define

$$S(U) = U \cup \{U' \mid U' \in \bigcup_{m=1}^{\infty} \mathfrak{S}^m(U)\}.$$

The principal object of the second part is to prove that

(i) $U_1, U_2 \in \mathfrak{U}_{2k-1}^i$ and $U_1 \neq U_2$ imply $S(U_1) \cap S(U_2) = \emptyset$,

(ii) $U_1 \in \mathfrak{U}_{2k-1}^i$ and $U_2 \in \mathfrak{U}_{2k-1+l}^i$ for some even $l \geq 2$ imply $S(U_2) \subseteq S(U_1)$ or $S(U_1) \cap S(U_2) = \emptyset$.

To prove (i) we take an arbitrary $V \in \bigcup_{m=1}^{\infty} \mathfrak{S}^m(U_1)$. If $V \in \mathfrak{S}^j(U_1)$, then there exists a sequence

$$U_1 = V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \ldots \leftarrow V_j = V \quad \text{of} \quad V_p^* \in \mathfrak{U}_{2k-1+n(p)}^i \quad (p = 0, 1, \ldots, j)$$

for some even numbers $n(p)$ $(p = 0, 1, \ldots, j)$ such that $n(p+1) \geq n(p)+2$. Especially we notice that $n(1) \geq n(0)+2 = 2$. Since $\mathfrak{U}_{2k-1+n(p)}^* < \mathfrak{U}_{2k-1+n(p)-1}$ and $\mathfrak{U}_m^i < \mathfrak{U}_m$ by (B), from (C) combined with the above remark we easily see that

$$V_j \subseteq S(V_j, \mathfrak{U}_{2k-1+n(j)}) \subseteq S(V_{j-1}, \mathfrak{U}_{2k-1+n(j-1)}) \subseteq S(V_{j-2}, \mathfrak{U}_{2k-1+n(j-2)}) \subset \ldots \subseteq S(V_1, \mathfrak{U}_{2k-1+n(1)}) \subseteq U' \quad \text{for some} \quad U' \in \mathfrak{U}_{2k-1+1}.$$

Since

$$U' \cap U_1 \supseteq S(V_1, \mathfrak{U}_{2k-1+n(1)}) \cap U_1 \neq \emptyset \quad \text{by the fact that} \quad U_1 \leftarrow V_1,$$

we obtain

(D) $V \subseteq U'$ for every $V \in \bigcup_{m=1}^{\infty} \mathfrak{S}^m(U_1)$, $U_1 \in \mathfrak{U}^i_{2k-1}$ and for some $U' \in \mathfrak{U}_{2k-1+l+1}$ with $U' \cap U_1 \neq \emptyset$.

It follows from (D) that $V_j \subseteq S(U_1, \mathfrak{U}_{2k-1+l})$ for every $V = V_j \in \bigcup_{m=1}^{\infty} \mathfrak{S}^m(U_1)$. Therefore we can conclude that

(E) $S(U_1) \subseteq S(U_1, \mathfrak{U}_{2k-1+l})$ for every $U_1 \in \mathfrak{U}_{2k-1}$.

In consequence, if $U_1, U_2 \in \mathfrak{U}^i_{2k-1}$ and $U_1 \neq U_2$, then by (B′) we can conclude that $S(U_1) \cap S(U_2) = \emptyset$. As is easily seen, it follows from (E) that

(F) $\{S(U) | U \in \mathfrak{U}^i_{2k-1}\} < \mathfrak{U}^*_{2k-1}$,

which will be used later.

Next we proceed to the case of (ii). If $S(U_1) \cap S(U_2) \neq \emptyset$ for $U_1 \in \mathfrak{U}^i_{2k-1}$ and $U_2 \in \mathfrak{U}^i_{2k-1+l}$, then there exist some $V_p \in \mathfrak{S}^p(U_1)$, $W_q \in \mathfrak{S}^q(U_2)$ with $V_p \cap W_q \neq \emptyset$ and consequently two sequences

$$U_1 = V_0 \leftarrow V_1 \leftarrow V_2 \leftarrow \ldots \leftarrow V_p \quad \text{of} \quad V_j \in \mathfrak{U}^i_{2k-1+n(j)} \quad (j = 0, 1, \ldots, p),$$
$$U_2 = W_0 \leftarrow W_1 \leftarrow W_2 \leftarrow \ldots \leftarrow W_q \quad \text{of} \quad W_j \in \mathfrak{U}^i_{2k-1+l+m(j)} \quad (j = 0, 1, \ldots, q).$$

We take $j \geqslant 0$ such that

$$2k-1+n(j) \leqslant 2k-1+l < 2k-1+n(j+1);$$

we notice that

(G) $2k-1+l+2 \leqslant 2k-1+n(j+1)$,

because l and $n(j+1)$ are even. Since $2k-1+n(j) = 2k-1+l$ implies $S(U_2) = S(V_j) \subseteq S(U_1)$ by (i), we assume

(H) $2k-1+n(j) < 2k-1+l$.

If $j = p$, *i. e.* $2k-1+n(p) < 2k-1+l$, then since by (D) there exists $W' \in \mathfrak{U}_{2k-1+l+1}$ such that $W' \supseteq W_q$ and $W' \cap U_2 \neq \emptyset$, we get $S(U_2, \mathfrak{U}_{2k-1+l}) \cap V_p \neq \emptyset$ from $V_p \cap W_q \neq \emptyset$, *i. e.*, $V_p \leftarrow U_2$. Hence $S(U_2) \subseteq S(V_p) \subseteq S(U_1)$.

If $j < p$, then by (D) there exist $W' \in \mathfrak{U}_{2k-1+l+1}$ and $V' \in \mathfrak{U}_{2k-1+n(j+1)}$ such that

(I) $W' \supseteq W_q$, $\quad W' \cap U_2 \neq \emptyset$, $\quad V' \supset V_p$, $\quad V' \cap V_{j+1} \neq \emptyset$.

(If $j+1 = p$, then we put $V' = V_p$.)

Since $V_j \leftarrow V_{j+1}$, there exists $V'' \in \mathfrak{U}_{2k-1+n(j+1)}$ with

(J) $V'' \cap V_{j+1} \neq \emptyset$, $\quad V'' \cap V_j \neq \emptyset$.

Therefore we get

(K) $V' \cup V_{j+1} \cup V'' \in \mathfrak{U}^*_{2k-1+n(j+1)} < \mathfrak{U}_{2k-1+l+1}$

from (G). Since $V' \cap W' \supseteq V' \cap W_q \neq \emptyset$ from (I), it follows from (I), (J) and (K) that

$$W' \cup V' \cup V_{j+1} \cup V'' = W'' \in \mathfrak{U}^*_{2k-1+l+1} < \mathfrak{U}_{2k-1+l}$$

and

$$W'' \cap U_2 \supseteq W' \cap U_2 \neq \emptyset, \qquad W'' \cap V_j \supseteq V'' \cap V_j \neq \emptyset.$$

Thus we deduce $V_j \leftarrow U_2$ from (H) and consequently $S(U_2) \subseteq S(V_j) \subseteq S(U_1)$.

3. We put $\mathfrak{S}^i_m = \{S(U) \mid U \in \mathfrak{U}^i_{2m-1}\}$ and define inductively open collections ${}_m\mathfrak{S}_{m+j}$ $(j = 1, 2, \ldots)$ by

$$\left.\begin{aligned} {}_m\mathfrak{S}^i_{m+1} &= \mathfrak{S}^i_m \cup \{S \mid S \in \mathfrak{S}^i_{m+1},\ S \underset{\neq}{\not\subset} S' \quad \text{for every } S' \in \mathfrak{S}^i_m\}, \\ {}_m\mathfrak{S}^i_{m+j+1} &= {}_m\mathfrak{S}^i_{m+j} \cup \{S \mid S \in \mathfrak{S}^i_{m+j+1},\ S \underset{\neq}{\not\subset} S' \text{ for every } S' \in {}_m\mathfrak{S}^i_{m+j}\} \end{aligned}\right\} (j = 1, 2, \ldots)$$

for a fixed m. Then $\mathfrak{T}^i_m = \bigcup_{j=1}^{\infty} {}_m\mathfrak{S}^i_{m+j}$ is a disjointed collection from (i), (ii) of 2. It follows from

$${}_{m+1}\mathfrak{S}^i_{m+1+j} < \bigcup_{k=0}^{j} \mathfrak{S}^i_{m+1+k} < {}_m\mathfrak{S}^i_{m+1+j}$$

that

(L) $$\mathfrak{T}^i_{m+1} = \bigcup_{j=1}^{\infty} {}_{m+1}\mathfrak{S}^i_{m+1+j} < \bigcup_{j=2}^{\infty} {}_m\mathfrak{S}^i_{m+j} < \bigcup_{j=1}^{\infty} {}_m\mathfrak{S}^i_{m+j} = \mathfrak{T}^i_m.$$

Since

$$\bigcup_{i=1}^{n+1} \mathfrak{S}^i_m > \bigcup_{i=1}^{n+1} \mathfrak{U}^i_{2m-1} > \mathfrak{U}_{2m}$$

by (B), $\bigcup_{i=1}^{n+1} \mathfrak{S}^i_m$ is an open covering of R; moreover it is a refinement of $\mathfrak{U}^*_{2m-1}$ by (F). In consequence $\{S(p, \bigcup_{i=1}^{n+1} \mathfrak{S}^i_m) | m = 1, 2, \ldots\}$ is a nbd basis for every point p of R; hence it follows from $\mathfrak{S}^i_m \subseteq \mathfrak{T}^i_m$ that $\{\mathfrak{T}^i_m | i = 1, \ldots, n+1;\ m = 1, 2, \ldots\}$ is an open basis of R. Combining this conclusion with (L), we get $n+1$ sequences $\mathfrak{T}^i_1 > \mathfrak{T}^i_2 > \ldots$ $(i = 1, \ldots, n+1)$ of disjointed collections such that $\{\mathfrak{T}^i_m\}$ is an open basis of R. Thus we conclude that $\dim R \leqslant n$ by Theorem 1.

From this theorem we easily obtain the following main theorem.

THEOREM 3. *In order that a T_1-topological space R be a metrizable space with $\dim R \leqslant n$ it is necessary and sufficient that there exists a sequence $\mathfrak{B}_1 > \mathfrak{B}^*_2 > \mathfrak{B}_2 > \mathfrak{B}^*_3 > \ldots$ of open coverings such that $\{S(p, \mathfrak{B}_m) | m = 1, 2, \ldots\}$ is a nbd basis for each point p of R and such that order $\mathfrak{B}_m \leqslant n+1$ $(m = 1, 2, \ldots)$.*

Proof. Since the necessity is contained in Theorem 2, we prove only the sufficiency. Let $\mathfrak{B}_1 = \{V_\alpha | \alpha \in A\}$; then we define U_α by

(A) $U_\alpha = \bigcup \{V | S(V, \mathfrak{B}_2) \subseteq V_\alpha, V \in \mathfrak{B}_2\}$.

Since $\mathfrak{B}_2^* < \mathfrak{B}_1$, $\mathfrak{U}_1 = \{U_\alpha | \alpha \in A\}$ is an open covering of R such that $\mathfrak{U}_1 > \mathfrak{B}_2$. Furthermore we notice that

(B_1) each set of $\mathfrak{B}_2$ intersects at most $n+1$ sets of $\mathfrak{U}_1$ by (A) and the condition: order $\mathfrak{B}_m \leqslant n+1$.

Next, we assume $\mathfrak{B}_3 = \{V_\beta | \beta \in B\}$ and define U_β by

$$U_\beta = \bigcup \{V | S(V, \mathfrak{B}_4) \subseteq V_\beta, V \in \mathfrak{B}_4\}.$$

Then $\mathfrak{U}_3 = \{U_\beta | \beta \in B\}$ is an open covering of R, and it follows from $\mathfrak{U}_3 < \mathfrak{B}_3$ that

$$\mathfrak{U}_1 > \mathfrak{B}_2 > \mathfrak{B}_3^* > \mathfrak{U}_3^* > \mathfrak{U}_3 > \mathfrak{B}_4.$$

We notice that

(B_2) each set of $\mathfrak{B}_4$ intersects at most $n+1$ sets of $\mathfrak{U}_3$.

Thus we can repeat this process and get a sequence

$$\mathfrak{U}_1 > \mathfrak{B}_2 > \mathfrak{U}_3^* > \mathfrak{U}_3 > \mathfrak{B}_4 > \mathfrak{U}_5^* > \mathfrak{U}_5 > \mathfrak{B}_6 > \ldots$$

of open coverings such that

(B_m) each set of $\mathfrak{B}_{2m}$ intersects at most $n+1$ sets of $\mathfrak{U}_{2m-1}$.

Hence by (B_m) $\mathfrak{U}_1 > \mathfrak{U}_3^* > \mathfrak{U}_3 > \mathfrak{U}_5^* > \ldots$ is a sequence such that each set of $\mathfrak{U}_{2m+1}$ intersects at most $n+1$ sets of $\mathfrak{U}_{2m-1}$ and such that $\{S(p, \mathfrak{U}_{2m-1}) | m = 1, 2, \ldots\}$ is a nbd basis of p. Therefore we conclude that $\dim R \leqslant n$ from Theorem 2.

First, let us apply our theorem to the notion "length of a multiplicative covering" due to Alexandroff and Kolmogoroff (see [1]).

Definition. We call a covering $\mathfrak{U}$ a *multiplicative covering* if every non-empty intersection $\bigcap_{i=1}^{k} U_i$ of elements U_i $(i = 1, \ldots, k)$ of $\mathfrak{U}$ is an element of $\mathfrak{U}$.

Definition. The maximal number n such that there exists a sequence $U_1 \underset{\neq}{\supset} U_2 \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} U_n$ of elements of a multiplicative covering $\mathfrak{U}$ is called the *length* of $\mathfrak{U}$.

Definition. We mean by the *rank* of an element U of a multiplicative covering $\mathfrak{U}$ the maximal number r such that there exists a sequence $U = U_1 \underset{\neq}{\supset} U_2 \underset{\neq}{\supset} \ldots \underset{\neq}{\supset} U_r$ of elements of $\mathfrak{U}$.

Theorem 4. *In order that a T_1-space R be a metrizable space with $\dim R \leqslant n$ it is necessary and sufficient that there exists a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^* > \mathfrak{U}_2 > \mathfrak{U}_3^* > \ldots$ of multiplicative coverings with length $\leqslant n+1$ such that $\{S(p, \mathfrak{U}_m) | m = 1, 2, \ldots\}$ is a nbd basis of p.*

Proof. If $\dim R \leqslant n$, then there exists, by Theorem 3, a sequence $\mathfrak{U}_1 > \mathfrak{U}_1^* > \mathfrak{U}_2 > \ldots$ of coverings of order $\leqslant n+1$ such that $\{S(p, \mathfrak{U}_m) | m = 1, 2, \ldots\}$ is a nbd basis of p. Since $\mathfrak{U}_m$ plus all the intersections of a finite number of elements of $\mathfrak{U}_m$ is obviously a multiplicative covering with length $\leqslant n+1$, the necessity is valid.

Let us assume the existence of a sequence satisfying the condition of the proposition. If we denote by $U_{r\alpha}$ $(\alpha \in A_r)$ all the elements of $\mathfrak{U}_1$ with rank r, then

$$\mathfrak{U}_1 = \{U_{r\alpha} | \alpha \in A_r,\ r = 1, \ldots, n+1\}.$$

We define $V_{r\alpha}^{(i)}$ $(i = 1, \ldots, n+1)$ by

$$V_{r\alpha}^{(1)} = U_{r\alpha}, \quad V_{r\alpha}^{(i)} = \{x | S(x, \mathfrak{U}_i) \subseteq V_{r\alpha}^{(i-1)}\}^\circ \quad (i = 2, 3, \ldots, n)\ (^9).$$

It follows directly from the above definition and $\mathfrak{U}_i^* < \mathfrak{U}_{i-1}$ that

(A) $V_{r\alpha}^{(n+1)} \subseteq \ldots \subseteq V_{r\alpha}^{(2)} \subseteq V_{r\alpha}^{(1)} = U_{r\alpha}$,

(B) $\mathfrak{U}_i < \{V_{r\alpha}^{(i)} | \alpha \in A_r,\ r = 1, \ldots, n+1\}$ $\quad (i = 1, \ldots, n+1)$,

(C) $S(V_{r\alpha}^{(i)}, \mathfrak{U}_i) \subseteq V_{r\alpha}^{(i-1)}$ $\quad (i = 2, \ldots, n+1)$.

Next we define $M_{r\alpha}$ $(r = 1, \ldots, n+1)$ by

$$M_{1\alpha} = V_{1\alpha}^{(1)} = U_{1\alpha},$$

(D) $$M_{r\alpha} = V_{r\alpha}^{(r)} - \overline{\cup \{S(V_{1\alpha}^{(r)}, \mathfrak{U}_{n+2}) | \alpha \in A_1\} \cup \{S(V_\alpha^{(r)}, \mathfrak{U}_{n+2}) | \alpha \in A_2\} \cup \ldots}$$
$$\overline{\cup \{S(V_{r-1,\alpha}^{(r)}, \mathfrak{U}_{n+2}) | \alpha \in A_{r-1}\}} \quad (r = 2, \ldots, n+1).$$

Let us show that

(E) $\mathfrak{U}_{n+2} < \mathfrak{M}_1 = \{M_{r\alpha} | \alpha \in A_r,\ r = 1, \ldots, n+1\}$.

Let U be an arbitrary set of $\mathfrak{U}_{n+2}$; then by using (B) for $i = n+1$ we get $V_{r\alpha}^{(n+1)}$ with $U \subseteq V_{r\alpha}^{(n+1)}$. $U \subseteq V_{r\alpha}^{(r)}$ follows from (A), and hence

$$\mathfrak{U}_{n+2} < \{V_{r\alpha}^{(r)} | \alpha \in A_r,\ r = 1, 2, \ldots, n+1\}.$$

Therefore we can find for every $U \in \mathfrak{U}_{n+2}$ the minimum number r such that $U \subseteq V_{r\alpha}^{(r)}$. To prove (E) we show that

$$U \cap S(V_{k\alpha}^{(r)}, \mathfrak{U}_{n+2}) = \emptyset$$

for this r and every k with $1 \leqslant k \leqslant r-1$ and for every $\alpha \in A_k$. If we assume the contrary: $U \cap S(V_{k\alpha}^{(r)}, \mathfrak{U}_{n+2}) \neq 0$, $1 \leqslant k \leqslant r-1$, $\alpha \in A_k$, then we have, from $\mathfrak{U}_{n+2}^* < \mathfrak{U}_r$, (A) and (C),

$$U \subseteq S(V_k^{(r)}, \mathfrak{U}_r) \subseteq V_{k\alpha}^{(r-1)} \subseteq V_{k\alpha}^{(k)},$$

(^9) A° denotes the interior of A.

which contradicts the character of r because $k < r$. Hence we must have $U \cap S(V_{k\alpha}^{(r)}, \mathfrak{U}_{n+2}) = \emptyset$ $(1 \leqslant k \leqslant r-1,\ \alpha \in A_k)$. This combined with $U \subseteq V_{r\alpha}^{(r)}$ for a definite $\alpha \in A_r$ implies $U \subseteq M_{r\alpha}$ by (D), proving (E).

Now, to show that

(F) order $\mathfrak{M}_1 \leqslant n+1$,

we prove

(G) $M_{r\alpha} \cap M_{r\beta} = \emptyset$ for $\alpha \neq \beta$.

In the case $\alpha, \beta \in A_1$, $\alpha \neq \beta$ implies clearly

$$M_{1\alpha} \cap M_{1\beta} = U_{1\alpha} \cap U_{1\beta} = \Phi,$$

because the ranks of $U_{1\alpha}$ and of $U_{1\beta}$ are 1.

To show the same assertion for $r > 1$, we prove that

(H) $U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma}$ for $\alpha, \beta \in A_r$, $\gamma \in A_{r'}$ implies $V_{r\alpha}^{(r)} \cap V_{r\beta}^{(r)} = V_{r'\gamma}^{(r)}$.

First, $V_{r'\gamma}^{(2)} \subseteq V_{r\alpha}^{(2)} \cap V_{r\beta}^{(2)}$ is obvious from the definition of $V_{r\alpha}^{(i)}$. Conversely, suppose that $x \in V_{r\alpha}^{(2)} \cap V_{r\beta}^{(2)}$; then there exist nbds $P(x), Q(x)$ of x such that

$$S(P(x), \mathfrak{U}_2) \subseteq U_{r\alpha}, \qquad S(Q(x), \mathfrak{U}_2) \subseteq U_{r\beta}.$$

Hence

$$S(P(x) \cap Q(x), \mathfrak{U}_2) \subseteq U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma}.$$

This means that $x \in V_{r'\gamma}^{(2)}$, proving $V_{r'\gamma}^{(2)} = V_{r\alpha}^{(2)} \cap V_{r\beta}^{(2)}$. Repeating this process, we conclude that $V_{r'\gamma}^{(r)} = V_{r\alpha}^{(r)} V \cap_{r\beta}^{(r)}$.

We now return to the proof of (G). By using (H) and (D), we have

$$M_{r\alpha} \cap M_{r\beta} \subseteq V_{r\alpha}^{(r)} \cap V_{r\beta}^{(r)} - \overline{\bigcup \{S(V_{1\alpha}^{(r)}, \mathfrak{U}_{n+2}) | \alpha \in A_1\} \cup \{S(V_{2\alpha}^{(r)}, \mathfrak{U}_{n+2}) | \alpha \in A_2\} \cup}$$
$$\overline{\ldots \cup \{S(V_{r-1,\alpha}^{(r)}, \mathfrak{U}_{n+2}) | \alpha \in A_{r-1}\}} \subseteq V_{r\alpha}^{(r)} \cap V_{r\beta}^{(r)} - S(V_{r\gamma}^{(r)}, \mathfrak{U}_{n+2}) = \emptyset$$

for r' determined by $U_{r\alpha} \cap U_{r\beta} = U_{r'\gamma}$, because $r' < r$ and consequently

$$S(V_{r'\gamma}^{(r)}, \mathfrak{U}_{n+2}) \subseteq \bigcup \{S(V_{1\alpha}^{(r)}, \mathfrak{U}_{n+2}) | \alpha \in A_1\} \cup \ldots \cup \{S(V_{r-1,\alpha}^{(r)}, \mathfrak{U}_{n+2}) | \alpha \in A_{r-1}\}.$$

Thus (G) is proved for $r = 1, \ldots, n+1$. Since $\mathfrak{M}_1 = \{M_{r\alpha} | \alpha \in A_r,\ r = 1, \ldots, n+1\}$, the assertion (F): order $\mathfrak{M}_1 \leqslant n+1$ follows directly from (G).

Since $\mathfrak{M}_1 < \mathfrak{U}_1$ is obvious, from (D) combined with (F) we obtain a covering $\mathfrak{M}_1$ satisfying

$$\mathfrak{U}_{n+2} < \mathfrak{M}_1 < \mathfrak{U}_1, \qquad \text{order } \mathfrak{M}_1 \leqslant n+1.$$

Repeating the same process, we get a sequence $\mathfrak{M}_m$ $(m = 1, 2, \ldots)$ of coverings of order $\leqslant n+1$ such that

$$\mathfrak{U}_{1+m(n+1)} < \mathfrak{M}_m < \mathfrak{U}_{1+(m-1)(n+1)}.$$

Therefore we have, from Theorem 3, $\dim R \leqslant n$.

§ 2. Dimension and metric function. We know that if $\varrho(x, y_i) < \varepsilon$ [10] $(i = 1, 2, 3)$ in Euclidean 1-space E_1, then $\varrho(y_i, y_j) < \varepsilon$ for some two points y_i, y_j of the three points y_1, y_2, y_3 and the same is also valid for seven points y_i $(i = 1, \ldots, 7)$ and a point x of E_2, and that the number of y_i having such character increases with the dimension n of E_n. To begin with, we shall characterize generally the dimension of a metric space with a similar property of metric function.

THEOREM 5. *In order that* $\dim R \leqslant n$ *for a metrizable space* R *it is necessary and sufficient to be able to define a metric* $\varrho(x, y)$ *agreeing with the topology of* R *such that for every* $\varepsilon > 0$ *and for every point* x *of* R,

$$\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon \qquad (i = 1, \ldots, n+2)$$

imply

$$\varrho(y_i, y_j) < \varepsilon \text{ for some } i, j \text{ with } i \neq j.$$

Proof. Necessity. 1. Let R be a metrizable space with $\dim R \leqslant n$; then by Theorem 2 there exists a sequence $\mathfrak{U}_1 > \mathfrak{U}_2^{**} > \mathfrak{U}_2 > \mathfrak{U}_3^{**} > \ldots$ $(\mathfrak{U}^{**} = (\mathfrak{U}^*)^*)$ of open coverings of R such that $\{S(p, \mathfrak{U}_m) | m = 1, 2, \ldots\}$ is a nbd basis for each point p of R and such that each $S^2(p, \mathfrak{U}_{m+1}^*)$ intersects at most $n+1$ sets of $\mathfrak{U}_m$. Now we define $S_{m_2, m_3, \ldots, m_p}(U)$ for $1 \leqslant m_1 < m_2 < \ldots < m_p$ and for $U \in \mathfrak{U}_{m_1}$ by

$$S_{m_2}(U) = \bigcup \{U' | S(U', \mathfrak{U}_{m_2}) \cap U \neq \emptyset,\ U' \in \mathfrak{U}_{m_2}\} = S^2(U, \mathfrak{U}_{m_2}),$$
$$S_{m_2, \ldots, m_p}(U) = \bigcup \{U' | S(U', \mathfrak{U}_{m_p}) \cap S_{m_2, \ldots, m_{p-1}}(U) \neq \emptyset,\ U' \in \mathfrak{U}_{m_p}\}$$
$$= S^2(S_{m_2, \ldots, m_{p-1}}(U), \mathfrak{U}_{m_p}),$$

and

$$S_{m_2, \ldots, m_p}(U) = U \quad \text{for} \quad p = 1.$$

Furthermore we define open coverings of R by

$$\mathfrak{S}_{m_1} = \mathfrak{U}_{m_1}, \qquad \mathfrak{S}_{m_1, \ldots, m_p} = \{S_{m_2, \ldots, m_p}(U) | U \in \mathfrak{U}_{m_1}\}.$$

We show first that

(A) $\dfrac{1}{2^{m_1}} + \ldots + \dfrac{1}{2^{m_p}} \geqslant \dfrac{1}{2^{l_1}} + \ldots + \dfrac{1}{2^{l_q}}$ implies $\mathfrak{S}_{m_1, \ldots, m_p} > \mathfrak{S}_{l_1, \ldots, l_q}$.

Since in the case $p \geqslant q$ and $m_i = l_i$ $(i = 1, \ldots, q)$ the validity of (A) is evident from the definition, we concern ourselves with the other cases only. We can easily prove the important proposition:

(B) $S_{k_2, \ldots, k_r}(U') \subseteq S(U', \mathfrak{U}_{k_1})$ for every $U' \in \mathfrak{U}_{k_1}$,

[10] We denote by $\varrho(x, y)$ the distance of x and y; see [5].

which will often be used in the remainder of this proof. For it follows from $k_1 < k_2 < \ldots < k_r$ that

$$\mathfrak{U}_{k_1} > \mathfrak{U}^{**}_{k_2} > \mathfrak{U}_{k_2} > \mathfrak{U}^{**}_{k_3} > \ldots > \mathfrak{U}^{**}_{k_r},$$

and hence

$$S_{k_2,\ldots,k_r}(U') \subseteq S^2(S_{k_2,\ldots,k_{r-1}}(U'), \mathfrak{U}_{k_r}) \subseteq S(S_{k_2,\ldots,k_{r-2}}(U'), \mathfrak{U}_{k_{r-2}})$$
$$\subseteq \ldots \subseteq S(S_{k_2}(U'), \mathfrak{U}_{k_2}) \subset S(U', \mathfrak{U}_{k_1}).$$

Therefore, if

$$\frac{1}{2^{m_1}} + \ldots + \frac{1}{2^{m_p}} > \frac{1}{2^{l_1}} + \ldots + \frac{1}{2^{l_q}}$$

and

$$m_1 = l_1,\ m_2 = l_2,\ \ldots,\ m_{i-1} = l_{i-1},\ m_i < l_i \text{ for a definite } i \text{ with } 2 \leqslant i \leqslant p, q,$$

then from (B) we have

(C) $S_{l_{i+1},\ldots,l_q}(U') \subseteq S(U', \mathfrak{U}_{l_i}) \subseteq U''$ for every $U' \in \mathfrak{U}_{l_i}$ and for some $U'' \in \mathfrak{U}_{m_i}$. If further this U' satisfies $S_{l_2,\ldots,l_{i-1}}(U) \leftarrow U'$ for $U \in \mathfrak{U}_{l_1}$ [11], then from (C) we have

(D) $U'' \cap S_{m_2,\ldots,m_{i-1}}(U) = U'' \cap S_{l_2,\ldots,l_{i-1}}(U) \neq \emptyset$.

Therefore by (C) and (D)

(E) $S_{l_{i+1},\ldots,l_q}(U') \subseteq U'' \subseteq S_{m_2,\ldots,m_i}(U) \subseteq S_{m_2,\ldots,m_p}(U)$ holds for every U' with $S_{l_2,\ldots,l_{i-1}}(U) \leftarrow U' \in \mathfrak{U}_{l_i}$. Hence it follows from (E) that

$$S_{l_2,\ldots,l_q}(U) = S_{l_2,\ldots,l_{i-1}}(U) \cup [\cup \{S_{l_{i+1},\ldots,l_q}(U') \mid S_{l_2,\ldots,l_{i-1}}(U) \leftarrow U' \in \mathfrak{U}_{l_i}\}] \subseteq S_{m_2,\ldots,m_p}(U),$$

proving $\mathfrak{S}_{l_1,\ldots,l_p} < \mathfrak{S}_{m_1,\ldots,m_p}$.

In the case of $m_1 < l_1$

$$S_{l_2,\ldots,l_q}(U') \subseteq S(U', \mathfrak{U}_{l_1}) \subseteq U'' \subseteq S_{m_2,\ldots,m_p}(U'')$$

for every $U' \in \mathfrak{U}_{l_1}$ and for some $U'' \in \mathfrak{U}_{m_1}$ follows directly from (C). This completes the proof of proposition (A).

2. Now we define a non-negatively valued function $\varrho(x, y)$ on $R \times R$ by

(F) $\varrho(x, y) = \inf\{1/2^{m_1} + \ldots + 1/2^{m_p} \mid y \in S(x, \mathfrak{S}_{m_1,\ldots,m_p})\}$,

$$\varrho(x, y) = 1 \quad \text{if} \quad y \notin S(x, \mathfrak{S}_{m_1,\ldots,m_p}) \quad \text{for every } m_i \quad (i = 1, \ldots, p).$$

Let us show that $\varrho(x, y)$ satisfies the axiom of metric function.

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[11] We use the notation $A \leftarrow U'$ in a somewhat different sense from that of the proof of Theorem 2, *i. e.*, $A \leftarrow U'$ for $U' \in \mathfrak{U}_l$ means $S(U', \mathfrak{U}_l) \cap A \neq \emptyset$ in this proof.

Since $\{S(p, \mathfrak{U}_m) \mid m = 1, 2, \ldots\}$ is a nbd basis of p, $\varrho(x, y)$ obviously agrees with the topology of R, *i. e.*, $\{S_\varepsilon(x) \mid \varepsilon > 0\}$ is a nbd basis of each point p of R.

To prove the triangle axiom $\varrho(x, y) + \varrho(y, z) \geqslant \varrho(x, z)$ we assume that $\varrho(x, y) = a \geqslant b = \varrho(y, z)$. For an arbitrary $\varepsilon > 0$ we can select $m_1, \ldots, m_p$; $l_1, \ldots, l_q$ such that

$$1 \leqslant m_1 < \ldots < m_p, \quad 1 \leqslant l_1 < \ldots < l_q,$$

$$a + \varepsilon > 1/2^{m_1} + \ldots + 1/2^{m_p} > a, \quad b + \varepsilon > 1/2^{l_1} + \ldots + 1/2^{l_q} > b$$

and such that

$$1/2^{m_1} + \ldots + 1/2^{m_p} > 1/2^{l_1} + \ldots + 1/2^{l_q}.$$

Since $y \in S(x, \mathfrak{S}_{m_1,\ldots,m_q})$, $z \in S(y, \mathfrak{S}_{l_1,\ldots,l_q})$ are obvious from (F), we assume

(G) $x, y \in S_{m_2,\ldots,m_p}(U)$, $U \in \mathfrak{U}_{m_1}$; $y, z \in S_{l_2,\ldots,l_p}(V)$, $V \in \mathfrak{U}_{l_1}$.

Moreover we notice that we can assume

(H) $p, q \geqslant 2$, $m_p > l_1$

without loss of generality.

(i) Let us consider first the case of $m_1 = l_1$. Since $S_{m_2,\ldots,m_p}(U) \subseteq S(U, \mathfrak{U}_{m_1})$ and $S_{l_2,\ldots,l_q}(V) \subseteq S(V, \mathfrak{U}_{l_1}) = S(V, \mathfrak{U}_{m_1})$ hold by (B), it follows from (G) and $\mathfrak{U}^{**}_{m_1} < \mathfrak{U}_{m_1-1}$ that

$$x, z \in S(U, \mathfrak{U}_{m_1}) \cup S(V, \mathfrak{U}_{m_1}) \subseteq W$$

for some $W \in \mathfrak{U}_{m_1-1}$. Hence $z \in S(x, \mathfrak{S}_{m_1-1})$, which implies

$$\varrho(x, z) \leqslant 1/2^{m_1-1} \leqslant 1/2^{m_1} + \ldots + 1/2^{m_p} + 1/2^{l_1} + \ldots + 1/2^{l_q} < a + b + 2\varepsilon$$

because $m_1 = l_1$.

(ii) To consider the case of $m_1 < l_1$ we notice that there exist two sequences

(I$_1$) $U = U_1 \leftarrow U_2 \leftarrow \ldots \leftarrow U_p$, $V = V_1 \leftarrow V_2 \leftarrow \ldots \leftarrow V_q$ with $U_i \in \mathfrak{U}_{m_i}$ $(i = 1, \ldots, p)$, $V_j \in \mathfrak{U}_{l_j}$ $(j = 1, \ldots, q)$, $y \in U_p \cap V_q$ and such that

(I$_2$) $x \in S_{m_2,\ldots,m_p}(U_1)$, $z \in S_{l_2,\ldots,l_q}(V_1)$.

By (H) we can take $i \geqslant 1$ such that $m_i < l_1 \leqslant m_{i+1}$.

a) In the case of $l_1 < m_{i+1}$ we can select $S_1, S_2 \in \mathfrak{U}^*_{l_1+1}$ such that

$$y \in S_1 \cap S_2, \quad S_1 \cap U_i \neq \emptyset, \quad S_2 \cap V_1 \neq \emptyset.$$

For it follows from m_{i+1}, $l_2 \geqslant l_1 + 1$ that $U_{i+1} \in \mathfrak{U}_{m_{i+1}} < \mathfrak{U}_{l_1+1}$ and $V_2 \in \mathfrak{U}_{l_2} < \mathfrak{U}_{l_1+1}$. Hence

$$y \in S_{m_{i+2},\ldots,m_p}(U_{i+1}) \subseteq S(U_{i+1}, \mathfrak{U}_{m_{i+1}}) \subseteq S(U_{i+1}, \mathfrak{U}_{l_1+1}) \subseteq S_1$$

for some $S_1 \in \mathfrak{U}^*_{l_1+1}$ and

$$y \in S_{l_2,\dots,l_q}(V_2) \subseteq S(V_2, \mathfrak{U}_{l_2}) \subseteq S_2$$

for some $S_2 \in \mathfrak{U}^*_{l_1+1}$ follows from (B). Then $S_1 \cap U_i \neq \emptyset$ and $S_2 \cap V_1 \neq \emptyset$ are obvious because $U_i \leftarrow U_{i+1}$, $V_1 \leftarrow V_2$. On the other hand, since $y \in S_1 \cap S_2 \neq \emptyset$ and $\mathfrak{U}^{**}_{l_1+1} < \mathfrak{U}_{l_1}$, $S_1 \cup S_2 \subseteq W$ holds for some $W \in \mathfrak{U}_{l_1}$. Hence $S(V_1, \mathfrak{U}_{l_1}) \cap U_i \neq \emptyset$, *i. e.*, $U_i \leftarrow V_1 \in \mathfrak{U}_{l_1}$. We can consider $S_{l_1,\dots,l_q}(U_i)$ because $U_i \in \mathfrak{U}_{m_i}$ and $m_i < l_1$, and hence from the above discussion we get

(J) $z \in S_{l_1,\dots,l_q}(U_i) \subseteq S_{m_2,\dots,m_i,l_1,\dots,l_q}(U_1)$.

By (I_2) there exists a sequence

(K) $U_1 \leftarrow U'_2 \leftarrow \dots \leftarrow U'_p \ni x$ with $U'_j \in \mathfrak{U}_{m_j}$ $(j = 2, \dots, p)$

It follows from (B) and from $l_1 < m_{i+1}$ that $x \in S(U'_{i+1}, \mathfrak{U}_{m_{i+1}}) \subseteq S'$ for some $S' \in \mathfrak{U}_{l_1}$. Since $S' \cap U'_i \neq \emptyset$ by $U'_i \leftarrow U'_{i+1}$, we get $x \in S_{m_2,\dots,m_i,l_1}(U_1)$. Therefore $x, z \in S_{m_2,\dots,m_i,l_1,\dots,l_q}(U_1)$, and hence $z \in S(x, \mathfrak{S}_{m_1,\dots,m_i,l_1,\dots,l_q})$. Thus we get

$$\varrho(x, y) \leqslant 1/2^{m_1} + \dots + 1/2^{m_i} + 1/2^{l_1} + \dots + 1/2^{l_q} < a + b + 2\varepsilon .$$

b) If $l_1 = m_{i+1}$, then we take k such that

(L) $0 \leqslant k \leqslant i$; $m_{i+1} - 1 = m_i$, $m_i - 1 = m_{i-1}$, $\dots$, $m_{i-k+2} - 1 = m_{i-k+1}$, $m_{i-k+1} - 1 > m_{i-k}$, where $k = 0$ means $m_{i+1} - 1 > m_i$, and $k = i$ means $m_{j+1} - 1 = m_j$ $(j = 1, 2, \dots, i)$.

In the case of $k < i$ it follows from (I_1) and (B) that

$$S(U_{i-k+1}, \mathfrak{U}_{m_{i-k+1}}) \cap U_{i-k} \neq \emptyset \quad \text{and} \quad y \in S(U_{i-k+1}, \mathfrak{U}_{m_{i-k+1}}) \cap S(V_1, \mathfrak{U}_{l_1}) \neq \emptyset ,$$

which implies

(M) $W \cap U_{i-k} \neq \emptyset$, $W \supseteq S(V_1, \mathfrak{U}_{l_1})$

for some $W \in \mathfrak{U}_{m_{i-k+1}-1}$ because $l_1 \geqslant m_{i-k+1}$ and $\mathfrak{U}^{**}_m < \mathfrak{U}_{m-1}$. Since we can consider $S_{m_2,\dots,m_{i-k},m_{i-k+1}-1}(U_1)$ because of $m_{i-k+1} - 1 > m_{i-k}$ and since $z \in S(V_1, \mathfrak{U}_{l_1}) \subseteq W$ by (I_1) and (B), we can conclude from (M) and (I_1) that

(N) $z \in S_{m_2,\dots,m_{i-k},m_{i-k+1}-1}(U_1)$

with respect to x, we select a sequence satisfying (K). Then

$$x \in S(U'_{i-k+1}, \mathfrak{U}_{m_{i-k+1}}) \underset{\neq}{\not\subset} (S_{m_2,\dots,m_{i-k}}(U_1))^c \ ^{(12)}$$

by (B), and hence there exists $W' \in \mathfrak{U}_{m_{i-k+1}-1}$ satisfying $x \in W' \underset{\neq}{\not\subset} (S_{m_2,\dots,m_{i-k}}(U_1))^c$. Hence $x \in S_{m_2,\dots,m_{i-k},m_{i-k+1}-1}(U_1)$. This combined with (N) implies $z \in S(x, \mathfrak{S}_{m_1,\dots,m_{i-k},m_{i-k+1}-1})$, and hence

$$\varrho(x,z) \leqslant 1/2^{m_1} + \dots + 1/2^{m_{i-k}} + 1/2^{m_{i-k+1}-1} = 1/2^{m_1} + \dots + 1/2^{m_{i+1}} + 1/2^{l_1} \leqslant a + b + 2\varepsilon$$

by $l_1 = m_{i+1}$ and (L).

[12] A^c denotes the complement set of A. Hence $B \underset{\neq}{\not\subset} A^c$ means $B \cap A \neq \emptyset$.

In the case of $k = i$ we select, by (I_1), (I_2) and (B), $P, Q \in \mathfrak{U}^*_{m_1}$ with $x \in P$, $z \in Q$, $P \cap Q \neq \emptyset$. Then $z \in S(x, \mathfrak{U}_{m_1-1})$ [13], which implies

$$\varrho(x, z) \leqslant 1/2^{m_1-1} = 1/2^{m_1} + \dots + 1/2^{m_{i+1}} + 1/2^{l_1} \leqslant a + b + 2\varepsilon .$$

Thus we get, in every case, $\varrho(x, z) \leqslant a + b + 2\varepsilon$ for an arbitrary $\varepsilon > 0$, proving

$$\varrho(x, z) \leqslant a + b = \varrho(x, y) + \varrho(y, z) .$$

3. Now it remains to prove that $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ $(i = 1, \dots, n+2)$ imply $\varrho(y_i, y_j) < \varepsilon$ for some distinct two points y_i, y_j. Since $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$, we can choose $n+2$ points x_i and a positive number δ such that

$$\varrho(x, x_i) < \delta < \varepsilon/2 , \qquad \varrho(x_i, y_i) < \varepsilon .$$

Let $m_1, \dots, m_p$ be positive integers satisfying

$$2\delta < 1/2^{m_1} + \dots + 1/2^{m_p} < \varepsilon ;$$

then there exist $S_i \in \mathfrak{S}_{m_1,\dots,m_p}$ $(i = 1, \dots, n+2)$ satisfying $x_i, y_i \in S_i$ because of $\varrho(x_i, y_i) < \varepsilon$. On the other hand, since $\delta < 1/2^{m_1+1} + \dots + 1/2^{m_p+1}$, we must have

(O) $x_i \in S(x, \mathfrak{S}_{m_1+1,\dots,m_p+1})$ $(i = 1, \dots, n+2)$

because of $\varrho(x_i, x) < \delta$. Let $S_i = S_{m_2,\dots,m_p}(U_i)$, $U_i \in \mathfrak{U}_{m_1}$; then by (B) there exists $S'_i \in \mathfrak{U}^*_{m_2}$ satisfying $S'_i \cap U_i \neq \emptyset$, $S'_i \ni x_i$. Hence it follows from (O) and $\mathfrak{S}_{m_1+1,\dots,m_p+1} < \mathfrak{U}^*_{m_1+1}$ that

$$S'_i \cap S(x, \mathfrak{U}^*_{m_1+1}) \neq \emptyset \qquad (i = 1, \dots, n+2) ,$$

which implies

$$S^2(x, \mathfrak{U}^*_{m_1+1}) \cap U_i \neq \emptyset \qquad (i = 1, \dots, n+2)$$

because $\mathfrak{U}^*_{m_2} < \mathfrak{U}^*_{m_1+1}$. Since by the first assumption $S^2(x, \mathfrak{U}^*_{m_1+1})$ intersects at most $n+1$ sets of $\mathfrak{U}_{m_1}$, we must have $U_i = U_j$ for some i, j with $i \neq j$. Then $y_j \in S(y_i, \mathfrak{S}_{m_1,\dots,m_p})$, and hence we conclude that

$$\varrho(y_i, y_j) \leqslant 1/2^{m_1} + \dots + 1/2^{m_p} < \varepsilon .$$

Sufficiency. We denote by $\varrho(x, y)$ a metric satisfying the condition of this theorem. Then we denote by M_1 a maximal subset of R such that $x, y \in M_1$ and $x \neq y$ imply $\varrho(x, y) \geqslant 1/2$. By the maximal property of M_1 $\mathfrak{U}'_1 = \{S_{1/2}(x) | x \in M_1\}$ is evidently an open covering of R. Let $S_{1/2^2}(x)$ intersect each of $S_{1/2}(x_i)$ for $x_i \in M_1$ $(i = 1, \dots, n+2)$; then it follows from the property of $\varrho(x, y)$ that $\varrho(x_i, x_j) < 1/2$ for some distinct points

[13] Since $\varrho(x, z) \leqslant a + b + 2\varepsilon$ is obvious in the case of $m_1 = 1$, we assume $m_1 > 1$.

x_i, x_j. This implies $x_i = x_j$ by the property of M_1. Therefore $S_{1/2^2}(x)$ for an arbitrary point x of R intersects at most $n+1$ sets of $\mathfrak{U}'_1$. Put

$$\mathfrak{U}_1 = \{\bigcup \{S_{1/2^3}(y) | y \in S_{1/2}(x)\} | x \in M_1\},$$

then order $\mathfrak{U}_1 \leqslant n+1$. Using the notation $\mathfrak{S}_r = \{S_r(x) | x \in R\}$, we have

$$\mathfrak{S}^*_{1/2^5} < \mathfrak{S}_{1/2^3} < \mathfrak{U}_1 < \mathfrak{S}_{1/2+1/2^3}.$$

Next we denote by M_2 a maximal subset of R such that $x, y \in M_2$ and $x \neq y$ imply $\varrho(x, y) \geqslant 1/2^6$. $\mathfrak{U}'_2 = \{S_{1/2^6}(x) | x \in M_2\}$ covers R, and each $S_{1/2^7}(x)$ intersects at most $n+1$ sets of $\mathfrak{U}'_2$ in the same way. Hence $\mathfrak{U}_2 = \{\bigcup\{S_{1/2^8}(y) | y \in S_{1/2^6}(x)\} | x \in M_2\}$ is an open covering with order $\leqslant n+1$ and satisfies

$$\mathfrak{U}_1 > \mathfrak{S}^*_{1/2^5} > \mathfrak{S}_{1/2^5} > \mathfrak{S}_{1/2^6+1/2^8} > \mathfrak{U}_2 > \mathfrak{S}_{1/2^8} > \mathfrak{S}^*_{1/2^{10}}.$$

By repeating such processes we get a sequence $\mathfrak{U}_1 > \mathfrak{U}^*_2 > \mathfrak{U}_2 > \ldots$ of open coverings of R such that order $\mathfrak{U}_m \leqslant n+1$ $(m = 1, 2, \ldots)$ and such that $\mathfrak{U}_m < \mathfrak{S}_{1/2^{1+(m-1)5}+1/2^{3+(m-1)5}}$ $(m = 1, 2, \ldots)$. Hence we conclude that $\dim R \leqslant n$ from Theorem 3.

As is easily seen from the proof of this theorem, we can state this theorem in the following form.

COROLLARY 1. *In order that* $\dim R \leqslant n$ *for a metrizable space* R *it is necessary and sufficient to be able to define a metric* $\varrho(x, y)$ *agreeing with the topology of* R *such that for every* $\varepsilon > 0$ *and for some* $\varphi(\varepsilon) > 0$, $\varrho(S_{\varphi(\varepsilon)}(x), y_i) < \varepsilon$ $(i = 1, \ldots, n+2)$ *imply* $\varrho(y_i, y_j) < \varepsilon$ *for some* i, j *with* $i \neq j$.

In the compact case we get the simpler conclusion.

COROLLARY 2. *In order that* $\dim R \leqslant n$ *for a compact metrizable space* R *it is necessary and sufficient to be able to define a metric* $\varrho(x, y)$ *agreeing with the topology of* R *such that for every* $\varepsilon > 0$, $\varrho(x, y_i) < \varepsilon$ $(i = 1, 2, \ldots, n+2)$ *imply* $\varrho(y_i, y_j) < \varepsilon$ *for some* i, j *with* $i \neq j$.

We can deduce the following theorem proved by J. de Groot (see [3]) from our theorem for the special case of $n = 0$.

COROLLARY 3. *A metrizable space* R *is* 0*-dimensional if and only if one can define a metric which satisfies*

$$\varrho(x, z) \leqslant \max[\varrho(x, y), \varrho(y, z)].$$

Proof. Let $\dim R = 0$; then by our theorem we can define a metric $\varrho(x, y)$ such that $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ $(i = 1, 2)$ imply $\varrho(y_1, y_2) < \varepsilon$. Hence if we assume $\varrho(x, z) = \varepsilon > \max[\varrho(x, y), \varrho(y, z)]$ for some $x, y, z \in R$, then $\varrho(S_{\varepsilon/2}(y), x) < \varepsilon$, $\varrho(S_{\varepsilon/2}(y), z) < \varepsilon$ and $\varrho(x, z) = \varepsilon$, which contradicts the character of $\varrho(x, y)$. Therefore we must have $\varrho(x, z) \leqslant \max[\varrho(x, y), \varrho(y, z)]$.

Conversely, let $\varrho(x, y)$ be a metric satisfying $\varrho(x, z) \leqslant \max[\varrho(x, y), \varrho(y, z)]$, and let us assume that $\varrho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ $(i = 1, 2)$. Then there exist $x_1, x_2 \in S_{\varepsilon/2}(x)$ such that $\varrho(x_i, y_i) < \varepsilon$ $(i = 1, 2)$. Since $\varrho(x_1, x_2) < \varepsilon$, we get $\varrho(y_1, y_2) \leqslant \max[\varrho(y_1, x_1), \varrho(x_1, y_2)] \leqslant \max[\varrho(y_1, x_1), \varrho(x_1, x_2), \varrho(x_2, y_2)] < \varepsilon$.

DEFINITION. A real-valued function ϱ of two points of a topological space R is a *non-Archimedean parametric* if

i) $\varrho(x, y) \geqslant 0$,

ii) $\varrho(x, y) = \varrho(y, x)$,

iii) $\{y | \varrho(x, y) < \varepsilon\}$ is open for every $\varepsilon < 0$,

iv) $\varrho(x, y) \leqslant \max[\varrho(x, z), \varrho(y, z)]$.

Now let us prove the following decomposition theorem for the metric function.

THEOREM 6. *In order that* $\dim R \leqslant n$ *for a metrizable space* R *it is necessary and sufficient to be able to define a metric* $\varrho(x, y)$ *agreeing with the topology of* R *such that*

$$\varrho(x, y) = \inf\{\varrho_0(x, z_1) + \varrho_0(z_1, z_2) + \ldots + \varrho_0(z_p, y) | z_i \in R\},$$

$$\varrho_0(x, y) = \min\{\varrho_i(x, y) | i = 1, \ldots, n+1\}$$

for some $n+1$ *non-Archimedean parametrics* $\varrho_i(x, y)$ $(i = 1, \ldots, n+1)$ [14].

Proof. Necessity. Let $\dim R \leqslant n$; then there exist $n+1$ 0-dimensional subspaces R_i such that $R = \bigcup_{i=1}^{n+1} R_i$ by the generalized decomposition theorem. We assign a metric $\varrho'(x, y)$ of R such that $\varrho'(x, y) \leqslant 1$. Since R_i $(i = 1, \ldots, n+1)$ are 0-dimensional, we get disjointed coverings $\mathfrak{U}^i_m$ $(i = 1, \ldots, n+1, m = 1, 2, \ldots)$ of R_i satisfying

$$\mathfrak{U}^i_{m+1} < \mathfrak{U}^i_m, \quad \mathfrak{U}^i_m < \mathfrak{S}_m = \{S_{1/2^m}(x) | x \in R\} \quad \text{in} \quad R_i.$$

Let $\mathfrak{U}^i_m = \{U_\alpha | \alpha \in A\}$; then for every point $x \in R_i$ we can find $\alpha \in A$ such that $x \in U_\alpha$ and $\varepsilon(x) > 0$ such that

$$S_{\varepsilon(x)}(x) \cap R_i \subseteq U_\alpha, \quad S_{\varepsilon(x)}(x) \subseteq S_\alpha,$$

where we denote by S_α a definite set $S_\alpha \in \mathfrak{S}_{m+1}$ for U_α such that $S_\alpha \supseteq U_\alpha$. Put $W_\alpha = \bigcup\{S_{\varepsilon(x)/2}(x) | x \in U_\alpha \cap R_i\}$; then we have a disjointed collection $\mathfrak{W}^i_m = \{W_\alpha | \alpha \in A\}$ satisfying $\mathfrak{W}^i_m < \mathfrak{S}_m$. By applying $\mathfrak{W}^i_m$ we define open disjointed collections $\mathfrak{V}^i_m$ $(i = 1, \ldots, n+1, m = 0, 1, \ldots)$ as follows:

$$\mathfrak{V}^i_0 = \{R\},$$

[14] This theorem is also a generalization of the above theorem of J. de Groot to the n-dimensional case.

if $\mathfrak{V}_m^i$ is defined, then we define $\mathfrak{V}_{m+1}^i$ by

$$\mathfrak{V}_{m+1}^i = \mathfrak{V}_m^i \wedge \mathfrak{W}_{m+1}^i \quad (i = 1, \ldots, n+1,\ m = 0, 1, \ldots).$$

Then it follows that $\mathfrak{V}_{m+1}^i < \mathfrak{V}_m^i < \mathfrak{S}_m$. Now we define a real-valued function ϱ_i of two points by

$$\varrho_i(x, y) = \inf\{1/2^{m-1} | y \in S(x, \mathfrak{V}_m^i)\}.$$

Then it is easy to see that ϱ_i is a non-Archimedean parametric. For $\varrho_i(x, y) \leqslant \max[\varrho_i(x, z), \varrho_i(z, y)]$ is evident from the disjointedness of $\mathfrak{V}_m^i$. Since

$$S_\varepsilon^i(x) = \{y | \varrho_i(x, y) < \varepsilon\} = \bigcup \{S(x, \mathfrak{V}_l^i) | \varepsilon > 1/2^{l-1}\},$$

$S_\varepsilon^i(x)$ is open for every $\varepsilon > 0$. Moreover, (i) and (ii) are clearly satisfied, and hence ϱ_i is a non-Archimedean parametric of R.

Since $y \in S(x, \mathfrak{V}_m^i)$ implies $\varrho'(x, y) < 1/2^{m-1}$ by $\mathfrak{V}_m^i < \mathfrak{S}_m$, we have

(A) $\varrho_i(x, y) \geqslant \varrho'(x, y)$ for every $x, y \in R$.

Now we can easily see that

(B) $\varrho(x, y) = \inf\{\varrho_0(x, z_1) + \ldots + \varrho_0(z_p, y) | z_i \in R\}$
$(\varrho_0(x, y) = \min\{\varrho_i(x, y) | i = 1, \ldots, n+1\})$

is a metric of R. It is enough to show only the agreement of ϱ with the topology of R. For a given $\varepsilon > 0$ and $x \in R$ we take m such that $\varepsilon > 1/2^{m-1}$ and R_i with $x \in R_i$. $y \in S(x, \mathfrak{V}_m^i)$ $(\neq \emptyset)$ generally implies $\varrho_i(x, y) < \varepsilon$ and consequently $\varrho(x, y) < \varepsilon$ by (B). On the other hand $\varrho(x, y) \geqslant \varrho'(x, y)$ is obvious from (A) and (B), and hence $\varrho(x, y) < \varepsilon$ implies $\varrho'(x, y) < \varepsilon$, which proves that $\{\{y | \varrho(x, y) < \varepsilon\} | \varepsilon > 0\}$ is a nbd basis of x, *i. e.*, ϱ agrees with the topology of R. Thus we deduce the necessity of the condition.

Sufficiency. Let $\varrho(x, y)$ be a metric of R satisfying the condition; then we easily see that $R = \bigcup_{i=1}^{n+1} R_i$ if we put $R_i = \{x | \varrho_i(x, x) = 0\}$. To see this we assume the existence of $x \in R$ such that $x \notin \bigcup_{i=1}^{n+1} R_i$. Then we must have $\varrho_i(x, x) = \varepsilon_i > 0$ $(i = 1, \ldots, n+1)$, and hence by the property of ϱ_i

$$\varrho_i(x, y) = \max[\varrho_i(x, y), \varrho_i(x, y)] \geqslant \varrho_i(x, x) = \varepsilon_i$$

for every $y \in R$. Therefore $\varrho_0(x, y) \geqslant \min \varepsilon_i > 0$, and hence $\varrho(x, z) \geqslant \min \varepsilon_i > 0$ for every $z \in R$, which is a contradiction.

Putting

$$S_{1/m}^i(x) = \{y | \varrho_i(x, y) < 1/m\},$$

we see that $S_{1/m}^i(x) \cap S_{1/m}^i(y) \neq \emptyset$ implies $S_{1/m}^i(x) = S_{1/m}^i(y)$. For, if we choose $z \in S_{1/m}^i(x) \cap S_{1/m}^i(y)$, then $\varrho_i(x, z) < 1/m$, $\varrho_i(y, z) < 1/m$, and hence

$\varrho_i(x, y) < 1/m$ by (iv). Therefore $\varrho_i(y, u) < 1/m$ implies $\varrho_i(x, u) < 1/m$, which proves $S_{1/m}^i(y) \subseteq S_{1/m}^i(x)$. In the same way we get $S_{1/m}^i(y) \supseteq S_{1/m}^i(x)$ and consequently $S_{1/m}^i(x) = S_{1/m}^i(y)$. Thus

$$\mathfrak{U}_m^i = \{S_{1/m}^i(x) \cap R_i | x \in R_i\} \quad (m = 1, 2, \ldots)$$

are open disjointed covering of R_i. Moreover, since $y \in S(x, \mathfrak{U}_m^i)$ implies $\varrho(x, y) \leqslant \varrho_i(x, y) < 1/m$, $\{\mathfrak{U}_m^i | m = 1, 2, \ldots\}$ is an open basis of R_i. Thus we conclude that $\dim R_i = 0$ by Theorem 1. This combined with $R = \bigcup_{i=1}^{n+1} R_i$ implies $\dim R \leqslant n$ by the generalized decomposition theorem.

§ 3. Imbedding of a metric space in a product of 1-dimensional spaces. We shall start with the following theorem, which is a generalization of the sufficiency part of Theorem 3.

THEOREM 7. *Let $n = n_1 + n_2 + \ldots + n_k$ for non-negative integers $(i = 1, \ldots, k)$. If there exist sequences*

$$\mathfrak{V}_{1,i} > \mathfrak{V}_{2,i}^* > \mathfrak{V}_{2,i} > \mathfrak{V}_{3,i}^* > \ldots \quad (i = 1, \ldots, k)$$

of open coverings of a T_1-space R such that order $\mathfrak{V}_{m,i} \leqslant n_i + 1$ $(m = 1, 2, \ldots)$ and such that $\{S(p, \mathfrak{V}_m) | m = 1, 2, \ldots\}$ for $\mathfrak{V}_m = \bigwedge_{i=1}^{k} \mathfrak{V}_{m,i}$ is a nbd basis of p, then R is a metrizable space with $\dim R \leqslant n$ and can be imbedded in a product of k metrizable spaces R_i $(i = 1, \ldots, k)$ with $\dim R_i \leqslant n_i$.

Proof. As is easily seen from the proof of Theorem 3, we can select sequences

$$\mathfrak{U}_{1,i} > \mathfrak{U}_{2,i}^{**} > \mathfrak{U}_{2,i} > \mathfrak{U}_{3,i}^{**} > \ldots \quad (i = 1, \ldots, k)$$

of open coverings such that $S(p, \mathfrak{U}_{m+1,i})$ intersects at most $n_i + 1$ sets of $\mathfrak{U}_{m,i}$ and such that $\{S(p, \mathfrak{U}_m) | m = 1, 2, \ldots\}$ for $\mathfrak{U}_m = \bigwedge_{i=1}^{k} \mathfrak{U}_{m,i}$ is a nbd basis of p. Let $\mathfrak{U}_{m,i} = \{U_\alpha | \alpha \in A\}$ for fixed m, i; then we put

(A) $V_\alpha = S(U_\alpha, \mathfrak{U}_{m+1,i})$;

$$W_{1/2^{m-1}} = R, \qquad W_{1/2^m} = S(V_\alpha^c, \mathfrak{U}_{m+2,i}),$$
$$W_{1/2^m + 1/2^{m+1}} = S(W_{1/2^m}, \mathfrak{U}_{m+3,i}), \qquad W_{1/2^{m+1}} = S(V_\alpha^c, \mathfrak{U}_{m+3,i}),$$
$$W_{1/2^m + 1/2^{m+1} + 1/2^{m+2}} = S(W_{1/2^m + 1/2^{m+1}}, \mathfrak{U}_{m+4,i}), \qquad W_{1/2^m + 1/2^{m+2}} = S(W_{1/2^m}, \mathfrak{U}_{m+4,i}),$$
$$W_{1/2^{m+1} + 1/2^{m+2}} = S(W_{1/2^{m+1}}, \mathfrak{U}_{m+4,i}), \qquad W_{1/2^{m+2}} = S(V_\alpha^c, \mathfrak{U}_{m+4,i}), \ldots$$

Defining $f_{\alpha,m,i}(x) = \inf\{r | x \in W_r\}$, we get continuous functions f_α $(\alpha \in A)$ satisfying

(B) $f_{a,m,i}(V_a^c) = 0, \quad f_{a,m,i}(U_a) = 1/2^{m-1}$ [15].

Clearly, for every $\varepsilon > 0$ there exists $l_i = l_i(\varepsilon)$ such that $y \in S(x, \mathfrak{U}_{l_i,i})$ implies

$$|f_{a,m,i}(x) - f_{a,m,i}(y)| < \varepsilon \quad (a \in A_{m,i},\ m = 1, 2, \ldots).$$

We consider a topological product

$$P_i = \prod \{I_a | a \in A_{m,i},\ m = 1, 2, \ldots\}$$

of

$$I_a = \{x | 0 \leqslant x \leqslant 1/2^{m-1}\} \quad (a \in A_{m,i}),$$

and consider $f_{a,m,i}$ a mapping of R into I_a. Then we define a continuous mapping F_i of R into P_i by

$$F_i(x) = \{f_{a,m,i}(x) | a \in A_{m,i},\ m = 1, 2, \ldots\} \quad (x \in R).$$

Now we proceed to prove that $F_i(R)$ ($\subseteq P_i$) is a metrizable space with $\dim F_i(R) \leqslant n_i$. Since

$$N_a = F_i(R) \cap \{\{p_a\} | p_a > 0\} \quad (a \in A_{m,i})$$

are open sets, and since

$$f_{a,m,i}(U_a) = 1/2^m, \quad \bigcup \{U_a | a \in A_{m,i}\} = R$$

by (B), $\mathfrak{N}_{m,i} = \{N_a | a \in A_{m,i}\}$ is an open covering of $F_i(R)$. First $\{S(p, \mathfrak{N}_{m,i}) | m = 1, 2, \ldots\}$ is a nbd basis of each point p of $F_i(R)$. Let $p = \{p_a | a \in A_{m,i},\ m = 1, 2, \ldots\} \in F_i(R)$; then for a given nbd

(C) $U(p) = \{\{q_a\} |\ |p_{a_j} - q_{a_j}| < \varepsilon,\ a_j \in A_{m_j,i}\ (j = 1, \ldots, h)\}$

of p we choose an integer l such that $y \in S(x, \mathfrak{U}_{l-1,i})$ implies

(D) $|f_{a,m,i}(x) - f_{a,m,i}(y)| < \varepsilon \ (a \in A_{m,i},\ m = 1, 2, \ldots)$.

If $q \in \{q_a\} \in S(p, \mathfrak{N}_{l,i})$, then $F_i(x) = p$, $F_i(y) = q$ and $p, q \in N_a$ for some $x, y \in R$ and $N_a \in \mathfrak{N}_{l,i}$. Since

$$f_{a,l,i}(x) = p_a > 0, \quad f_{a,l,i}(y) = q_a > 0,$$

it must be $x, y \in V_a$ for $a \in A_{l,i}$. Hence it follows from (A) and $\mathfrak{U}_{l-1,i}^{**} < \mathfrak{U}_{l,i}$ that $y \in S(x, \mathfrak{U}_{l-1,i})$. In consequence

$$|p_{a_j} - q_{a_j}| = |f_{a_j,m_j,i}(x) - f_{a_j,m_j,i}(y)| < \varepsilon \quad (j = 1, \ldots, h)$$

by (D), *i. e.*, $q \in U(p)$ follows from (C). Thus we conclude that $S(p, \mathfrak{N}_{l,i}) \subseteq U(p)$, and hence $\{S(p, \mathfrak{N}_{m,i}) |\ m = 1, 2, \ldots\}$ is a nbd basis of p.

Next let us show that

$$\text{order } \mathfrak{N}_{m,i} \leqslant n_i + 1, \quad \mathfrak{N}_{m+1,i}^* < \mathfrak{N}_{m,i}.$$

[15] $f(V) = a$ means $f(x) = a$ ($x \in V$).

If $\bigcap_{j=1}^{h} N_{a_j} \neq \emptyset$ and $a_j \in A_{m,i}$ ($j = 1, \ldots, h$), then we can choose $p = \{p_a\} \in F_i(R)$ and $x \in R$ such that

(E) $p \in \bigcap_{j=1}^{h} N_{a_j}, \quad F_i(x) = p$.

Since $f_{a_j,m,i}(x) = p_{a_j} > 0$ ($j = 1, \ldots, h$) follows from (E), we have

(F) $x \in V_{a_j}, \quad a_j \in A_{m,i}$ ($j = 1, \ldots, h$).

On the other hand, since each $S(p, \mathfrak{U}_{m+1,i})$ intersects at most $n_i + 1$ sets of $\mathfrak{U}_{m,i}$, we have

$$\text{order } \{V_a | a \in A_{m,i}\} \leqslant n_i + 1$$

from (A). Therefore we obtain $h \leqslant n_i + 1$ from (F), which means order $\mathfrak{N}_{m,i} \leqslant n_i + 1$.

We have learned from the above discussion that $N_a \cap N_{a'} \neq \emptyset$ implies $V_a \cap V_{a'} \neq \emptyset$ for fixed m, i and for arbitrary $a, a' \in A_{m+1,i}$. Hence it follows from $N_a \subseteq F_i(V_a)$ that

(G) $S(N_a, \mathfrak{N}_{m+1,i}) \subseteq F_i(S(V_a, \mathfrak{V}_{m+1,i}))$

for every $a \in A_{m+1,i}$, where we denote by $\mathfrak{V}_{m+1,i}$ the covering $\{V_a | a \in A_{m+1,i}\}$ of R. Let N_a be an arbitrary set of $\mathfrak{N}_{m+1,i}$, then there exists, by $\mathfrak{U}_{m+1,i}^{**} < \mathfrak{U}_{m,i}$, $U_\beta \in \mathfrak{U}_{m,i}$ such that $S(V_a, \mathfrak{V}_{m+1,i}) \subseteq U_\beta$. Thus we have

$$S(N_a, \mathfrak{N}_{m+1,i}) \subseteq F_i(S(V_a, \mathfrak{V}_{m+1,i})) \subseteq F_i(U_\beta) \subseteq N_\beta \in \mathfrak{N}_{m,i}$$

from (G) proving $\mathfrak{N}_{m+1,i}^* < \mathfrak{N}_{m,i}$. In consequence we can deduce from Theorem 3 the metrizability of $F_i(R)$ and $\dim F_i(R) \leqslant n_i + 1$.

Now we define a mapping $F(x)$ of R into $F_1(R) \times F_2(R) \times \ldots \times F_k(R)$ by

$$F(x) = (F_1(x), \ldots, F_k(x)) \in \prod_{i=1}^{k} F_i(R) \quad (x \in R).$$

First $F(x)$ is one-to-one. If $x, y \in R$ and $x \neq y$, then $y \notin S(x, \mathfrak{U}_{m,i})$ for some $\mathfrak{U}_{m,i}$ because $\{S(p, \bigwedge_{i=1}^{k} \mathfrak{U}_{m,i}) | m = 1, 2, \ldots\}$ is a nbd basis of p. Hence $x \in U_a \in \mathfrak{U}_{m+1,i}$, $y \notin V_a$ for some $a \in A_{m+1,i}$. This means that $f_{a,m+1,i}(x) > 0$, $f_{a,m+1,i}(y) = 0$, *i. e.*, $F_i(x) \neq F_i(y)$. Therefore $F(x)$ is one-to-one.

It remains to prove that $F(x)$ is homeomorphic. Since $F(x)$ is evidently continuous by the continuity of $F_i(x)$, we shall show the continuity of the inverse mapping. For a given nbd $U(x)$ of $x \in R$ we select $\mathfrak{U}_m = \bigwedge_{i=1}^{k} \mathfrak{U}_{m,i}$ satisfying $S(x, \mathfrak{U}_m) \subseteq U(x)$. Choosing $U_{a_i} \in \mathfrak{U}_{m+1,i}$ ($i = 1, \ldots, k$) such that $x \in \bigcap_{i=1}^{k} U_{a_i}$, we have

(H) $x \in \bigcap_{i=1}^{k} V_{a_i} \subseteq S(x, \mathfrak{U}_m) \subseteq U(x)$

from (A) and $\mathfrak{U}^{**}_{m+1,i} < \mathfrak{U}_{m+1}$, and hence $f_{a_i,m+1,i}(x) = 1/2^m > 0$. Hence if

$$|f_{a_i,m+1,i}(x) - f_{a_i,m+1,i}(y)| < 1/2^{m+1} \quad (i = 1, \ldots, k),$$

then $f_{a_i,m+1,i}(y) > 0$ $(i = 1, \ldots, k)$. Therefore $y \in V_{a_i}$ $(i = 1, \ldots, k)$, *i. e.*, $y \in U(x)$ by (H). This proves the continuity of $F(x)$, and consequently R is homeomorphic with the subspace $F(R)$ of the product space $\prod_{i=1}^{k} F_i(R)$ with $\dim F_i(R) \leqslant n_i$ $(i = 1, \ldots, k)$. From the generalized product theorem due to Katětov and to Morita (see [4] and [5]) we have $\dim R \leqslant n_1 + \ldots + n_k = n$.

THEOREM 8. *Every metric space R with $\dim R \leqslant n$ can be topologically imbedded in a topological product of $n+1$ at most 1-dimensional metric spaces.*

Proof. If $\dim R \leqslant n$, then it is easily shown that

(A) we can define a covering $\mathfrak{V}$ and open collections $\mathfrak{U}_i$ $(i = 1, \ldots, n+1)$ to every covering $\mathfrak{U}$ of R such that $\mathfrak{V} < \bigcup_{i=1}^{n+1} \mathfrak{U}_i < \mathfrak{U}$ and such that each $S^2(p, \mathfrak{V})$ intersects at most one of sets belonging to $\mathfrak{U}_i$ for a fixed i. For $R = \bigcup_{i=1}^{n+1} R_i$ for some R_i with $\dim R_i = 0$, and hence there exists a disjointed collection $\mathfrak{V}_i$ of R_i satisfying $\mathfrak{V}_i < \mathfrak{U}$. For every point x of R_i wede note by $\varepsilon(x)$ a positive number such that

$$S_{\varepsilon(x)}(x) \cap R_i \subseteq V_a \in \mathfrak{V}_i, \qquad S_{\varepsilon(x)}(x) \subseteq U_a \in \mathfrak{U}$$

for U_a defined by V_a so that $V_a \subseteq U_a$. Then $\mathfrak{V}'_i = \{\bigcup \{S_{\varepsilon(x)/2}(x) | x \in V_a\} | V_a \in \mathfrak{V}_i\}$ is an open collection of R with $\bigcup_{i=1}^{n+1} \mathfrak{V}'_i < \mathfrak{U}$. Selecting a covering $\mathfrak{W}$ with $\mathfrak{W}^* < \bigcup_{i=1}^{n+1} \mathfrak{V}'_i$, we can define an open collection $\mathfrak{U}_i$ by

$$\mathfrak{U}_i = \{\bigcup \{W | S(W, \mathfrak{W}) \subseteq V'_a\} | V'_a \in \mathfrak{V}'_i\}.$$

It is easy to see from the disjointedness of $\mathfrak{W}'_i$ that $\bigcup_{i=1}^{n+1} \mathfrak{U}_i$ covers R and that each set of $\mathfrak{W}$ intersects at most one of sets of $\mathfrak{U}_i$. Choosing a covering $\mathfrak{V}$ with $\mathfrak{V}^{**} < \mathfrak{W}$, we have open collections and a covering satisfying the required condition (A).

We denote by $\mathfrak{S}_1 > \mathfrak{S}^*_2 > \mathfrak{S}_2 > \mathfrak{S}^*_3 > \ldots$ a sequence of coverings such that $\{S(p, \mathfrak{S}_m) | m = 1, 2, \ldots\}$ is a nbd basis for each point of R, and take a covering $\mathfrak{V}$ and collections $\mathfrak{U}_{1,i}$ $(i = 1, \ldots, n+1)$ satisfying (A) for $\mathfrak{S}_2$, *i. e.*,

$$\mathfrak{V} < \bigcup_{i=1}^{n+1} \mathfrak{U}_{1,i} < \mathfrak{S}_2$$

and $S^2(p, \mathfrak{V})$ intersects at most one set of $\mathfrak{U}_{1,i}$. Let $\mathfrak{U}_{1,i} = \{U_a | a \in A\}$ and define $\mathfrak{N}_{1,i}$ by

$$\mathfrak{N}_{1,i} = \{S(U_a, \mathfrak{V}),\ R - \bigcup_{a \in A} \overline{U}_a | a \in A\} \quad \text{for a fixed } i;$$

then $\mathfrak{N}_{1,i}$ is a covering of order $\leqslant 2$. Moreover, it follows from $\mathfrak{S}^*_2 < \mathfrak{S}_1$ and $\bigcup_{i=1}^{n+1} \mathfrak{U}_{1,i} \supseteq R$ that $\bigwedge_{i=1}^{n+1} \mathfrak{N}_{1,i} < \mathfrak{S}_1$.

Now we notice that

(B) every covering $\mathfrak{P}$ of order $\leqslant 2$ has a locally finite star-refinement $\mathfrak{Q}'$ with order $\leqslant 2$.

To show this we put $\mathfrak{P} = \{P_\delta | \delta \in D\}$ and denote by $\mathfrak{P}'$ a star-refinement of $\mathfrak{P}$. Then

(C) $\mathfrak{M} = \{M_\delta = \bigcup \{P' | S(P', \mathfrak{P}') \subseteq P_\delta,\ P' \in \mathfrak{P}'\} | \delta \in D\}$

is a locally finite refinement of $\mathfrak{P}$ of order $\leqslant 2$. We define an open set L_δ for every $\delta \in D$ such that

(D) $M_\delta - \bigcup_{\delta \neq \delta' \in D} M_{\delta'} \subseteq L_\delta \subseteq \overline{L}_\delta \subseteq M_\delta$

and put

(E) $Q_\delta = L_\delta - \bigcup_{\delta' \neq \delta} \overline{L}_{\delta'}$, $\mathfrak{Q} = \{Q_\delta,\ M_a \cap M_\beta | \delta, a, \beta \in D,\ a \neq \beta\}$.

It is easy to see that $\mathfrak{Q}$ is an open covering satisfying $\mathfrak{Q}^{\Delta} < \mathfrak{P}$ ([16]) order $\mathfrak{Q} \leqslant 2$. To show that $\mathfrak{Q}$ covers R we take an arbitrary point p of R. If $p \in M_\delta$ and $p \notin M_{\delta'}$ $(\delta' \neq \delta)$, then $p \in L_\delta$ by (D). Since $\{\overline{L}_\delta | \delta \in D\}$ is locally finite, it follows from (D) that there exists a nbd $U(p)$ of p such that $U(p) \cap L_{\delta'} = \emptyset$ for every δ': $\delta' \neq \delta$. Hence $p \notin \bigcup_{\delta' \neq \delta} \overline{L_{\delta'}}$ and consequently $p \in Q_\delta$ by (E). Therefore from (C), (D), (E) we get that $\mathfrak{Q}$ covers R and that $S(p, \mathfrak{Q}) = Q_\delta \subseteq M_\delta \subseteq P_\delta$. If $p \in M_a \cap M_\beta$, then $p \notin M_\gamma$ for every γ with $\gamma \neq a, \beta$. It follows from (E) that either $p \notin Q_a$ or $p \notin Q_\beta$. Therefore either

$$S(p, \mathfrak{Q}) = Q_a \cup (M_a \cap M_\beta) \subseteq M_a \subseteq P_a$$

or

$$S(p, \mathfrak{Q}) = Q_\beta \cup (M_a \cap M_\beta) \subseteq M_\beta \subseteq P_a,$$

which shows that $\mathfrak{Q}^{\Delta} < \mathfrak{P}$ and order $\mathfrak{Q} \leqslant 2$. Repeating such a process we have a locally finite Δ-refinement $\mathfrak{Q}'$ of $\mathfrak{Q}$ with order $\leqslant 2$. $\mathfrak{Q}'$ satisfies the required condition of (B).

To show the existence of sequences $\mathfrak{N}_{1,i} > \mathfrak{N}^*_{2,i} > \mathfrak{N}_{2,i} > \mathfrak{N}^*_{3,i} > \ldots$ $(i = 1, \ldots, n+1)$ of coverings of order $\leqslant 2$ such that $\bigwedge_{i=1}^{n+1} \mathfrak{N}_{m,i} < \mathfrak{S}_m$,

([16]) $\mathfrak{Q}^{\Delta} = \{S(p, \mathfrak{Q}) | p \in R\}$.

we assume the existence of such $\mathfrak{N}_{l,i}$ for $l \leqslant m$. Then there exists by (B) a locally finite covering $\mathfrak{N}_i$ of order $\leqslant 2$ such that $\mathfrak{N}_i^* < \mathfrak{N}_{m,i}$. Let us select a covering $\mathfrak{M}$ with $\mathfrak{M}^{**} < \bigwedge_{i=1}^{n+1} \mathfrak{N}_i$. Next we select by (A) a covering $\mathfrak{Q}$ and open collections $\mathfrak{P}_i$ $(i = 1, \ldots, n+1)$ such that

(F) $\mathfrak{Q} < \bigcup_{i=1}^{n+1} \mathfrak{P}_i < \mathfrak{M} \wedge \mathfrak{S}_{m+2}$

and such that each $S^2(p, \mathfrak{Q})$ intersects at most one of sets belonging to $\mathfrak{P}_i$ for a fixed i. We put $\mathfrak{P}_i = \{P_\beta | \beta \in B\}$, $\mathfrak{N}_i = \{N_\gamma | \gamma < \tau\}$ and denote by $\gamma(\beta)$ the first ordinal γ satisfying

(G) $\overline{S(P_\beta, \mathfrak{Q})} \subseteq N_\gamma \in \mathfrak{N}_i$

for $\beta \in B$. Then we define a covering $\mathfrak{N}_{m+1,i}$ by

(H) $\mathfrak{N}_{m+1,i} = \{K_\gamma, S(P_\beta, \mathfrak{Q}) | \gamma < \tau, \beta \in B\}$,

where we put

(I) $K_\gamma = N_\gamma - \bigcup \{\overline{P}_\beta \,|\, \gamma = \gamma(\beta)\} \cup \{\overline{S(P_\beta, \mathfrak{Q})} \,|\, \gamma \neq \gamma(\beta)\}$.

It easily follows that

(J) $\mathfrak{N}_{m+1,i} < \mathfrak{N}_i$ and order $\mathfrak{N}_{m+1,i} \leqslant 2$.

Since $\mathfrak{N}_{m+1,i} < \mathfrak{N}_i$ is obvious, let us prove the latter assertion. We denote by x an arbitrary point of R. First we consider the case of $x \notin S(P_\beta, \mathfrak{Q})$ $(\beta \in B)$. Then either $x \in \overline{S(P_\beta, \mathfrak{Q})}$ for some $\beta \in B$ or $x \notin \overline{S(P_\beta, \mathfrak{Q})}$ for every $\beta \in B$. If the former is the case, then $x \notin \overline{S(P_{\beta'}, \mathfrak{Q})}$ for every β' with $\beta' \neq \beta$ because $S^2(x, \mathfrak{Q})$ intersects at most one set of $\mathfrak{P}_i$. Hence it follows from (G), (I) that $x \in K_{\gamma(\beta)}$ and $x \notin K_\gamma$ $(\gamma \neq \gamma(\beta))$. If the latter is the case, then it follows from (I) and order $\mathfrak{N}_i \leqslant 2$ that $x \in K_\gamma$ for some (at most two) γ such that $x \in N_\gamma$. Next we consider the case of $x \in S(P_\beta, \mathfrak{Q})$. Then $x \notin S(P_{\beta'}, \mathfrak{Q})$ for every β' with $\beta \neq \beta'$ as in the above discussion. Since $x \notin K_\gamma$ for $\gamma \neq \gamma(\beta)$ is obvious from (G), by (H) x is contained in at most two sets of $\mathfrak{N}_{m+1,i}$. Thus in every case x is contained in at most two sets of $\mathfrak{N}_{m+1,i}$, *i. e.*, $\mathfrak{N}_{m+1,i}$ is a covering of order $\leqslant 2$.

Let $K_{\gamma_i} \in \mathfrak{N}_{m+1,i}$ $(i = 1, \ldots, n+1)$ and let x be an arbitrary point of R; then $x \in P_\beta \in \mathfrak{P}_i$ for some i because $\bigcup_{i=1}^{n+1} \mathfrak{P}_i$ covers R. Hence $x \notin K_{\gamma_i}$ by (I), which shows $\bigcap_{i=1}^{n+1} K_{\gamma_i} = \emptyset$. Therefore every set of $\bigwedge_{i=1}^{n+1} \mathfrak{N}_{m+1,i}$ is contained in $S(P_\beta, \mathfrak{Q})$ with $P_\beta \in \mathfrak{P}_i$ for some i, which implies

(K) $\bigwedge_{i=1}^{n+1} \mathfrak{N}_{m+1,i} < (\bigvee_{i=1}^{n+1} \mathfrak{P}_i)^* < \mathfrak{S}_{m+2}^* < \mathfrak{S}_{m+1}$

by (F). It follows from (J) and $\mathfrak{N}_i^* < \mathfrak{N}_{m,i}$ that $\mathfrak{N}_{m+1,i}^* < \mathfrak{N}_{m,i}$. This combined with (J), (K) completes the induction, and hence we get sequences

$$\mathfrak{N}_{1,i} > \mathfrak{N}_{2,i}^* > \mathfrak{N}_{2,i} > \mathfrak{N}_{3,i}^* > \ldots \quad (i = 1, \ldots, n+1)$$

such that $\bigwedge_{i=1}^{n+1} \mathfrak{N}_{m,i} < \mathfrak{S}_m$, order $\mathfrak{N}_{m,i} \leqslant 2$. Hence by Theorem 7 we can imbed R into a topological product of $n+1$ metrizable spaces R_i with $\dim R_i \leqslant 1$.

§ 4. Imbedding n-dimensional spaces in $E_{2n+1} \times N(\Omega)$.

Definition. We call a covering $\mathfrak{U}$ *star-finite (star-countable)* if every set of $\mathfrak{U}$ intersects finitely (countably) many sets of $\mathfrak{U}$.

An open basis consisting of an enumerable number of star-finite (star-countable) open coverings is called a *σ-star-finite* (*σ-star-countable*) *open basis*.

Remark. A regular space R has a σ-star-finite basis if and only if R has a σ-star-countable basis. Moreover K. Morita has proved the following theorem: A regular space having a σ-star-finite (σ-star-countable) basis can be imbedded in the topological product $N(\Omega) \times I^\omega$ of a generalized Baire 0-dimensional space $N(\Omega)$ [17] and Hilbert cube I^ω, and the converse is also true.

Remark. A metric space having a σ-star-finite basis need not have the star-finite property or the star-countable property [18]. For example, $N(\Omega) \times \{x | 0 < x < 1\}$ has obviously a σ-star-finite basis, but it has not the star-countable propetry if the cardinal number of Ω is greater than $\aleph_0$. For if we put

$$S(\alpha_1, \alpha_2, \ldots, \alpha_k) = \{p | p = (\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots) \in N(\Omega)\},$$

then it is easily seen that the open covering

$$\{N(\Omega) \times \{x | 1/2 < x < 1\}, S(\alpha_1) \times \{x | 1/2^2 < x < 1/2 + 1/2^2\}, \ldots, S(\alpha_1, \ldots, \alpha_k) \times \\ \times \{x | 1/2^{k+1} < x < 1/2^k + 1/2^{k+1}\}, \ldots \,|\, \alpha_i \in \Omega \ (i = 1, 2, \ldots)\}$$

of this space has no star-countable refinement and accordingly no star-finite refinement. To see this we assume that $\mathfrak{U}$ is a star-countable refinement of this covering. Then $\bigcup_{n=1}^{\infty} S^n(U, \mathfrak{U})$ for an arbitrary $U \in \mathfrak{U}$ consisting of countably many sets of $\mathfrak{U}$. We can select $S(\alpha_1, \ldots, \alpha_k) \times I \subseteq U$. It follows from the connectedness of $\{x | 0 < x < 1\}$ that

$$\bigcup_{n=1}^{\infty} S^n(U, \mathfrak{U}) \supseteq S^n(\alpha_1, \ldots, \alpha_k) \times \{x | 0 < x < 1\}.$$

---

(17) This notion is due to [5]. For any two sequences of elements from an abstract set Ω $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots)$, we define the metric $d'(\alpha, \beta)$ by

$$d'(\alpha, \beta) = 1/\min\{k | \alpha_k \neq \beta_k\}, \quad d'(\alpha, \alpha) = 0.$$

Then the set $N(\Omega)$ of all such sequences turns out to be a zero-dimensional space.

(18) We say that R has the star-finite (star-countable) property if only if every open covering of R has a star-finite (star-countable) open refinement.

Hence $\bigcup_{n=1}^{\infty} S^n(U, \mathfrak{U})$ contains every set of $\mathfrak{U}$ contained in $S(a_1, \ldots, a_k, a_{k+1}) \times$ $\times \{x | 1/2^{k+2} < x < 1/2^{k+1} + 1/2^{k+2}\}$ for some $a_{k+1} \in \Omega$, i. e., $\bigcup_{n=1}^{\infty} S^n(U, \mathfrak{U})$ contains non-enumerably many sets of $\mathfrak{U}$, which is a contradiction.

THEOREM 9. *Suppose that R is a regular space having a σ-star-finite (σ-star-countable) basis and $\dim R \leqslant n$. Then R is homeomorphic to a subset of $N(\Omega) \times I_{2n+1}$, where I_{2n+1} is a $(2n+1)$-dimensional Euclidean cube and $N(\Omega)$ is the generalized Baire 0-dimensional space for a set Ω whose cardinal number is not less than the cardinal number of an open basis of R.*

Proof. 1. There exists, as is seen from the above Morita's theorem, a sequence $\mathfrak{N}_1 > \mathfrak{N}_2 > \mathfrak{N}_3 > \ldots$ of star-finite open coverings $\mathfrak{N}_m$ of R such that $\{S(p, \mathfrak{N}_m) | m = 1, 2, \ldots\}$ is a nbd basis of every point p of R. We define a disjointed covering $\mathfrak{S}_m$ of R by $\mathfrak{S}_m = \{S^\infty(N, \mathfrak{N}_m) | N \in \mathfrak{N}_m\}$, where $S^\infty(N, \mathfrak{N}_m) = \bigcup_{n=1}^{\infty} S^n(N, \mathfrak{N}_m)$. Let $\mathfrak{S}_m = \{S_a | a \in A_m'\}$ and $S_a \cap S_\beta = \emptyset$ $(a \neq \beta)$; then for every $a \in A_m$ S_a is a countable sum of sets of $\mathfrak{N}_m$, i. e.,

$$S_a = \bigcup \{N_{a,i}^{(m)} | i = 1, 2, \ldots\}, \qquad N_{a,i}^{(m)} \in \mathfrak{N}_m \qquad (i = 1, 2, \ldots).$$

Since $\mathfrak{N}_m$ is locally finite, there exists an open covering $\mathfrak{P}_m$ of R such that

$$\mathfrak{P}_m = \{P_{a,i}^{(m)} | a \in A_m,\ i = 1, 2, \ldots\}, \qquad \overline{P_{a,i}^{(m)}} \subseteq N_{a,i}^{(m)}.$$

Next, we define a sequence of open coverings by

$$\mathfrak{U}_{i,m} = \{N_{a,i}^{(m)},\ S_a - \overline{P_{a,i}^{(m)}} | a \in A_m\},$$

$$\mathfrak{U}_1 = \mathfrak{U}_{1,1},\ \mathfrak{U}_2 = \mathfrak{U}_1 \wedge \mathfrak{U}_{2,1} \wedge \mathfrak{U}_{1,2},\ \ldots,\ \mathfrak{U}_m = \mathfrak{U}_{m-1} \wedge \mathfrak{U}_{m,1} \wedge \mathfrak{U}_{m-1,2} \wedge \ldots \wedge \mathfrak{U}_{1,m},\ \ldots$$

Then $\mathfrak{U}_1 > \mathfrak{U}_2 > \mathfrak{U}_3 > \ldots$, and $\{S(p, \mathfrak{U}_m) | m = 1, 2, \ldots\}$ is a nbd basis of each point p of R, and

(A) $\mathfrak{U}_m$ is finite in every S_a $(a \in A_k,\ k \geqslant m)$.

Let $\Omega = \bigcup_{n=1}^{\infty} A_m$; then it is clear from the disjointedness of $\mathfrak{S}_m$ that $|\Omega| \leqslant$ the cardinal number of any open basis of R. We define a continuous mapping $c(x)$ of R into $N(\Omega)$ by

$$c(x) = (a_1, a_2, \ldots) \qquad (x \in S_{a_m},\ a_m \in A_m;\ m = 1, 2, \ldots)$$

and denote by $M(R)$ the totality of a continuous mapping φ of R into $N(\Omega) \times I_{2n+1}$ such that $\varphi(x) = \big(c(x), \Phi(x)\big)$ $(x \in R)$ for a continuous mapping $\Phi(x)$ of R into I_{2n+1}.

Moreover we define the following notions, which will be needed later on.

$$T_a = c(S_a) = \{(a_1, a_2, \ldots) | a_m = a\} \cap c(R) \qquad \text{for} \quad a \in A_m,$$
$$\mathfrak{T}_a = \{T_a \times S_{1/m}(x) | x \in I_{2n+1}\}$$

where we denote by $S_{1/m}(x)$ the spherical nbd of radius $1/m$ around x in I_{2n+1}. We mean by a *star-decomposition* a disjointed covering of R consisting of open sets contained in $\bigcup_{m=1}^{\infty} \mathfrak{S}_m$. Let C: $\{S_\gamma | \gamma \in C\ (\subseteq \bigcup_{m=1}^{\infty} A_m)\}$ be a star-decomposition; then for every $\gamma \in C$ we denote by $m(\gamma)$ such a number that $\gamma \in A_{m(\gamma)}$.

We denote by $M(R, m)$ the totality of mappings of $M(R)$ satisfying

$$f^{-1}(\mathfrak{T}_\gamma) = \{f^{-1}(T) | T \in \mathfrak{T}_\gamma\} < \mathfrak{U}_m \qquad (\gamma \in C)$$

for some star-decomposition C: $\{S_\gamma | \gamma \in C\}$ of R. Finally we define *C-neighborhood* $N_C(f)$ of $f \in M(R)$ by

$$N_C(f) = \{g | g \in M(R),\ \sup\{d(\pi f(x), \pi g(x)) | x \in S_\gamma\} < 1/m(\gamma)\}$$

for a star-decomposition C: $\{S_\gamma | \gamma \in C\}$, where π and d denote the projection of $N(\Omega) \times I_{2n+1}$ onto I_{2n+1} and the metric of I_{2n+1} respectively.

2. First we prove

(B) $N_C(f) \cap M(R, m) \neq \emptyset$ for every $f \in M(R)$, every star-decomposition C and every positive integer m.

Take $l(\gamma) = \max\big(6m(\gamma), m\big)$ for every $\gamma \in C$ and put

$$D_\gamma = \{\delta | \delta \in A_{l(\gamma)},\ T_\delta \subseteq T_\gamma \text{ (or } S_\delta \subseteq S_\gamma \text{ as the same)}\}.$$

Since we can cover I_{2n+1} by a finite subcovering of $\{S_{1/l(\gamma)}(x) | x \in I_{2n+1}\}$, we denote by $\{S_{1/l(\gamma)}(x_i) | i = 1, 2, \ldots, a(\gamma)\}$ such a covering; then $\mathfrak{T}_\delta' = \{T_\delta \times S_{1/l(\gamma)}(x_i) |\ i = 1, 2, \ldots, a(\gamma)\}$ is a finite subcovering of $\mathfrak{T}_\delta = \{T_\delta \times S_{1/l(\gamma)}(x) | x \in I_{2n+1}\}$ $(\delta \in A_{l(\gamma)})$. Since $f^{-1}(\mathfrak{T}_\delta') = \{f^{-1}(T') | T' \in \mathfrak{T}_\delta'\}$ and $\mathfrak{U}_m$ are, by $l(\gamma) \geqslant m$ and (A), finite open coverings of S_δ, we have an open finite covering $\mathfrak{B}_\delta$ of S_δ satisfying order $\mathfrak{B}_\delta \leqslant n+1$, $\mathfrak{B}_\delta^d < \mathfrak{U}_m \wedge f^{-1}(\mathfrak{T}_\delta')$. $\mathfrak{B} = \bigcup \{\mathfrak{B}_\delta | \delta \in D_\gamma,\ \gamma \in C\}$ is an open covering of R of order $\leqslant n+1$.

Let us consider fixed $\gamma \in C$ and $\delta \in D_\gamma$, and assume that $V_1, \ldots, V_s$ are all the numbers of $\mathfrak{B}_\delta$. Then we select vertices $x(V_i)$ $(i = 1, \ldots, s)$ in I_{2n+1} for which it is true that $d\big(\pi f(V_i), x(V_i)\big) < 1/3m(\gamma)$ $(i = 1, \ldots, s)$, the $x(V_i)$ are in a general position in E_{2n+1}, i. e., no $m+2$ of the vertices $x(V_i)$ $(m = 0, 1, \ldots, 2n)$ lie in an m-dimensional linear subspace of E_{2n+1}. We define a barycentric mapping Φ_δ of S_δ into I_{2n+1} by

$$\text{(C)}\quad \Phi_\delta(p) = \frac{\sum_{i=1}^{s} \varrho(p, V_i^c)\, x(V_i)}{\sum_{i=1}^{s} \varrho(p, V_i^c)} \quad (p \in S_\delta),$$

where we consider $x(V_i)$ $(i = 1, \ldots, s)$ as vectors and denote by

$$\varrho(x, V_i^c \inf\{\varrho(p, q) | q \in V_i^c\}\ (^{19}).$$

Thus we get a continuous mapping

$$\Phi(p) = \Phi_\delta(p) \quad (p \in S_\delta,\ \delta \in D_\gamma,\ \gamma \in C)$$

of R into I_{2n+1}. We now prove that the mapping $\varphi(p) = (c(p), \Phi(p)) \in M(R)$ is contained in the common part of $N_C(f)$ and $M(R, m)$.

To prove $\varphi \in N_C(f)$ we take an arbitrary point $p \in S_\gamma$ for $\gamma \in C$. Then $p \in S_\delta$ for some $\delta \in D_\gamma$. Assume that V_i are so numbered that $\{V_1, \ldots, V_t\}$ is the set of all the $V_i \in \mathfrak{V}_\delta$ which contain p. Then $\varrho(p, V_i^c) = 0$ for $i > t$. From

$$\delta(\pi f(V_i)) \leqslant 2l(\gamma) \leqslant 1/3m(\gamma) \quad \text{and} \quad d(\pi f(V_i), x(V_i)) < 1/3m(\gamma)$$

we get

$$d(x(V_i), \pi f(p)) < 2/3m(\gamma) \quad (i = 1, 2, \ldots, t).$$

A fortiori, the centre of gravity $\Phi(p)$ of the $x(V_i)$ satisfies

$$d(\Phi(p), \pi f(p)) < 2/3m(\gamma) < 1/m(\gamma).$$

Therefore $\varphi \in N_C(f)$.

Next in order to show that $\varphi \in M(R, m)$ we fix $\gamma \in C$ and $\delta \in D_\gamma$ and suppose that $V_{i_1}, \ldots, V_{i_t}$ are all the members of $\mathfrak{V}_\delta$ containing a given point p of S_δ. Consider the linear $(t-1)$-space $L_\delta(x)$ in I_{2n+1} spanned by the vertices $x(V_{i_1}), \ldots, x(V_{i_t})$; then $t \leqslant n+1$ and $\Phi_\delta(p) \in L_\delta(p)$ are obvious from (C).

Since there are only a finite number of linear subspaces $L_\delta(p)$, there exists a positive number $h(\delta) > l(\delta)$ such that any two of these linear subspaces $L_\delta(p)$ and $L_\delta(p')$ either meet or are at a distance $\geqslant 2/h(\delta)$ from each other.

Putting $E_\delta = \{\varepsilon | \varepsilon \in A_{h(\delta)},\ T_\varepsilon \subseteq T_\delta\}$, we consider a star-decomposition

$$E: \{S_\varepsilon | \varepsilon \in E_\delta,\ \delta \in D_\gamma,\ \gamma \in C\}.$$

If $\varphi(p), \varphi(p') \in T_\varepsilon \times S_{1/h(\delta)}(x)$ for $p, p' \in R$, then it follows that $c(p)$, $c(p') \in T_\varepsilon$ and

$$d(\Phi(p), x) < 1/h(\delta), \quad d(\Phi(p'), x) < 1/h(\delta);$$

[19] $\varrho(p, q)$ denotes the metric of R.

hence $p, p' \in S_\varepsilon \subseteq S_\delta$. Therefore we get $d(\Phi(p), \Phi(p')) < 2/h(\delta)$, which implies $L_\delta(p) \cap L_\delta(p') \neq \emptyset$. If we suppose that $L_\delta(p')$ is spanned by $x(V_{j_1}), \ldots, x(V_{j_u})$, $u \leqslant n+1$, then since $x(V_{i_1}), \ldots, x(V_{i_t}), x(V_{j_1}), \ldots, x(V_{j_u})$ are in a general position in E_{2n+1}, it follows that at least one of $x(V_{j_1}), \ldots, x(V_{j_u})$ is also one of $x(V_{i_1}), \ldots, x(V_{i_t})$. Hence p and p' are contained in a common member V_i of $\mathfrak{V}_\delta$, i. e., $p' \in S(p, \mathfrak{V}_\delta)$. It follows from $\mathfrak{V}_\delta^\Delta < \mathfrak{U}_m$ that $\varphi^{-1}(T_\varepsilon \times S_{1/h(\delta)}(x)) \subseteq U$ for some $U \in \mathfrak{U}_m$. Thus we get $\varphi^{-1}(\mathfrak{T}_\varepsilon) < \mathfrak{U}_m$ for every $\varepsilon \in E$, proving $\varphi \in M(R, m)$. We now prove that

(D) for a given $\varphi \in N_C(f) \cap M(R, m)$ there exists a star-decomposition C' satisfying $N_{C'}(\varphi) \subseteq N_C(f) \cap M(R, m)$.

Since $\varphi \in N_C(f)$ implies

$$\sup\{d(\pi f(p), \pi\varphi(p)) | p \in S_\gamma\} = \alpha_\gamma < 1/m(\gamma) \quad (\gamma \in C),$$

we take a positive integer $h(\gamma)$ for $\gamma \in C$: $1/h(\gamma) < 1/m(\gamma) - \alpha_\gamma$ and define a star-decomposition D by

$$D: \{S_\delta | \delta \in D_\gamma,\ \gamma \in C\} \quad (D_\gamma = \{\delta | \delta \in A_{h(\gamma)},\ T_\delta \subseteq T_\gamma\}).$$

Let $\psi \in N_D(\varphi)$; then taking $p \in S_\delta$, $\delta \in D_\gamma$ for a given point p of S_γ, we get $d(\pi\varphi(p), \pi\psi(p)) < 1/h(\gamma)$. Therefore

$$\sup\{d(\pi f(p), \pi\psi(p)) | p \in S_\gamma\} \leqslant \alpha_\gamma + 1/h(\gamma) < 1/m(\gamma),$$

proving $\psi \in N_C(f)$, i. e., it holds that

(E) $N_D(\varphi) \subseteq N_C(f)$.

Moreover, since $\varphi \in M(R, m)$, we have $\varphi^{-1}(\mathfrak{T}_\beta) < \mathfrak{U}_m$ $(\beta \in B)$ for some star-decomposition B, where $\mathfrak{T}_\beta = \{T_\beta \times S_{1/m(\beta)}(x) | x \in I_{2n+1}\}$ as above defined. Putting $D_\beta = \{\delta | \delta \in A_{2m(\beta)},\ T_\delta \subseteq T_\beta\}$ for every $\beta \in B$, we have a star-decomposition E: $\{S_\varepsilon | \varepsilon \in D_\beta,\ \beta \in B\}$. Let $\psi \in N_E(\varphi)$; then we easily see that $\varepsilon \in D_\beta$ implies

$$\psi^{-1}(T_\varepsilon \times S_{1/2m(\beta)}(x)) \subseteq \varphi^{-1}(T_\beta \times S_{1/m(\beta)}(x))$$

for every $x \in I_{2n+1}$. For if we assume the contrary, then there exists a point p of R such that

$$\psi(p) \in T_\varepsilon \times S_{1/2m(\beta)}(x), \quad \varphi(p) \notin T_\beta \times S_{1/m(\beta)}(x).$$

Hence $p \in S_\varepsilon$, $\pi\psi(p) \in S_{1/2m(\beta)}(x)$ and $\pi\varphi(p) \notin S_{1/m(\beta)}(x)$, and hence $d(\pi\psi(p), \pi\varphi(p)) \geqslant 1/2m(\beta)$, which contradicts $\psi \in N_\varepsilon(\varphi)$. Thus we must have

$$\psi^{-1}(T_\varepsilon \times S_{1/2m(\beta)}(x)) \subseteq \varphi^{-1}(T_\beta \times S_{1/m(\beta)}(x)) \subseteq U$$

for some $U \in \mathfrak{U}_m$. Therefore $\psi^{-1}(\mathfrak{T}_\varepsilon) < \mathfrak{U}_m$ $(\varepsilon \in E)$, proving $\psi \in M(R, m)$, i. e., $N_E(\varphi) \subseteq M(R, m)$. This combined with (E) shows that $C' = D \wedge E$

$= \{S_\delta \cap S_\varepsilon | \delta \in D,\ \varepsilon \in E\}$ is a star-decomposition satisfying $N_{C'}(\varphi) \subseteq N_C(f) \cap M(R, m)$.

3. We can select by (B) and (D) two sequences $C_1 > C_2 > C_3 > \ldots$, $D_1 > D_2 > D_3 > \ldots$ of star-decompositions and a sequence $f_1, f_2, f_3, \ldots$ of elements of $M(R)$ such that

$$\gamma \in C_m \quad \text{implies} \quad m(\gamma) > m,$$

$$N_{C_1}(f_1) \subseteq M(R, 1),$$

$$D_1\colon \{T_\delta | \delta \in D_{1,\gamma},\ \gamma \in C_1\} \qquad \big(D_{1,\gamma} = \{\delta | \delta \in A_{2m(\gamma)},\ T_\delta \subseteq T_\gamma\}\ (\gamma \in C_1)\big),$$

$$N_{C_2}(f_2) \subseteq N_{D_1}(f_1) \cap M(R, 2),$$

$$D_2\colon \{T_\delta | \delta \in D_{2,\gamma},\ \gamma \in C_2\} \qquad \big(D_{2,\gamma} = \{\delta | \delta \in A_{2m(\gamma)},\ T_\delta \subseteq T_\gamma\}\ (\gamma \in C_2)\big),$$

. . . .

$$N_{C_h}(f_h) \subseteq N_{D_{h-1}}(f_{h-1}) \cap M(R, h),$$

$$D_h\colon \{T_\delta | \delta \in D_{h,\gamma},\ \gamma \in C_h\} \qquad \big(D_{h,\gamma} = \{\delta | \delta \in A_{2m(\gamma)},\ T_\delta \subseteq T_\gamma\}\ (\gamma \in C_h)\big),$$

. . . .

Then, since $f_h \in N_{D_k}(f_k)$ $(h \geqslant k)$, we have

$$d\big(\pi f_k(p),\ \pi f_h(p)\big) < 1/2m(\gamma) < 1/k \qquad (h \geqslant k)$$

for some $\gamma \in C_k$, and hence $\{\pi f_h(p) | h = 1, 2, \ldots\}$ uniformly converges to a continuous mapping $\Phi(p)$ of R into I_{2n+1}.

Let us show that

$$\big(c(p),\ \Phi(p)\big) = \varphi(p) \in \bigcap_{m=1}^{\infty} M(R, m).$$

Since $f_h \in N_{D_k}(f_k)$ $(h \geqslant k)$, if we take, for given $\gamma \in C_k$ and $p \in S_\gamma$, $\delta \in D_{k,\gamma}\colon p \in S_\delta$, then

$$d\big(\pi f_h(p),\ \pi f_k(p)\big) < 1/2m(\gamma) \qquad (h \geqslant k);$$

hence

$$d\big(\Phi(p),\ \pi f_k(p)\big) \leqslant 1/2m(\gamma) < 1/m(\gamma) \qquad (p \in S_\gamma).$$

Therefore

$$\varphi(p) \in N_{C_k}(f_k) \subseteq M(R, k), \qquad i.\ e., \qquad \varphi(p) \in \bigcap_{m=1}^{\infty} M(R, m).$$

Thus we get a homeomorphic mapping φ of R into $N(\Omega) \times I_{2n+1}$.

COROLLARY 4. *Let R be a metric space having the local Lindelöff property* [20] *such that* $\dim R \leqslant n$. *Then R is homeomorphic to a subset of* $N(\Omega) \times I_{2n+1}$.

Proof. Since every metric space with the local Lindelöff property has a σ-star-countable basis, this proposition is a direct consequence of Theorem 9.

THEOREM 10. *In order that a metric space R with a σ-star-finite (countable) basis have dimensions $\leqslant n$ and have an open basis whose cardinal number is not greater than $\mathfrak{n}$ it is necessary and sufficient that R be homeomorphic with a subset of $N(\Omega) \times M_{2n+1}^n$, where Ω is a set with $|\Omega| = \mathfrak{m}$, and M_{2n+1}^n is the set of points in I_{2n+1} at most n of whose coordinates are rational.*

Proof. Since it is well known that $\dim M_{2n+1}^n = n$, $\dim N(\Omega) \times M_{2n+1}^n = n$ from the generalized product theorem (see [4] and [5]). Hence the sufficiency is obvious.

The proof of the necessity is analogous to that of Theorem 9. Let $L_1, L_2, \ldots$ be a sequence of n-dimensional linear subspaces in I_{2n+1}; then we shall prove generally that R is homeomorphic with a subset of $N(\Omega) \times (I_{2n+1} - \bigcup_{m=1}^{\infty} L_m)$. If $L_1, L_2, \ldots$ are all the linear spaces in I_{2n+1} of the form $x_{i_1} = r_1, \ldots, x_{i_{n+1}} = r_{n+1}$, the r's being rational, then we get the necessity part of this proposition. To show this we generally use the same notation as the above, but we replace $M(R, m)$ in the above proof by

$$N(R, m) = \{\varphi | \varphi \in M(R),\ \varphi^{-1}(\mathfrak{T}_\gamma) < \mathfrak{U}_m\ (\gamma \in C),$$
$$\overline{\pi\varphi(S_\gamma)} \cap L_m = \emptyset \ \ (\gamma \in C) \text{ for some star-decomposition } C\}.$$

The part 1 of the above proof (of Theorem 9) is suitable for the present proof too.

We now prove $N_C(f) \cap N(R, m) \neq \emptyset$ for every $f \in M(R)$, every star-decomposition C and every positive integer m. We define D_γ $(\gamma \in C)$ and $\mathfrak{B}_\delta$ $(\delta \in D_\gamma,\ \gamma \in C)$ in the same way as in the proof of Theorem 9 [21] and consider fixed $\gamma \in C$ and $\delta \in D_\gamma$. Assume that $V_1, \ldots, V_s$ are all the members of $\mathfrak{B}_\delta$. Then we select vertices $x(V_i)$ $(i = 1, \ldots, s)$ in I_{2n+1} and $p_0, p_1, \ldots, p_n$ in L_m for which it is true that $d\big(\pi f(V_i),\ x(V_i)\big) < 1/3m(\gamma)$, the $x(V_i)$ and p_j are in a general position in E_{2n+1}. Defining $\varphi(p) \in M(R)$ by (C) in the above proof, we see $\varphi \in N_C(f)$ in the same way.

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[20] We mean by Lindelöff property the property that every open covering has a countable subcovering. If every point of R has a nbd whose closure has the Lindelöff property, then R is said to have the local Lindelöff property.

[21] From now on we omit "in the proof of Theorem 9" for brevity.

To show that $\varphi \in N(R, m)$ we consider fixed $\gamma \in C$ and $\delta \in D_\gamma$ and suppose that $V_{i_1}, \ldots, V_{i_t}$ are all the members of $\mathfrak{V}_\delta$ containing a given point p of S_δ. We denote by $L_\delta(p)$ the linear $(t-1)$-space in I_{2n+1} spanned by the vertices $x(V_{i_1}), \ldots, x(V_{i_t})$. Then $\Phi_\delta(p) \in L_\delta(p) \subseteq I_{2n+1} - L_m$ and there exists $h(\delta) > 0$ such that $L_\delta(p) \frown L_\delta(p') = \emptyset$ implies $d(L_\delta(p), L_\delta(p')) \geqslant 2/h(\delta)$. Defining a star-decomposition E in the same way, we have $\varphi^{-1}(\mathfrak{T}_\varepsilon) < \mathfrak{U}_m$ $(\varepsilon \in E)$ and

$$\overline{\pi\varphi(S_\varepsilon)} = \overline{\Phi_\delta(S_\varepsilon)} \subseteq \bigcup \{L_\delta(p) | p \in S_\delta\} \subseteq I_{2n+1} - L_m,$$

proving $\varphi \in N(R, m)$.

Next, in order to show that for every $\varphi \in N_C(f) \frown N(R, m)$ there exists a star-decomposition C' satisfying $N_{C'}(\varphi) \subseteq N_C(f) \frown N(R, m)$, we shall prove $N_E(\varphi) \subseteq N(R, m)$ for some star-decomposition E. Since $\varphi \in N(R, m)$,

$$\varphi^{-1}(\mathfrak{T}_\beta) < \mathfrak{U}_m \quad (\beta \in B), \qquad \delta(\overline{\pi\varphi(S_\beta)}, L_m) \geqslant 1/l(\beta) > 0 \qquad (\beta \in B)$$

for some star-decomposition B and positive integers $l(\beta)$ $(\beta \in B)$. Letting

$$\max(2m(\beta), l(\beta)) = k(\beta), \qquad E_\beta = \{\varepsilon | \varepsilon \in A_{k(\beta)},\ T_\varepsilon \subseteq T_\beta\} \qquad (\beta \in B),$$

we have a star-decomposition E: $\{S_\varepsilon | \varepsilon \in E_\beta,\ \beta \in B\}$. For an arbitrary $\psi \in N_E(\varphi)$ $\psi^{-1}(\mathfrak{T}_\varepsilon) < \mathfrak{U}_m$ $(\varepsilon \in E)$ is proved in the same way. Moreover $p \in S_\varepsilon$ implies

$$d(\pi\psi(p), \pi\varphi(p)) < 1/k(\beta) - \eta(\varepsilon) \leqslant 1/l(\beta) - \eta(\varepsilon)$$

for some $\eta(\varepsilon) > 0$. Therefore

$$d(\pi\psi(S_\varepsilon), L_m) \geqslant \eta(\varepsilon) > 0 \qquad (\varepsilon \in E),$$

which means $\overline{\pi\psi(S_\varepsilon)} \frown \overline{L_m} = \emptyset$ $(\varepsilon \in E)$. Hence $\psi \in N(R, m)$, *i. e.*, $N_E(\varphi) \subseteq N(R, m)$. Thus we can conclude that $N_{C'}(\varphi) \subseteq N_C(f) \frown N(R, m)$ for $C' = D \wedge E$. Since we can prove $\bigcap_{m=1}^{\infty} N(R, m) \neq \emptyset$ in the same way, we have $\varphi(p) \in \bigcap_{m=1}^{\infty} N(R, m)$ which topologically maps R into $N(\Omega) \times M^n_{2n+1}$.

Definition. We say that the *p-dimensional density* of a subset S of a metric space is zero if and only if for every $\varepsilon > 0$ there exists a decomposition $S = \bigcup \{A_{i,\gamma} | \gamma \in C,\ i = 1, 2, \ldots\}$ such that $\delta(A_{i,\gamma}) < \varepsilon$ $(\gamma \in C,$ $i = 1, 2, \ldots)$, $\sum_{i=1}^{\infty} [\delta(A_{i,\gamma})]^p < \varepsilon$ $(\gamma \in C)$ and such that $\bigcup_{i=1}^{\infty} A_{i,\gamma} = S_\gamma$ is open in S for every $\gamma \in C$ and $S_\gamma \frown S_{\gamma'} = \emptyset$ $(\gamma \neq \gamma')$ [22].

[22] This notion and the following theorems are deeply related with Hausdorff's p-dimensional measure and Szpilrajn's theorem respectively. See [7]. We denote by $\delta(A)$ the diameter of A.

Theorem 11. *Every metric space R of $(n+1)$-density zero has dimension $\leqslant n$* [23].

Proof. Let us show that $\operatorname{ind\,dim} R \leqslant n$. We consider an arbitrary pair F, G of closed sets with $\varrho(F, G) > 0$. If we can show the existence of an open set U with $\dim(\overline{U} - U) \leqslant n-1$, then $\operatorname{ind\,dim} R \leqslant n$ is proved [24]. Select a positive integer m with $1/m < \varrho(F, G)$, and let $R = \bigcup \{A_{i,\gamma} | \gamma \in C,\ i = 1, 2, \ldots\}$ be a decomposition of R such that

$$\delta(A_{i,\gamma}) < 1/m^2, \qquad \sum_{i=1}^{\infty} [\delta(A_{i,\gamma})]^{n+1} < 1/m^2$$

and such that $S_\gamma = \bigcup \{A_{i,\gamma} | i = 1, 2, \ldots\}$ is open for every $\gamma \in C$. We put

$$u_{i,\gamma} = \sup\{\varrho(F, x) | x \in A_{i,\gamma}\}, \qquad v_{i,\gamma} = \inf\{\varrho(F, x) | x \in A_{i,\gamma}\}.$$

Then it is easily seen that $u_{i,\gamma} - v_{i,\gamma} \leqslant \delta(A_{i,\gamma})$. We define a non-negatively valued function $d_\gamma(r)$ for every $\gamma \in C$ by

$$d_{i,\gamma}(r) = \begin{cases} 0 & (0 \leqslant r < v_{i,\gamma} \text{ or } u_{i,\gamma} < r) \\ [\delta(A_{i,\gamma})]^n & (v_{i,\gamma} \leqslant r \leqslant u_{i,\gamma}), \end{cases}$$

$$d_\gamma(r) = \sum_{i=1}^{\infty} d_{i,\gamma}(r).$$

It follows from

$$\int_0^{1/m} d_{i,\gamma}(r)\, dr \leqslant [\delta(A_{i,\gamma})]^{n+1}$$

that

$$\int_1^{1/m} d_\gamma(r)\, dr = \int_0^{1/m} \sum_{i=1}^{\infty} d_{i,\gamma}(r)\, dr = \sum_{i=1}^{\infty} \int_0^{1/m} d_{i,\gamma}(r)\, dr \leqslant \sum_{i=1}^{\infty} [\delta(A_{i,\gamma})]^{n+1} < 1/m^2 \qquad (\gamma \in C)$$

since considering $d_{i,\gamma}(r) \geqslant 0$ we may interchange integration and summation by Lebesgue's theorem. This implies $d_r(r(\gamma)) < 1/m$ for some $r(\gamma)$ with $0 < r(\gamma) \leqslant 1/m$. We denote by $S(F, r)$ the set of all the points satisfying $\varrho(F, x) < r$ and by $S(r)$ the boundary of $S(F, r)$. Then $\left[\delta\left(A_{i,\gamma} \frown S(r(\gamma))\right)\right]^n \leqslant d_{i,\gamma}(r(\gamma))$ combined with $d_\gamma(r(\gamma)) < 1/m$ implies $\sum_{i=1}^{\infty} \left[\delta\left(A_{i,\gamma} \frown S(r(\gamma))\right)\right]^n < 1/m$. We notice that $\bigcup \{S_\gamma \frown S(F, r(\gamma)) | \gamma \in C\} = U$

[23] This theorem is an extension of Szpilrajn's theorem "every metric space of $(n+1)$-measure zero has $\dim \leqslant n$ to a non-separable case". See [7].

[24] $\operatorname{ind\,dim} R \leqslant n$ if and only if R has a σ-locally finite open basis $\mathfrak{U}$ such that the boundary of each set of $\mathfrak{U}$ has $\operatorname{ind\,dim} \leqslant n-1$. See [5].

is evidently an open set of R satisfying $F \subseteq U \subseteq G^c$. Since $\overline{U} - U = \bigcup\{A_{i,\gamma} \cap S(r(\gamma)) | \gamma \in C, i = 1, 2, \ldots\}$, the n-dimensional density of $\overline{U} - U$ is zero. The above arguement is also valid for $n = 0$; hence $\overline{U} - U = \emptyset$ for a space R of 1-dimensional density zero, proving $\operatorname{ind dim} R \leqslant 0$. Thus we can inductively establish this theorem.

THEOREM 12. *If a metric space R has dimension $\leqslant n$ and has a σ-star-finite (countable) basis, then it is homeomorphic to a subset S of $N(\Omega) \times I_{2n+1}$ such that $(n+1)$-density of S is zero.*

Proof. We define the distance $d''(x, y)$ between two points $x = (x_1, x_2)$, $y = (y_1, y_2)$ of $N(\Omega) \times I_{2n+1}$ for $x_1, y_1 \in N(\Omega)$, $x_2, y_2 \in I_{2n+1}$ by $d''(x, y) = d'(x_1, y_1) + d(x_2, y_2)$. Replacing $M(R, m)$ in the proof of Theorem 9 by

$O(R, m) = \Big\{\varphi | \varphi \in M(R), \varphi^{-1}(\mathfrak{T}_\gamma) < \mathfrak{U}_m \ (\gamma \in C)$ for some star-decomposition C and there exist decompositions $\overline{\varphi(R)} \cap T_\gamma = \bigcup_{i=1}^{\infty} A_{i,\gamma} \ (\gamma \in C)$ such that

$$\delta(A_{i,\gamma}) < 1/m, \ \sum_{m=1}^{\infty} [\delta(A_{i,\gamma})]^{n+1} < 1/m \ (\gamma \in C, \ i = 1, 2, \ldots)\Big\},$$

we can analogously prove $\bigcap_{m=1}^{\infty} O(R, m) \neq \emptyset$. Part 1 of the proof of Theorem 9 is suitable for the present proof.

To prove $N_C(f) \cap O(R, m) \neq \emptyset$ for every $f \in M(R)$, every star-decomposition C and every positive integer m, we define $\varphi \in M(R)$ by (C) in the proof of Theorem 9. Since $\pi(\varphi(R) \cap T_\delta) = \pi\varphi(S_\delta)$ for a fixed δ is contained in an n-dimensional polytope in I_{2n+1}, $\pi(\overline{\varphi(R)} \cap T_\delta)$ is also contained in an n-dimensional polytope because

$$\pi(\overline{\varphi(R)} \cap T_\delta) \subseteq \pi\overline{\varphi(S_\delta)} \subseteq \overline{\pi\varphi(S_\delta)}.$$

It is well-known that the $(n+1)$-dimensional measure of an n-dimensional polytope is zero (see [7]), and hence, by the compactness of $\pi(\overline{\varphi(R)} \cap T_\delta)$, there exist open sets $K_{i,\delta}$ $(i = 1, 2, \ldots, p(\delta))$ of I_{2n+1} such that

$$\pi(\overline{\varphi(R)} \cap T_\delta) \subseteq \bigcup_{i=1}^{p(\delta)} K_{i,\delta}, \quad \delta(K_{i,\delta}) < 1/m, \quad \sum_{i=1}^{p(\delta)} [\delta(K_{i,\delta})]^{n+1} < 1/m.$$

We can select a positive integer $h(\delta)$ satisfying

$$\sum_{i=1}^{p(\delta)} [\delta(K_{i,\delta}) + 1/h(\delta)]^{n+1} < 1/m, \quad \delta(K_{i,\delta}) + 1/h(\delta) < 1/m \quad (i = 1, \ldots, p(\delta)).$$

$$L(p) \cap L(p') = \emptyset \quad \text{implies} \quad d(L(p), L(p')) \geqslant 2/h(\delta).$$

Then we consider a star-decomposition

$$E\colon \{S_\varepsilon | \varepsilon \in E_\delta, \ \delta \in D_\gamma, \ \gamma \in C\} \text{ for } E_\delta = \{\varepsilon | \varepsilon \in A_{h(\delta)}, \ T_\varepsilon \subseteq T_\delta\}.$$

$\varphi^{-1}(\mathfrak{T}_\varepsilon) < \mathfrak{U}_m$ is proved in the same way. Moreover it follows from

$$\delta(T_\varepsilon) \leqslant \frac{1}{h(\delta)+1}$$

that

$$\delta(T_\varepsilon \times K_{i,\delta}) \leqslant \delta(K_{i,\delta}) + 1/h(\delta) < 1/m,$$

$$\sum_{i=1}^{p(\delta)} [\delta(T_\varepsilon \times K_{i,\delta})]^{n+1} \leqslant \sum_{i=1}^{p(\delta)} [\delta(K_{i,\delta}) + 1/h(\delta)]^{n+1} < 1/m.$$

Since $\overline{\varphi(R)} \cap T_\varepsilon \subseteq \bigcup_{i=1}^{p(\delta)} (T_\varepsilon \times K_{i,\delta})$ is obvious, we have $\varphi \in O(R, m)$.

Next let us prove that $\varphi \in N_C(f) \cap O(R, m)$ implies $N_{C'}(\varphi) \subseteq N_C(f) \cap O(R, m)$ for a suitable star-decomposition C'. We have, by $\varphi \in O(R, m)$, a star-decomposition B such that $\varphi^{-1}(\mathfrak{T}_\beta) < \mathfrak{U}_m$ and such that

$$\overline{\varphi(R)} \cap T_\beta = \bigcup_{i=1}^{\infty} A_{i,\beta}, \quad \delta(A_{i,\beta}) < 1/m, \quad \sum_{i=1}^{\infty} [\delta(A_{i,\beta})]^{n+1} < 1/m \quad (\beta \in B, \ i = 1, 2, \ldots)$$

for some $A_{i,\beta}$. This implies

$$\pi(\overline{\varphi(R)} \cap T_\beta) = \bigcup_{i=1}^{\infty} \pi(A_{i,\beta}), \quad \delta(\pi(A_{i,\beta})) < 1/m, \quad \sum_{i=1}^{\infty} [\delta(\pi(A_{i,\beta}))]^{n+1} < 1/m \quad (\beta \in B, \ i = 1, 2, \ldots).$$

Hence, by the compactness of $\pi(\overline{\varphi(R)} \cap T_\beta)$, there exist open sets $H_{i,\beta}$ $(\beta \in B, \ i = 1, \ldots, q(\beta))$ of I_{2n+1} satisfying

$$\pi(\overline{\varphi(R)} \cap T_\beta) \subseteq \bigcup_{i=1}^{q(\beta)} H_{i,\beta}, \quad \delta(H_{i,\beta}) < 1/m, \quad \sum_{i=1}^{q(\beta)} [\delta(H_{i,\beta})]^{n+1} < 1/m \quad (\beta \in B, \ i = 1, \ldots, q(\beta)).$$

We choose a positive integer $h(\beta)$ for every $\beta \in B$ satisfying

$$\sum_{i=1}^{q(\beta)} [\delta(H_{i,\beta}) + 5/h(\beta)]^{n+1} < 1/m, \quad \delta(H_{i,\beta}) + 5/h(\beta) < 1/m \quad (i = 1, \ldots, q(\beta)).$$

Letting

$$k(\beta)=\max(2m(\beta),h(\beta)), \qquad E_\beta=\{\varepsilon|\varepsilon\in A_{k(\beta)},\ T_\varepsilon\subseteq T_\beta\} \qquad (\beta\in B)$$

we have a star-decomposition E: $\{S_\varepsilon|\varepsilon\in E_\beta,\ \beta\in B\}$.

To prove $N_E(\varphi)\subseteq O(R,m)$, we consider a given $\psi\in N_E(\varphi)$. Then $\psi^{-1}(\mathfrak{T}_\varepsilon)<\mathfrak{U}_m$ $(\varepsilon\in E)$ is proved in the same way. On the other hand, for any $x\in\overline{\psi(R)}\cap T_\varepsilon$ there exists $y\in\psi(R)\cap T_\varepsilon=\psi(S_\varepsilon)$ with $d(\pi(x),\pi(y))<1/h(\beta)$. Let $\psi(\beta)=y$, $p\in S_\varepsilon$; then it follows from

$$d(\pi\psi(p),\pi\varphi(p))<1/k(\beta)\leqslant 1/h(\beta)$$

for $\beta\in B$ with $\varepsilon\in E_\beta$ that $d(\pi(x),\pi\varphi(p))<2/h(\beta)$. That is to say for any $x\in\overline{\psi(R)}\cap T_\varepsilon$ we can select $z\in\varphi(k)\cap T_\varepsilon$ satisfying $d(\pi(x),\pi(x))<2/h(\beta)$. Hence letting

$$B_{i,\varepsilon}=\{x|x\in\overline{\psi(R)}\cap T_\varepsilon,\ d(\pi(x),\pi(z))<2/h(\beta)\ \text{for some}\ \pi(z)\in H_{i,\beta}\}$$

we have $\overline{\psi(R)}\cap T_\varepsilon=\bigcup_{i=1}^{q(\beta)}B_{i,\varepsilon}$. For given $x_1,x_2\in B_{i,\varepsilon}$ we take z_1,z_2 with

$$d(\pi(x_1),\pi(z_1))<2/h(\beta), \qquad d(\pi(x_2),\pi(z_2))<2/h(\beta), \qquad \pi(z_1),\pi(z_2)\in H_{i,\beta}.$$

Therefore $d(\pi(x_1),\pi(x_2))<\delta(H_{i,\beta})+4/h(\beta)$, which implies $d''(x_1,x_2)<\delta(H_{i,\beta})+5/h(\beta)$ since $x_1,x_2\in T_\varepsilon$. Thus we have

$$\delta(B_{i,\varepsilon})\leqslant\delta(H_{i,\beta})+5/h(\beta)<1/m,$$

$$\sum_{i+1}^{q(\beta)}[\delta(B_{i,\varepsilon})]^{n+1}\leqslant\sum_{i=1}^{q(\beta)}[\delta(H_{i,\beta})+5/h(\beta)]^{n+1}<1/m,$$

proving $\psi\in O(R,m)$, *i. e.*, $N_E(\varphi)\subseteq O(R,m)$. This combined with $N_D(\varphi)\subseteq N_C(f)$ for a suitable D implies $N_{C'}(\varphi)\subseteq N_C(f)\cap O(R,m)$ for $C'=D\wedge E$. Since we can prove $\bigcap_{m=1}^{\infty}O(R,m)\neq\emptyset$ in the same way, we get $\varphi(p)\in\bigcap_{m=1}^{\infty}O(R,m)$, which topologically maps R on $\varphi(R)$ of $(n+1)$-dimensional density zero.

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Addendum. Recently we have proved by applying Theorem 3 that every metric space can be topologically imbedded in a product of an enumerable number of metric spaces R_i of $\dim R_i\leqslant 1$ $(i=1,2,\ldots)$. See *On imbedding a metric space in a product of one-dimensional spaces*, Proc. of Japan Acad. 33 (1957), p. 445-449.